A fair rule in minimum cost spanning tree problems[1]

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Abstract

We study minimum cost spanning tree problems and define a cost sharing rule that satisfies many more properties than other rules in the literature. Furthermore, we provide an axiomatic characterization based on monotonicity properties.

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1   Introduction

Many problems involving network formation have been studied in operations research and economics literature. Two particular issues have been extensively explored in operations research, namely efficient algorithm designs and computational complexity, whereas the economic literature focuses on aspects such as cost sharing within networks and the design of mechanisms which attempt to explain how networks are formed.

In this paper we focus on the cost sharing aspect. Our contribution can be considered against the background of the well-known literature on cost allocation. We assume that there are no external forces (for example, the market) which determine final allocation. Agreements can be reached directly between individual agents, or indirectly by leaving the final decision to a neutral referee. In both cases the important issue is to achieve a "fair allocation" of cost.

In particular we study minimum cost spanning tree problems ($mcstp$). Consider a group of agents located at different geographical points who want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

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There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network. This example appears in Dutta and Kar (2004). Bergantiños and Lorenzo (2004) studied a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Other examples include communication networks, such as telephone, Internet, or cable television.

The literature on *mcstp* starts by defining algorithms for constructing minimal cost spanning trees (*mt*). We can mention, for instance, the papers of Kruskal (1956) and Prim (1957). However, constructing an *mt* is only part of the problem. Another important issue is how to allocate the cost associated with *mt* among agents.

Bird (1976) associated a cooperative game with any *mcstp*. Moreover, for cases when the *mcstp* has a single *mt*, Bird proposed a rule called the Bird rule. Granot and Huberman (1981, 1984) studied the core and the nucleolus of this cooperative game. Sharkey (1995) has surveyed most of this literature. More recently, Kar (2002) studied the corresponding Shapley value (which we denote as K); Dutta and Kar (2004) extended the Bird rule to more than one *mt* (an extension we denote as B), and proposed a new rule (which we denote as DK).

We will now discuss the allocation proposed by K, B, and DK in a very simple example. The first non-trivial case in *mcstp* occurs when two agents wish to be connected to the source and the optimal choice is for one of the agents to connect through the other. The following example describes a particular case of such a situation.

Example 1.1 *There are two agents. The connection cost between agent* 1 *and the source is* 10, *between agent* 1 *and* 2 *is* 2, *and between agent* 2 *and the source is* $10+x$,

where $x \geq 0$. This situation can be represented by the following figure:
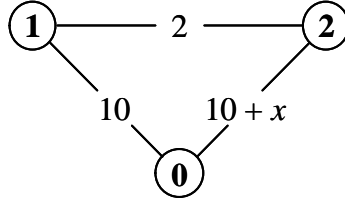
where 0 is the source.

If $x = 0$, the agents are symmetric. The three rules propose symmetric allocation $(6, 6)$.

If $x > 0$, the agents are asymmetric. The unique mt is $\{(0, 1), (1, 2)\}$. We can proceed in one of two ways. First of all, we can ignore x because the arc $(0, 2)$ will not be constructed. Hence, $(6, 6)$ is still valid. Secondly, since the problem is asymmetric we can use the information provided by x. Hence, $(6, 6)$ is wrong.

The three rules proceed according to the second alternative. The following table shows the allocation proposed by each of the rules when $x > 0$:

	Agent 1	Agent 2
B	10	2
DK	2	10
K	$6 - \frac{x}{2}$	$6 + \frac{x}{2}$

B and DK propose the same allocation independently of x (as long as $x > 0$). Moreover, minor changes in the cost of the arc $(0, 2)$ produce major changes in the proposal. We consider this to be unfair, and claim that the rule should be a continuous function of the cost.

K is a continuous function of the cost. However, if we take $x = 100$, then $K = (-44, 56)$. This means that agent 2 pays 44 units to agent 1 in addition to the cost of the network. Again, we believe that this allocation is unfair. We claim that the rule should be positive, *i.e.* agents should not make a profit on the transaction.

Our conclusion, in Example 1.1, is that $(6, 6)$ is a better allocation than those proposed by K, B, and DK, even when the problem is asymmetric. Note that $(6, 6)$

can be obtained through a two-stage procedure. First of all, we argue that when $x > 0$, the allocation should coincide with the allocation when $x = 0$. Secondly, we argue that $(6, 6)$ is the correct allocation when $x = 0$.

Something similar happens with bargaining problems. Consider the bargaining problems (d, S) and (d, S') where $d = (0, 0)$, $S = \{(x_1, x_2) : x_1 + x_2 \leq 1\}$, and $S' = \{(x_1, x_2) : x_1 + x_2 \leq 1 \text{ and } x_1 \leq 0.5\}$. Even though both problems are different – because of the property of independence of irrelevant alternatives – the Nash solution to both problems is $(0.5, 0.5)$. In Example 1.1 we consider x to be irrelevant.

In this paper we generalize this idea to the entire class of *mcstp*. Given an *mcstp* modelled by a matrix C, we first associate a matrix C^* with C; secondly, we compute an allocation in C^*; and thirdly, we define the allocation in C as the allocation obtained in C^*.

Take an *mcstp* defined through a matrix C. Given an *mt t*, Bird (1976) defined the minimal network associated with C and t. It is known that this minimal network does not depend on the chosen *mt*. Hence, it makes sense to define the matrix C^*, referred to as an *irreducible matrix*, as the minimal network associated with some *mt t*.

We introduce a procedure to associate a corresponding irreducible matrix C^* with each arbitrary matrix C. In Propositions 3.1, 3.2, and 3.3 we present new results regarding irreducible matrices and the procedure. These results will be crucial to the rest of the proofs in this paper.

In Proposition 3.4 we prove that B and K coincide in irreducible matrices. Thus, we define the rule φ in the matrix C as the Bird rule (or the Kar rule) of its irreducible matrix C^*.

Our next step is to explain why φ is a fair rule. We draw up a list of "basic properties" and prove that φ satisfies many more basic properties than the other three rules. The list of basic properties includes, from our point of view, those properties that provide the best way to proceed with this type of problems. For instance, assume that two agents are symmetric. The best thing that a fair rule can

do is to allocate the same cost to both agents. The list of basic properties include properties which have already been used in the existing literature on *mcstp*, and others introduced in this paper.

Our final step is to present two characterizations of φ. If we restrict ourselves to irreducible problems, φ is the only rule satisfying *Symmetry* (*SYM*) and *Independence of Other Costs* (*IOC*). *IOC* says that the amount paid by an agent only depends on the cost of the arcs to which he belongs.

We also provide a characterization of φ for the entire class of *mcstp*. This characterization uses *Solidarity* (*SOL*), *Population Monotonicity* (*PM*) and *Equal Share of Extra Costs* (*ESEC*).

SOL says that if a network connection cost increases, no agent should pay less.

PM says that if we add new agents, no agent will be worse off.

The idea of *ESEC* is the following: consider a problem where the most expensive connection cost for any agent is the cost of connecting to the source. Moreover, the connection cost to the source is the same for all agents. Assume that this connection cost increases by $x > 0$. *ESEC* says that if agent i pays f_i in the original problem, he must pay $f_i + \frac{x}{n}$ when the cost increases (where n is the number of agents).

SOL and *PM* are standard properties often used often in economic models. We believe that these properties are very natural and that any fair rule should satisfy both. *ESEC* is a property defined explicitly for *mcstp*. We believe that it is a natural property with a clear meaning. We do not claim that every fair rule should satisfy *ESEC*. However, we do see it as a property that selects a rule from among the set of "fair rules" (rules that satisfy *SOL* and *PM*).

Feltkamp, Tijs and Muto (1994) introduced a rule for *mcstp* called the *Equal Remaining Obligations* rule (*ERO*). They introduced *ERO* through Kruskal's algorithm. In Bergantiños and Vidal-Puga (2004b) we proved that φ coincides with *ERO*, and moreover, presented other alternative definitions for φ.

This rule has been studied in other papers. Brânzei, Moretti, Norde and Tijs (2004) and Bergantiños and Vidal-Puga (2004a) obtained other axiomatic charac-

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terizations of φ using an additivity property. On the other hand, in Bergantiños and Vidal-Puga (2006c) we proved that φ is the Shapley value of the cooperative game (N, v^+) where $v^+(S)$ represents the cost of connecting agents of S to the source and assuming that the rest of agents are already connected. Moreover, in Bergantiños and Vidal-Puga (2006a) we proved that φ can be obtained as the equilibrium payoff of a non-cooperative game.

Our paper is organized as follows. In Section 2 we introduce the *mcstp*, along with the rules and properties considered in the paper. In Section 3 we introduce the rule φ and study the irreducible form of an *mcstp*. In Section 4 we study the properties satisfied by φ and provide the axiomatic characterizations. Finally, in Section 5 we briefly comment on some of the results obtained for φ in other papers. Most of the proofs are in Section 6 (Appendix).

2 The minimum cost spanning tree problem

This section is divided into three subsections. In the first subsection, we introduce the problem. In the second subsection, we introduce some rules of the literature, and finally, in the third subsection, we present some properties of the rules.

2.1 The problem

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite subset $N \subset \mathcal{N}$, let Π_N denote the set of all orders in N. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.*

$$ Pre(i, \pi) = \{ j \in N \mid \pi(j) < \pi(i) \} . $$

For notational simplicity, given $\pi \in \Pi_N$, we denote the agent $i \in N$ with $\pi(i) = s$ as π_s. Moreover, given $\pi \in \Pi_N$ and $S \subset N$, let π_S denote the order induce by π among agents in S.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$. Our interest lies on networks where each node in N is (directly or indirectly) connected to the source.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ in N represents the cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for all $i, j \in N_0$ and that $c_{ii} = 0$ for all $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given $C, C' \in \mathcal{C}^N$ we say that $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimal cost spanning tree problem*, more briefly referred to as an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

A *network* g over N_0 is a subset of $\{(i, j) \mid i, j \in N_0\}$. The elements of g are called *arcs*.

Given a network g and a pair of different nodes i and j, a *path* from i to j (in g) is a sequence of different arcs $\{(i_{s-1}, i_s)\}_{s=1}^p$ that satisfy $(i_{s-1}, i_s) \in g$ for all $s \in \{1, 2, ..., p\}$, $i = i_0$ and $j = i_p$. We say that $i, j \in N$ are *linked* (in g) if there exists a path from i to j which does not include the source. If $(i, j) \in g$, we say that i and j are *directly linked* (in g). We say that the node i is *connected to the source* (in g) if there exists a path from i to the source.

A *tree* is a network where there is a unique path from i to the source for all $i \in N$. If t is a tree, we usually write $t = \{(i^0, i)\}_{i \in N}$, where i^0 represents the first node in the unique path in t from i to the source.

We denote the set of all networks over N_0 as $\mathcal{G}^N$ and the set of networks over N_0 in such a way that every agent in N is connected to the source as $\mathcal{G}_0^N$.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), more briefly referred to as an *mt*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well established in the literature on *mcstp* that an *mt* exists, even though it does not necessarily have to be unique. Given an *mcstp* (N_0, C) we denote the cost associated with any *mt* t in (N_0, C) as $m(N_0, C)$.

Given an *mcstp* (N_0, C), we denote the *mcstp* induced by C in $S \subset N$ as (S_0, C).

Bird (1976) associated a cooperative game (N, v_C) with each *mcstp* (N_0, C) where $v_C(S) = m(S_0, C)$ for each $S \subset N$.

We will now introduce some well-known results of cooperative games which will be used throughout the paper. We introduce them considering the cooperative game as a cost sharing problem.

We define the *core* of the cooperative game (N, v) as

$$core(N, v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \leq v(S), \forall S \subset N \right\}.$$

We say that (N, v) is *concave* if, for all $S, T \subset N$ and $i \in N$ such that $S \subset T$ and $i \notin T$,

$$v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T).$$

The *Shapley value* (Shapley, 1953) of the cooperative game (N, v) is defined as

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi_N} [v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi))].$$

It is well-known that the Shapley value belongs to the core when the cooperative game is concave.

2.2 Rules

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source. We will now briefly introduce some of the rules studied in the literature.

A (*cost allocation*) *rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each *mcstp* (N_0, C) and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

Notice that we implicitly assume that the agents build an *mt*. As far as we know, all the rules proposed in the literature make this assumption.

Given an *mcstp*, Prim (1957) provides an algorithm for solving the problem of connecting all agents to the source such that the total cost of creating the network is minimal. The idea of this algorithm is simple: starting from the source we construct a network by sequentially adding arcs with the lowest cost and without introducing cycles.

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take an arc $(0, i)$ such that $c_{0i} = \min_{j \in N} \{c_{0j}\}$. If there are several arcs satisfying this condition, select just one. Now, $S^1 = \{0, i\}$ and $g^1 = \{(0, i)\}$.

Stage $p + 1$: Assume that we have defined $S^p \subset N_0$ and $g^p \in \mathcal{G}^N$. We now define S^{p+1} and g^{p+1}. Take an arc (j, i) with $j \in S^p$ and $i \in N_0 \backslash S^p$ such that $c_{ji} = \min_{k \in S^p, l \in N_o \backslash S^p} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Now, $S^{p+1} = S^p \cup \{i\}$ and $g^{p+1} = g^p \cup \{(j, i)\}$.

This process is completed in n stages. We say that g^n is a tree obtained following Prim's algorithm. Notice that this algorithm leads to a tree, but that this is not always unique.

We will now introduce three rules from the literature: the Bird rule, the Kar rule, and Dutta-Kar's rule.

The Bird rule (Bird, 1976) and Dutta-Kar's rule (Dutta and Kar, 2004) are defined through Prim's algorithm. We first assume that there is a unique *mt* t.

Given $i \in N$, let i^0 be the first node in the unique path in t from i to the source. The *Bird rule* (B) is defined for each $i \in N$ as

$$B_i(N_0, C) = c_{i^0 i}.$$

The idea of this rule is simple. Agents connect sequentially to the source following Prim's algorithm and each agent pays the corresponding connection cost.

Dutta-Kar's rule (DK) is defined in a more elaborate way. The agents connect to the source via Prims's algorithm, but with a pivotal switch in the allocation cost at each step. See Dutta and Kar (2004) for a formal definition.

Now assume that there is more than one mt. In this case, the Bird rule and Dutta-Kar's rule can be defined as an average of the trees associated with Prim's algorithm.

Dutta and Kar (2004) proceeded as follows. Given $\pi \in \Pi_N$ they defined $B^\pi(N_0, C)$ as the allocation obtained when they applied the previous protocol to (N_0, C) and solved the indifferences by selecting the first agent given by π. Then they defined

$$B(N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} B^\pi(N_0, C).$$

They defined $DK(N_0, C)$ in a similar way.

The game theory approach can also be used for defining rules. Bird (1976) associated a cooperative game (N, v_C) with each $mcstp$ (N_0, C). Later, several authors defined rules using this cooperative game. For instance, Granot and Huberman (1981, 1984) studied the core and the nucleolus, and Kar (2002) studied the Shapley value.

The *Kar rule* (K) is defined as

$$K(N_0, C) = Sh(N, v_C).$$

2.3 Properties

We will now introduce several properties of rules. Some of these properties are known in the literature while others are introduced in this paper.

Given a rule ψ, we consider the following properties:

Core Selection (CS) For all $mcstp$ (N_0, C) and all $S \subset N$, we have

$$\sum_{i \in S} \psi_i(N_0, C) \leq m(S_0, C).$$

This property implies that no group of agents would be better off by constructing their own network instead of paying what the rule ψ proposes for each of them. Notice that CS is equivalent to saying that $\psi(N_0, C) \in core(N, v_C)$.

Cost Monotonicity (CM) For all $mcstp$ (N_0, C) and (N_0, C') such that $c_{ij} < c'_{ij}$ for some $i \in N$, $j \in N_0$ and otherwise $c_{kl} = c'_{kl}$, we have

$$\psi_i(N_0, C) \leq \psi_i(N_0, C').$$

This property implies that if a particular connection cost increases for agent i but the rest of the connection costs remain the same, then agent i cannot be better off.

Solidarity (SOL) For all $mcstp$ (N_0, C) and (N_0, C') such that $C \leq C'$, we have

$$\psi(N_0, C) \leq \psi(N_0, C').$$

This property implies that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off. Notice that SOL demands agents' contribution to move in the same direction irrespective of their locations on minimum cost spanning trees.

This property is called Strong Cost Monotonicity (SCM) in Bergantiños and Vidal-Puga (2006a, 2006b).

Population Monotonicity (PM) For all $mcstp$ (N_0, C), $S \subset N$, and $i \in S$, we have

$$\psi_i(N_0, C) \leq \psi_i(S_0, C).$$

This property implies that if new agents join a "society" no agent from the "initial society" can be worse off.

Continuity (CON) For all $N \subset \mathcal{N}$, $\psi(N_0, \cdot)$ is a continuous function of $\mathcal{C}^N$.

This property implies that minor changes in agents' connection costs cannot lead to major changes in the amount they have to pay.

Positivity (POS) For all $mcstp$ (N_0, C) and all $i \in N$, we have

$$\psi_i(N_0, C) \geq 0.$$

This property implies that agents should not make a profit.

Separability (SEP) For all $mcstp$ (N_0, C) and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$, we have

$$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

Two subsets of agents, S and $N \setminus S$, can be connected to the source either separately or jointly. If there are no savings when they are jointly connected to the source, this property implies that agents will pay the same in both circumstances.

SEP appears in Megiddo (1978), Granot and Huberman (1981), and Granot and Maschler (1998). They used the name *Decomposition*. They studied its relationship with the core and the nucleolus of (N, v_C).

Symmetry (SYM) For all $mcstp$ (N_0, C) and all pair of symmetric agents $i, j \in N$,

$$\psi_i(N_0, C) = \psi_j(N_0, C).$$

We say that $i, j \in N$ are *symmetric* if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

Independence of Other Costs (IOC) For all $mcstp$ (N_0, C) and (N_0, C'), and all $i \in N$ such that $c_{ij} = c'_{ij}$ for all $j \in N_0 \setminus \{i\}$, we have

$$\psi_i(N_0, C) = \psi_i(N_0, C').$$

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This property implies that the amount paid by agent i depends only on the cost of the arcs to which he belongs.

Equal Share of Extra Costs ($ESEC$) Let (N_0, C) and (N_0, C') be two *mcstp*. Let $c_0, c_0' \geq 0$. Assuming $c_{0i} = c_0$ and $c_{0i}' = c_0'$ for all $i \in N$, $c_0 < c_0'$, and $c_{ij} = c_{ij}' \leq c_0$ for all $i, j \in N$, we have

$$\psi_i(N_0, C') = \psi_i(N_0, C) + \frac{c_0' - c_0}{n}$$

for all $i \in N$.

This property is interpreted as follows: a group of agents N faces a problem (N_0, C) in which all of them have the same connection cost to the source $(c_{i0} = c_0)$ and in which this cost is greater than the connection costs between agents $(c_{ij} \leq c_0)$. Under these circumstances, an optimal network implies that any one agent connects directly to the source, and that the rest connect to the source through this agent. Moreover, they agree that the correct solution is $\psi(N_0, C)$. Assume that an error was made and that the connection cost to the source is $c_0' > c_0$. $ESEC$ states that agents should share this extra cost $c_0' - c_0$ equally.

We say that two *mcstp* (N_0, C) and (N_0, C') are *tree-equivalent* if there exists a tree t such that, firstly, t is an *mt* for both (N_0, C) and (N_0, C'), and secondly, $c_{ij} = c_{ij}'$ for all $(i, j) \in t$.

Independence of Irrelevant Trees (IIT) If two *mcstp* (N_0, C) and (N_0, C') are tree-equivalent,

$$\psi(N_0, C) = \psi(N_0, C').$$

Remark 2.1 *Dutta and Kar (2004) defined the property of* Tree Invariance. *This property says that the rule must depends only on the set of mt. Both B and DK satisfy Tree Invariance.*

Notice that if a rule satisfies IIT it also satisfies Tree Invariance. However, Tree Invariance does not imply IIT. This can be easily checked in Example 1.1 by taking $x = 0$ and $x = 100$.

CON, SYM, and POS are standard properties. CS, CM, PM, and SEP already appeared in other papers from the literature on $mcstp$, whereas SOL, IOC, $ESEC$, and IIT are introduced in this paper.

Certain relationships exist between these properties. It is not difficult to see that SOL implies CM and that PM implies CS. The reciprocal statements are false.

PM implies SEP. Let ψ be a rule satisfying PM and $S \subset N$ as in the definition of SEP. Under PM we know that $\psi_i(N_0, C) \leq \psi_i(S_0, C)$ for all $i \in S$ and $\psi_i(N_0, C) \leq \psi_i((N \setminus S)_0, C)$ for all $i \in N \setminus S$. Since $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$, it is not difficult to see that ψ satisfies SEP. The reciprocal is false.

In Section 3 (Proposition 3.5) we prove that SOL implies IIT.

3 The irreducible form

This section is devoted to the study of the irreducible form of an $mcstp$, which already appeared in Bird (1976). The irreducible form has the property that, if we reduce the cost of any arc, then the cost of connecting agents to the source is also reduced.

We obtain new results regarding the irreducible form. We also present a procedure to associate each $mcstp$ with its irreducible form. The procedure and the new results will be crucial in proving the main results of the paper.

We prove that the Bird rule and the Kar rule coincide in irreducible forms. This allows us to define the rule φ for a general $mcstp$ as the Kar rule (or the Bird rule) of the irreducible form of the original problem.

We also study the rules that only depend on the irreducible form. We prove that they coincide with the rules satisfying IIT. Finally, we obtain that if a rule does not only depend on the irreducible form, it does not satisfy SOL. This allows us to argue that if we decide to use the information from an $mcstp$ which is not in the irreducible form, we will almost certainly miss something.

Given an *mcstp* (N_0, C) and an *mt* t, Bird (1976) defined the *minimal network* (N_0, C^t) associated with t as follows: $c_{ij}^t = \max_{(k,l) \in g_{ij}} \{c_{kl}\}$, where g_{ij} denotes the unique path in t from i to j. Bird (1976) used this minimal network to define the irreducible core of an *mcstp*, which is a subset of the core.

Even though this definition is dependent on the choice of *mt* t, it is independent of the chosen t. Proof of this can be found, for instance, in Aarts and Driessen (1993).

We define the *irreducible form* of an *mcstp* (N_0, C) as the minimal network (N_0, C^*) associated with a particular *mt* t. Sometimes we write C^t instead of C^* to indicate the *mt* t. If (N_0, C^*) is an *irreducible form*, we say that C^* is an *irreducible matrix*.

Remark 3.1 *We see that the definition of the irreducible form associated with a particular mt only depends on this mt. Hence, if two mcstp (N_0, C) and (N_0, C') are tree-equivalent, we have $C^* = C'^*$.*

On the other hand, given an mt t in (N_0, C), t is also an mt in (N_0, C^*). Hence, C and C^* are tree-equivalent.

It is well-known that (N_0, C^*) is an irreducible form if and only if, by reducing the cost of an arc, is the cost of connecting agents to the source also reduced. Thus, we have the following result:

Lemma 3.1 *a) For all $i, j \in N_0$ there exists an mt t in (N_0, C^*) such that $(i, j) \in t$.*
  b) For all mcstp (N_0, C), $C^* \leq C$.

In the next proposition we prove that an *mcstp* is irreducible if and only if we can find a "linear tree" such that the direct link between two nodes represents the maximum cost of the arcs that connect them in the linear tree.

Proposition 3.1 (N_0, C^*) *is irreducible if and only if there exists a tree t in (N_0, C^*) that satisfies the following two conditions:*
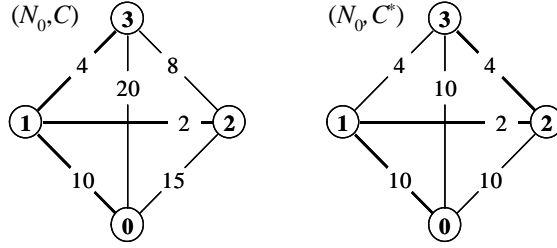
(A1) $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ *where* $\pi_0 = 0$ *(the source).*

(A2) *Given* $\pi_p, \pi_q \in N_0$ *with* $p < q$, $c^*_{\pi_p \pi_q} = \max\limits_{s|p<s\leq q} \left\{ c^*_{\pi_{s-1}\pi_s} \right\}$.

Moreover, t *is an mt.*

Proof. See Appendix. ∎

Example 3.1 *The following figures represent an mcstp* (N_0, C) *with* $N = \{1, 2, 3\}$ *and its associated irreducible form* (N_0, C^*):

In this case, $t = \{(0, 1), (1, 2), (2, 3)\}$ *satisfies* (A1) *and* (A2).

We now introduce a procedure to associate an irreducible matrix $C^* \in \mathcal{C}^N$ with each arbitrary matrix $C \in \mathcal{C}^N$, and an *mt* t that satisfies (A1) and (A2). This procedure will be crucial to most of the proofs of our results.

Let $t' = \{(i^0, i)\}_{i \in N}$ be an *mt* in (N_0, C) and let $\pi \in \Pi_N$. For notational convenience, we denote $\pi_0 = 0$.

We say that the nodes in C *connect to the source via* t' *in the order* π *following Prim's algorithm* if t' is obtained through Prim's algorithm and in stage p, the arc selected is (π_p^0, π_p), for each p. This is the equivalent of stating that $p < q$ for all $\pi_p, \pi_q \in N_0$ such that $\pi_p = \pi_q^0$ and, moreover, for each $\pi_s \in N$,

$$c_{\pi_s^0 \pi_s} = \min_{(p,q)|p<s\leq q} c_{\pi_p \pi_q}. \tag{1}$$

In Example 3.1, the only *mt* is $t' = \{(0, 1), (1, 2), (1, 3)\}$. Moreover, the nodes in C connect to the source via t' following Prim's algorithm only in the order [123]. In C^* there are several *mt*, and the nodes can connect to the source following Prim's algorithm in the orders [123], [213], [312], and [321].

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Assume the nodes in C connect to the source via t' in the order π following Prim's algorithm.

From (1) it is not difficult to check that, given $\pi_p = \pi_q^0$,

$$c_{\pi_s^0 \pi_s} \leq c_{\pi_q^0 \pi_q} \quad \forall s \mid p < s \leq q. \tag{2}$$

We define C' as follows: for all $\pi_p, \pi_q \in N_0$ with $p < q$,

$$\boxed{c'_{\pi_p \pi_q} = \max_{s \mid p < s \leq q} \left\{ c_{\pi_s^0 \pi_s} \right\}.} \tag{3}$$

Hence, for all $\pi_s \in N$,

$$c'_{\pi_{s-1} \pi_s} = c_{\pi_s^0 \pi_s}, \text{ and} \tag{4}$$

$$c'_{\pi_s^0 \pi_s} = c_{\pi_s^0 \pi_s}. \tag{5}$$

Proposition 3.2 *Given a particular mcstp (N_0, C), the matrix C' obtained as above is the irreducible matrix associated with C, i.e. $C' = C^*$. Moreover, $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ is an mt in (N_0, C^*) that satisfies (A1) and (A2).*

Proof. See Appendix. ∎

In the next proposition we provide three properties of irreducible matrices, that will be used frequently throughout the rest of the paper. Parts (a) and (b) are new results. Part (c) is already known.

Proposition 3.3 *Let (N_0, C^*) be an irreducible form and let $\pi \in \Pi_N$ be such that $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ with $\pi_0 = 0$ is an mt in (N_0, C^*) that satisfies (A1) and (A2). Take $S \subset N$. We can assume that $S = \{\pi_{s(1)}, ..., \pi_{s(|S|)}\}$ with $s(q-1) < s(q)$ for all $q \in \{1, ..., |S|\}$ and $s(0) = 0$. Hence:*

(a) $t' = \left\{ \left(\pi_{s(q-1)}, \pi_{s(q)} \right) \right\}_{q=1}^{|S|}$ *is an mt of (S_0, C^*) and $v_{C^*}(S) = \sum_{q=1}^{|S|} c^*_{\pi_{s(q-1)} \pi_{s(q)}}$;*

(b) $v_{C^*}(S) - v_{C^*}\left(S \setminus \left\{ \pi_{s(p)} \right\} \right) = \min \left\{ c^*_{\pi_{s(p-1)} \pi_{s(p)}}, c^*_{\pi_{s(p)} \pi_{s(p+1)}} \right\}$ *if $p < |S|$ and* $v_{C^*}(S) - v_{C^*}\left(S \setminus \left\{ \pi_{s(|S|)} \right\} \right) = c^*_{\pi_{s(|S|-1)} \pi_{s(|S|)}}$;

(c) (N, v_{C^*}) *is concave.*

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Proof. See Appendix. ∎

If we compute the rules K, B, and DK in Example 3.1 we obtain $K(N_0, C) = (-0.5, 6.5, 10)$, $B(N_0, C) = (10, 2, 4)$, and $DK(N_0, C) = (2, 4, 10)$. Moreover, $K(N_0, C^*) = B(N_0, C^*) = (5, 5, 6)$ and $DK(N_0, C^*) = (4, 4, 8)$. Hence, B and K coincide in this example for the irreducible form. We now prove that this result holds in general.

Proposition 3.4 *If (N_0, C^*) is an irreducible form, $K(N_0, C^*) = B(N_0, C^*)$.*

Proof. See Appendix. ∎

Proposition 3.4 results in the following definition:

Definition 3.1 *Given an mcstp (N_0, C) we define the rule φ as*

$$\varphi(N_0, C) = K(N_0, C^*) = B(N_0, C^*)$$

where C^* is the irreducible matrix associated with C.

As already mentioned in the discussion in relation to Example 1.1, we define the rule through the irreducible form. Moreover, we define it as K and B because the fact that these rules, which are very different in general, coincide in irreducible matrices might be of significance. In the next section we will provide more arguments that justify this approach.

We say that a rule ψ *depends only on the irreducible form* if and only if $\psi(N_0, C) = \psi(N_0, C^*)$ for all *mcstp* (N_0, C). In the next proposition we study this class of rules (notice that φ is one of them).

Proposition 3.5 *(a) A rule ψ depends only on the irreducible form if and only if ψ satisfies IIT.*
(b) SOL implies IIT.

Proof. See Appendix. ∎

When we define a rule through the irreducible form there is certain important information regarding the problem that we choose to ignore. We also believe that something is missing when we use information that is not in the irreducible form. Proposition 3.5 provides a strong argument supporting this. Any rule that does not depend on the irreducible form does not satisfy SOL either.

From our point of view, an interesting issue is whether it is possible to define a rule such that: (i) it does not depend on the irreducible form; (ii) it satisfies good properties; and (iii) no rule depending only on the irreducible form satisfies these properties. In the next section we will see that this is not a trivial question. In particular, no rule has yet been studied that satisfies (i)-(ii)-(iii).

4   Properties and axiomatic characterizations

In this section we prove that φ satisfies all the properties stated in the paper but IOC. If we compare φ with the other rules, it satisfies many more properties.

We also present an axiomatic characterization in irreducible problems and an axiomatic characterization in the general class of $mcstp$.

Lemma 4.1 (a) *No rule satisfies IOC.*

(b) φ *satisfies IOC in irreducible problems.*

Proof. See Appendix. ∎

We now prove a property of irreducible matrices, which will be used later.

Lemma 4.2 *Given $C, C' \in \mathcal{C}^N$ such that $C \leq C'$, then $C^* \leq C'^*$.*

Proof. See Appendix. ∎

In the next theorem we prove that φ satisfies all the properties mentioned in the paper but IOC.

Theorem 4.1 (a) φ satisfies CS, CM, SOL, PM, CON, POS, SEP, SYM, $ESEC$, and IIT.

Proof. See Appendix. ∎

In the next table we summarize the properties satisfied by the above-mentioned rules. Some of the results for K, B, and DK are well-known in the literature while others can be found in Bergantiños and Vidal-Puga (2006b).

	K	B	DK	φ
CS	no	YES	YES	YES
CM	YES	no	YES	YES
PM	no	no	no	YES
CON	YES	no	no	YES
POS	no	YES	YES	YES
SEP	no	no	no	YES
SYM	YES	YES	YES	YES
IOC	no	no	no	no
SOL	no	no	no	YES
$ESEC$	no	no	no	YES
IIT	no	no	no	YES

This table clearly shows that φ satisfies many more properties than the other rules.

From our point of view, this table contains two kinds of properties. Some of these properties are what we referred to in the introduction as basic properties. These are CS, CM, PM, CON, POS, SEP, and SYM. We call them basic properties because the statements pertaining to these properties propose the best thing that a fair rule should do in particular circumstances.

IOC could also be considered a basic property. However, no rule satisfies it.

SOL proposes a reasonable way to proceed, but in some cases other ways are also reasonable. For instance, in the first example, SOL implies that the solution to the

problem with $x = 0$ and $x = 80$ should be the same. This means that the solution in which agent 1 pays 5 (half of the cost of arc $(0,1)$) and agent 2 pays 7 (half of the cost of arc $(0,1)$ plus the cost of the arc $(1,2)$) is not possible under SOL.

$ESEC$ also proposes a reasonable way to proceed (to divide the extra cost equally among the agents). However, we do not claim that this is clearly the best way to proceed. For instance, we may also find it reasonable to divide this extra cost in proportion to what the agents paid before this cost arose.

IIT is computationally nice in the sense that it makes it easier to compute the rule. Note that, under IIT, we only need to know an mt in order to compute the rule.

K, B, and φ coincide in irreducible problems. In general, however, they are different. B and K use information which is not in the irreducible form, unlike φ.

This table, together with Proposition 3.5, would suggest that we should be careful when defining rules which do not depend exclusively on the irreducible form.

We now present two characterizations of φ. The first only applies to the subclass of irreducible forms.

Proposition 4.1 *In the class of irreducible forms, φ is the unique rule that satisfies SYM and IOC.*

Proof. See Appendix. ∎

The properties used in Proposition 4.1 are independent. The equal division rule $Ed_i (N_0, C) = \frac{m(N_0, C)}{n}$ for all $i \in N$ satisfies SYM but fails IOC. B^{π^N}, where π^N is the order induced in N by the index of the agents (i.e. $\pi_p^N < \pi_q^N$ if and only if $p < q$), satisfies IOC but fails SYM.

In the next theorem we provide a characterization of φ in the class of all $mcstp$.

Theorem 4.2 *φ is the unique rule satisfying IIT, SEP, and $ESEC$.*

Proof. See Appendix. ∎

Since SOL implies IIT, PM implies SEP, and φ satisfies PM and SOL, the following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.1 *φ is the only rule that satisfies SOL, PM, and $ESEC$.*

Even though Corollary 4.1 is a trivial consequence of Theorem 4.2, we state it explicitly because we believe that PM and SOL are more appealing properties than SEP and IIT.

Theorem 4.2 and Corollary 4.1 are tight characterization results. Let ψ be defined as

$$\psi_i(N_0, C) = \frac{1}{|\Pi'_N|} \sum_{\pi \in \Pi'_N} \left[v_{C^*}\left(Pre\left(i, \pi\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi\right)\right) \right]$$

for all $i \in N$, where Π'_N is the subset of orders in which the agents with the cheapest cost to the source connect first, *i.e.*

$$\Pi'_N = \left\{ \pi \in \Pi_N \mid c_{0\pi_q} \leq c_{0\pi_p} \text{ when } q < p \right\}.$$

This rule satisfies PM (and hence SEP) and $ESEC$, but fails IIT (and hence SOL).

The equal division rule satisfies SOL (and hence IIT) and $ESEC$, but fails SEP (and hence PM).

Let $\psi(N_0, C) = B^{\pi^N}(N_0, C^*)$. This rule satisfies SOL (and hence IIT) and PM (and hence SEP), but fails $ESEC$.

5 Other results for φ

Kruskal (1956) introduced an algorithm for computing the mt of an $mcstp$. Feltkamp, Tijs and Muto (1994) defined the ERO rule through Kruskal's algorithm. Initially, each agent has an obligation of 1 and the network is empty. Applying Kruskal's

algorithm, the obligation of each agent decreases when for each arc added to the network. This obligation is $\frac{1}{n_i}$, where n_i is the number of agents linked to agent i. At each step of the algorithm, each agent pays the proportion of the cost of the additional arc resulting from the difference between his obligation before the arc was added, and his obligation after the arc was added. See Feltkamp *et al* (1994) for a formal definition.

In Bergantiños and Vidal-Puga (2004b) we proved that φ coincides with ERO. Moreover, two other definitions of φ were presented. In the first definition we proposed a method for dividing the cost of an mt t among the agents, taking into account the position of agents in t. We proved that this procedure is independent of the chosen mt, and that the final allocation coincides with that proposed by φ.

B and DK assign the whole cost of each arc to one agent following some specific protocol. Thus, we can consider B and DK to be rules assigning indivisible goods (cost of the arcs). This procedure can lead to unfair allocations when x is very small, as can be seen in Example 1.1. A classical way of ensuring fairness in an order-dependent allocation is to take the average over the set of all orders. In general, this approach is incompatible with efficiency. Nevertheless, if (N_0, C) is a general problem, it is possible to generate an efficient and fair allocation for C by averaging over the orders in C^*. This is the second definition in Bergantiños and Vidal-Puga (2004b).

Additivity is quite a standard property in the literature of cost allocation. The natural formulation of additivity in $mcstp$ is $\psi(C + C') = \psi(C) + \psi(C')$ for all C and C' matrices. This property, however, is very demanding and no rule satisfies it. Brânzei, Moretti, Norde, and Tijs (2004) and Bergantiños and Vidal-Puga (2004a) claimed additivity only for some subclasses of problems. Brânzei *et al* (2004) characterized φ with an additivity property, *Equal Treatment* (which is different from SYM) and *Upper Bound Contributions*. Bergantiños and Vidal-Puga (2004a) characterized the rule with a similar additivity property, PM, and SYM.

Bird (1976) associated the cooperative game (N, v_C) with each $mcstp$ (N_0, C).

It should be noted that $v_C(S)$ is the cost of connecting agents in S assuming that agents of $N \backslash S$ are not present. In Bergantiños and Vidal-Puga (2006c) we associated a different cooperative game (N, v^+) with each $mcstp$ (N_0, C), where $v^+(S)$ is the cost of connecting agents in S assuming that agents of $N \setminus S$ are already connected to the source. We proved that φ is the Shapley value of (N, v^+).

In Bergantiños and Vidal-Puga (2006a), we proved that φ can be obtained as the equilibrium payoff in a non-cooperative game.

6   Appendix

In this section we prove the results stated in the paper.

6.1   Proof of Proposition 3.1 and Proposition 3.2

In Claim 1 we prove the first part of Proposition 3.1. In Claim 2 we prove Proposition 3.2. In Claim 3 we prove the second part of Proposition 3.1.

Claim 1. If C^* and t satisfy $(A1)$ and $(A2)$, then C^* is irreducible.

We prove that we cannot reduce the cost of an arc without reducing $m(N_0, C^*)$. We assume wlog that $\pi_s = s$ for all $s = 0, 1, ..., n$. This means that $t = \{(i-1, i)\}_{i=1}^{n}$.

Take $k, k' \in N_0$ $(k < k')$ and $C' \in \mathcal{C}^N$ such that $c'_{kk'} < c^*_{kk'}$, otherwise $c'_{ij} = c^*_{ij}$. We need to prove that $m(N_0, C') < m(N_0, C^*)$. It is enough to prove that there exists a tree t' such that $c(N_0, C', t') < m(N_0, C^*)$.

Since C^* and t satisfy $(A1)$ and $(A2)$, using Prim's algorithm we can deduce that t is an mt.

Under $(A2)$, $c^*_{kk'} = c^*_{(i-1)i}$ for some i with $k < i \leq k'$. We define $t' = (t \setminus \{(i-1, i)\}) \cup \{(k, k')\}$. Thus, t' is a tree and

$$
\begin{aligned}
c(N_0, C', t') &= m(N_0, C^*) - c^*_{(i-1)i} + c'_{kk'} \\
&= m(N_0, C^*) - c^*_{kk'} + c'_{kk'} \\
&< m(N_0, C^*).
\end{aligned}
$$

Hence, C^* is irreducible.

Claim 2. Proposition 3.2.

We first prove that t satisfies both $(A1)$ and $(A2)$.

$(A1)$ Trivial.

$(A2)$ Given $\pi_p, \pi_q \in N_0$ with $p < q$,

$$c'_{\pi_p \pi_q} \overset{(3)}{=} \max_{s | p < s \leq q} \left\{ c_{\pi_s^0 \pi_s} \right\} \overset{(4)}{=} \max_{s | p < s \leq q} \left\{ c'_{\pi_{s-1} \pi_s} \right\}.$$

Under Claim 1, C' is an irreducible matrix and t is an mt in C'.

By (4) and (5),

$$c \left(N_0, C', t' \right) = \sum_{s=1}^{n} c'_{\pi_s^0 \pi_s} = \sum_{s=1}^{n} c'_{\pi_{s-1} \pi_s} = c \left(N_0, C', t \right).$$

Hence, t' is an mt in C' which satisfies that $c'_{i^0 i} = c_{i^0 i}$ for all $i \in N$. By Remark 3.1, $C^* = (C')^* = C'$.

Claim 3. If C^* is irreducible, then there exits an mt t that satisfies $(A1)$-$(A2)$

Under Claim 2, we can find an mt t in $(N_0, (C^*)^*)$ that satisfies $(A1)$ and $(A2)$. But we already know $(C^*)^* = C^*$, which completes the proof.

6.2 Proof of Proposition 3.3

(a) We compute t' following Prim's algorithm in (S_0, C^*). Since t satisfies $(A2)$ and $s(q-1) < s(q)$ for all $q \in \{1, ..., |S|\}$ we deduce that $c^*_{0\pi_{s(1)}} \leq c^*_{0\pi_{s(q)}}$ for all $q \in \{1, ..., |S|\}$. Thus, the arc $\left(0, \pi_{s(1)} \right)$ will be the first arc in Prim's algorithm.

Since t satisfies $(A2)$ we deduce that $c^*_{0\pi_{s(2)}} \leq c^*_{0\pi_{s(q)}}$ and $c^*_{\pi_{s(1)} \pi_{s(2)}} \leq c^*_{\pi_{s(1)} \pi_{s(q)}}$ for all $q \in \{2, ..., |S|\}$. Hence,

$$\min \left\{ c^*_{0\pi_{s(2)}}, c^*_{\pi_{s(1)} \pi_{s(2)}} \right\} \leq \min \left\{ c^*_{0\pi_{s(q)}}, c^*_{\pi_{s(1)} \pi_{s(q)}} \right\}$$

for all $q \in \{2, ..., |S|\}$. Under $(A2)$, $c^*_{\pi_{s(1)} \pi_{s(2)}} \leq c^*_{0\pi_{s(2)}}$. Thus, the arc $\left(\pi_{s(1)}, \pi_{s(2)} \right)$ will be the second arc in Prim's algorithm.

If we continue with this process, following Prim's algorithm, we obtain $t' = \left\{\left(\pi_{s(q-1)}, \pi_{s(q)}\right)\right\}_{q=1}^{|S|}$ as a tree. This means that t' is an mt in (S_0, C^*) and hence,

$$v_{C^*}(S) = m(S_0, C^*) = c(S_0, C^*, t') = \sum_{q=1}^{|S|} c^*_{\pi_{s(q-1)}\pi_{s(q)}}.$$

(b) Assume first $p < |S|$. Under (a), it is easy to see that

$$v_{C^*}(S) - v_{C^*}\left(S \setminus \left\{\pi_{s(p)}\right\}\right) = c^*_{\pi_{s(p-1)}\pi_{s(p)}} + c^*_{\pi_{s(p)}\pi_{s(p+1)}} - c^*_{\pi_{s(p-1)}\pi_{s(p+1)}}.$$

Since t satisfies $(A2)$ we deduce

$$v_{C^*}(S) - v_{C^*}\left(S \setminus \left\{\pi_{s(p)}\right\}\right) = \min\left\{c^*_{\pi_{s(p-1)}\pi_{s(p)}}, c^*_{\pi_{s(p)}\pi_{s(p+1)}}\right\}.$$

Under Part (a), it is trivial to see that

$$v_{C^*}(S) - v_{C^*}\left(S \setminus \left\{\pi_{s(|S|)}\right\}\right) = c^*_{\pi_{s(|S|-1)}\pi_{s(|S|)}}.$$

(c) This is an immediate consequence of (b).

6.3 Proof of Proposition 3.4

Let $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^{n}$ be an mt in C^* satisfying $(A1)$ and $(A2)$. For all $i \in N$, we know that

$$B_i(N_0, C^*) = \frac{1}{n!} \sum_{\pi \in \Pi_N} B_i^{\pi}(N_0, C^*) \text{ and}$$

$$K_i(N_0, C^*) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \left[v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))\right].$$

We prove that $B = K$ using an induction argument over the number of agents. If $n = 1$, it is clear that $B = K$. Assume that $B = K$ when there are fewer than n agents.

We prove it when there are n agents. We first give an outline of the proof.

We consider two cases. If $(0, \pi_1)$ is not the most expensive arc in t (Case I), we divide the $mcstp$ in two problems with less than n agents each. Under separability

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and the induction hypothesis, $B = K$. If $(0, \pi_1)$ is the most expensive arc in t (Case II), the proof is made through the orders. Given $i \in N$, let Π_N^i be the set of orders $\sigma \in \Pi_N$ such that $\sigma_1 = i$. We prove that the average of the vectors marginal contributions over Π_N^i is $c_{0\pi_1}^*$ for agent i and $K_j\left((N \setminus \{i\})_0, C^i\right)$ for $j \neq i$, where $\left((N \setminus \{i\})_0, C^i\right)$ is the irreducible problem obtained from C^* assuming that i is also a source. We also prove that the average of B_i^σ over Π_N^i is $c_{0\pi_1}^*$ for agent i and $B_j\left((N \setminus \{i\})_0, C^i\right)$ for $j \neq i$. Under the induction hypothesis, $B = K$.

Case I *There exists an arc* $(\pi_{p-1}, \pi_p) \in t$ *with* $p > 1$ *such that* $c_{\pi_{p-1}\pi_p}^* \geq c_{0\pi_1}^*$.

We take $S = \{\pi_1, ..., \pi_{q-1}\}$ where $q > 1$ satisfies $c_{\pi_{q-1}\pi_q}^* \geq c_{0\pi_1}^*$ and $c_{\pi_{s-1}\pi_s}^* < c_{0\pi_1}^*$ for all $s \in \{2, ..., q-1\}$. Under condition $(A2)$, this implies $c_{\pi_{q-1}\pi_q}^* = c_{0\pi_q}^*$.

Under Proposition 3.3(a),

$$
\begin{aligned}
v_{C^*}(N) &= \sum_{s=1}^{q-1} c_{\pi_{s-1}\pi_s}^* + \sum_{s=q'}^{n} c_{\pi_{s-1}\pi_s}^* \\
&= v_{C^*}(S) + c_{\pi_{q-1}\pi_q}^* + \sum_{s=q+1}^{n} c_{\pi_{s-1}\pi_s}^* \\
&= v_{C^*}(S) + c_{0\pi_q}^* + \sum_{s=q+1}^{n} c_{\pi_{s-1}\pi_s}^* \\
&= v_{C^*}(S) + v_{C^*}(N \setminus S).
\end{aligned}
$$

Making some computations we can prove that

$$
K_i(N_0, C^*) = \begin{cases} K_i(S_0, C^*) & \text{if } i \in S \\ K_i\left((N \setminus S)_0, C^*\right) & \text{if } i \in N \setminus S. \end{cases}
$$

We now prove that the last expression also holds with B instead of K. We first prove that, given $\sigma \in \Pi_N$,

$$
B_i^\sigma(N_0, C^*) = \begin{cases} B_i^{\sigma_S}(S_0, C^*) & \text{if } i \in S \\ B_i^{\sigma_{N \setminus S}}\left((N \setminus S)_0, C^*\right) & \text{if } i \in N \setminus S. \end{cases}
$$

Take $i, j \in S$, $i \neq j$ and $k \in N \setminus S$. Since C^* is an irreducible matrix, π is the order satisfying $(A1)$ and $(A2)$, and $S = \{\pi_1, ..., \pi_{q-1}\}$, we have

$$c_{ij}^* < c_{0i}^* = c_{0j}^* = c_{0\pi_1}^* \leq c_{\pi_{q-1}\pi_q}^* = c_{0\pi_q}^* \leq c_{ik}^* = c_{jk}^*.$$

Let $j \in S$ be such that $\sigma(j) \leq \sigma(i)$ for all $i \in S \setminus \{j\}$. If we apply Prim's algorithm to (N_0, C^*) solving indifferences by σ we obtain:

- Agent j is the first agent of S to be connected.

  For each $i \in S \setminus \{j\}$ and $k \in N \setminus S$, $c_{0i}^* = c_{0j}^* = c_{0\pi_1}^* \leq c_{ik}^* = c_{jk}^*$ and $\sigma(j) \leq \sigma(i)$ for each $i \in S \setminus \{j\}$. If $j \in S^{p+1}$ and $j \notin S^p$, then $S^p \subset N \setminus S$, which means that agent j is the first agent in S to be connected.

- Agents in $S \setminus \{j\}$ connect to the source through agents in S.

  For each $i \in S \setminus \{j\}$ and $k \in (N \setminus S)_0$, $c_{ij}^* < c_{ik}^*$. Assume that $i \in S^{p+1}$, $i \notin S^p$, and $g^{p+1} = g^p \cup \{(k, i)\}$. Then, $k \in S$.

Now it is not difficult to see that $B_i^\sigma(N_0, C^*) = B_i^{\sigma_S}(S_0, C^*)$ for all $i \in S$.

We now prove that $B_i^\sigma(N_0, C^*) = B_i^{\sigma_{N \setminus S}}((N \setminus S)_0, C^*)$ for all $i \in N \setminus S$. Let i_1 be the first agent of $N \setminus S$ selected when we apply Prim's algorithm to (N_0, C^*) solving indifferences by σ. This means that $c_{0i_1}^* = c_{\pi_{q-1}\pi_q}^*$ and $\sigma(i_1) \leq \sigma(i)$ for all $i \in N \setminus S$ such that $c_{0i}^* = c_{\pi_{q-1}\pi_q}^*$. Thus, i_1 is also the first agent of $N \setminus S$ selected when we apply Prim's algorithm to $((N \setminus S)_0, C^*)$ solving indifferences by $\sigma_{N \setminus S}$. Moreover, $B_{i_1}^\sigma(N_0, C^*) = B_{i_1}^{\sigma_{N \setminus S}}((N \setminus S)_0, C^*) = c_{\pi_{q-1}\pi_q}^*$.

Assume that $i_1, ..., i_r$ are the first agents selected when we apply Prim's algorithm to (N_0, C^*) by σ and $((N \setminus S)_0, C^*)$ by $\sigma_{N \setminus S}$. We also assume that $B_i^\sigma(N_0, C^*) = B_i^{\sigma_{N \setminus S}}((N \setminus S)_0, C^*)$ for all $i \in \{i_1, ..., i_r\}$.

Let i be the $(r+1)th$ selected agent in $N \setminus S$ when we apply Prim's algorithm to (N_0, C^*) solving indifferences by σ. Assume that agent i is selected in Stage $p + 1$. This means that $i \in S^{p+1}$, $i \notin S^p$, and $g^{p+1} = g^p \cup \{(k, i)\}$. If $k \notin S$, then

agent i is also selected when we apply Prim's algorithm to $\left((N \setminus S)_0, C^*\right)$ solving indifferences by $\sigma_{N \setminus S}$. Moreover, $B_i^\sigma(N_0, C^*) = B_i^{\sigma_{N \setminus S}}\left((N \setminus S)_0, C^*\right) = c_{ki}^*$. Assume that $k \in S$. Thus, $c_{ki}^* = c_{\pi_{s-1}\pi_s}^*$ where $s \geq q$ and, hence, $c_{\pi_{s-1}\pi_s}^* \geq c_{\pi_{q-1}\pi_q}^*$. Since $c_{jk}^* < c_{\pi_{q-1}\pi_q}^*$ when $j, k \in S$ and $c_{0k}^* \leq c_{\pi_{q-1}\pi_q}^*$, we deduce that $c_{0i}^* = c_{ki}^*$. Thus, it is possible to select the arc $(0, i)$ instead of the arc (k, i). Now, agent i is also selected when we apply Prim's algorithm to $\left((N \setminus S)_0, C^*\right)$ solving indifferences by $\sigma_{N \setminus S}$ and $B_i^\sigma(N_0, C^*) = B_i^{\sigma_{N \setminus S}}\left((N \setminus S)_0, C^*\right) = c_{0i}^*$.

We now prove that

$$
B_i\left(N_0, C^*\right) = \begin{cases} B_i\left(S_0, C^*\right) & \text{if } i \in S \\ B_i\left((N \setminus S)_0, C^*\right) & \text{if } i \in N \setminus S. \end{cases}
$$

Take $i \in S$ (the case $i \in N \setminus S$ is similar and we omit it).

$$
\begin{aligned}
B_i\left(N_0, C\right) &= \frac{1}{n!} \sum_{\sigma \in \Pi_N} B_i^\sigma\left(N_0, C\right) \\
&= \frac{1}{n!} \sum_{\sigma \in \Pi_N, \sigma_S = \sigma'} \left(\sum_{\sigma' \in \Pi_S} B_i^{\sigma'}\left(S_0, C\right) \right) \\
&= \frac{1}{n!} \frac{n!}{s!} \sum_{\sigma' \in \Pi_S} B_i^{\sigma'}\left(S_0, C\right) = B_i\left(S_0, C\right).
\end{aligned}
$$

Since S and $N \setminus S$ have fewer than n agents each, under the induction hypothesis we deduce that $B\left(N_0, C^*\right) = K\left(N_0, C^*\right)$.

Case II $c_{0\pi_1}^* > c_{\pi_{s-1}\pi_s}^*$ for all $s \in \{2, ..., n\}$.

For each $i \in N$, we have $i = \pi_q$ for some $q \in \{1, ..., n\}$. Since t satisfies $(A2)$,

$$
c_{0i}^* = \max_{s \in \{1, ..., q\}} \left\{ c_{\pi_{s-1}\pi_s}^* \right\} = c_{0\pi_1}^*. \tag{6}
$$

Moreover, given $S \subset N$ with $i \in S$, we define the $mcstp$ $\left(S_i^{-i}, C^i\right)$ as the $mcstp$ where the set of agents is $S^{-i} = S \setminus \{i\}$, the source of this problem is agent i, and the cost matrix is C^* $\left(C^i = C_S^*\right)$.

Moreover, given $\sigma \in \Pi_N$, we denote as σ^{-i} the order induced by σ among agents in N^{-i}.

We now proceed with a series of claims.

Claim 1 *For all $i \in N$ and $j \in N^{-i}$,*

$$B_j \left(N_i^{-i}, C^i \right) = \frac{1}{(n-1)!} \sum_{\sigma \in \Pi_N^i} B_j^\sigma \left(N_0, C^* \right)$$

where $\Pi_N^i = \{ \sigma \in \Pi_N \mid \sigma_1 = i \}$.

Proof. Take $j \in N^{-i}$. Since t satisfies $(A2)$ and $c_{0\pi_1}^* > c_{\pi_{s-1}\pi_s}^*$ for all $s \in \{2, ..., n\}$, we deduce that $c_{ij}^* < c_{0j}^*$. By the definition of Prim's algorithm, for all $\sigma \in \Pi_N^i$,

$$B_j^\sigma \left(N_0, C^* \right) = B_j^{\sigma^{-i}} \left(N_i^{-i}, C^i \right).$$

Note that $\sigma \in \Pi_N^i$ if and only if $\sigma^{-i} \in \Pi_{N^{-i}}$. Hence,

$$B_j \left(N_i^{-i}, C^i \right) = \frac{1}{(n-1)!} \sum_{\sigma \in \Pi_{N^{-i}}} B_j^\sigma \left(N_i^{-i}, C^i \right) = \frac{1}{(n-1)!} \sum_{\sigma \in \Pi_N^i} B_j^\sigma \left(N_0, C^* \right).$$

∎

Claim 2 *For all $S \subset N$ and all $i \in S$,*

$$m \left(S_0, C^* \right) = m \left(S_i^{-i}, C^i \right) + c_{0\pi_1}^*.$$

Proof. Assume $S = \left\{ \pi_{s(1)}, ..., \pi_{s(|S|)} \right\}$ such that $s(q-1) < s(q)$ for all $q \in \{1, ..., |S|\}$ ($s(0) = 0$). Under Proposition 3.3(a), $t' = \left\{ \left(\pi_{s(q-1)}, \pi_{s(q)} \right) \right\}_{q=1}^{|S|}$ is an mt in (S_0, C^*). Under (6), $c_{0\pi_{s(1)}}^* = c_{0\pi_1}^*$.

The graph $t^i = t' \setminus \left\{ \left(0, \pi_{s(1)} \right) \right\}$ is a tree in $\left(S_i^{-i}, C^i \right)$ and $c \left(S_i^{-i}, C^i, t^i \right) = m \left(S_0, C^* \right) - c_{0\pi_1}^*$. Thus, $m \left(S_i^{-i}, C^i \right) \leq m \left(S_0, C^* \right) - c_{0\pi_1}^*$.

Assume $m \left(S_i^{-i}, C^i \right) < m \left(S_0, C^* \right) - c_{0\pi_1}^*$. Thus, there exists a tree $\widehat{t^i}$ in $\left(S_0^{-i}, C^i \right)$ which satisfies $c \left(S_i^{-i}, C^i, \widehat{t^i} \right) < c \left(S_i^{-i}, C^i, t^i \right)$. Hence, $\widehat{t} = \widehat{t^i} \cup \{(0, i)\}$ is a well-defined tree in (S_0, C^*). Under (6),

$$c \left(S_0, C^*, \widehat{t} \right) = c_{0i}^* + c \left(S_i^{-i}, C^i, \widehat{t^i} \right) < c_{0\pi_1}^* + c \left(S_i^{-i}, C^i, t^i \right) = m \left(S_0, C^* \right)$$

which is a contradiction. Thus, $m \left(S_i^{-i}, C^i \right) = m \left(S_0, C^* \right) - c_{0\pi_1}^*$. ∎

Claim 3 *For all $i \in N$ and $j \in N^{-i}$,*

$$K_j \left(N_i^{-i}, C^i \right) = \frac{1}{(n-1)!} \sum_{\sigma \in \Pi_N^i} \left[v_{C^*} \left(Pre\left(j, \sigma\right) \cup \{j\}\right) - v_{C^*} \left(Pre\left(j, \sigma\right)\right) \right].$$

Proof. Under Claim 2, for all $\sigma \in \Pi_N^i$ and $j \in N^{-i}$, we can prove that

$$v_{C^*} \left(Pre\left(j, \sigma\right) \cup \{j\}\right) - v_{C^*} \left(Pre\left(j, \sigma\right)\right) = v_{C^i} \left(Pre\left(j, \sigma^{-i}\right) \cup \{j\}\right) - v_{C^i} \left(Pre\left(j, \sigma^{-i}\right)\right).$$

But $\sigma \in \Pi_N^i$ if and only if $\sigma^{-i} \in \Pi_{N^{-i}}$. Hence, Claim 3 holds. ∎

Claim 4 *For all $i \in N$ and $\sigma \in \Pi_N^i$,*

$$B_i^{\sigma} \left(N_0, C^* \right) = v_{C^*} \left(Pre\left(i, \sigma\right) \cup \{i\}\right) - v_{C^*} \left(Pre\left(i, \sigma\right)\right) = c_{0\pi_1}^*.$$

Proof. It is a trivial consequence of (6). ∎

Claim 5 *For all $j \in N$, $B_j \left(N_0, C^* \right) = K_j \left(N_0, C^* \right)$.*

Proof. Take $j \in N$. Under Claims 1 and 4, making some computations we can prove that

$$B_j \left(N_0, C^* \right) = \frac{1}{n} c_{0\pi_1}^* + \frac{1}{n} \sum_{i \in N \setminus \{j\}} B_j \left(N_i^{-i}, C^i \right).$$

Under Claims 3 and 4, making some computations we can prove that

$$K_j \left(N_0, C^* \right) = \frac{1}{n} c_{0\pi_1}^* + \frac{1}{n} \sum_{i \in N \setminus \{j\}} K_j \left(N_i^{-i}, C^i \right).$$

Under the induction hypothesis we know that $B \left(N_i^{-i}, C^i \right) = K \left(N_i^{-i}, C^i \right)$ for all $i \in N$. Thus, $B_j \left(N_0, C^* \right) = K_j \left(N_0, C^* \right)$. ∎

6.4   Proof of Proposition 3.5

(a) Let (N_0, C) be an *mcstp*. Assume first that a rule ψ satisfies *IIT*. Since C and C^* are tree-equivalent, we have $\psi \left(N_0, C \right) = \psi \left(N_0, C^* \right)$.

Assume now that ψ depends only on the irreducible form. Given that (N_0, C) and (N_0, C') are two tree-equivalent problems, we have to prove $\psi \left(N_0, C \right) = \psi \left(N_0, C' \right)$.

Since (N_0, C) and (N_0, C') are tree-equivalent problems, there exists some mt $t = \{(i^0, i)\}_{i \in N}$ in both (N_0, C) and (N_0, C') such that $c_{i^0 i} = c'_{i^0 i}$ for all $i \in N$. Under Remark 3.1, $C^* = C'^*$. Since ψ depends only on the irreducible form, $\psi(N_0, C) = \psi(N_0, C')$.

(b) If ψ satisfies SOL, $\psi(N_0, C^*) \leq \psi(N_0, C)$ because $C^* \leq C$ (Lemma 3.1(b)). Since $m(N_0, C) = m(N_0, C^*)$, $\psi(N_0, C) = \psi(N_0, C^*)$. Under ($a$), we conclude that ψ satisfies IIT.

6.5   Proof of Lemma 4.1

(a) Let ψ be a rule satisfying IOC. Given $N = \{1, 2\}$, $x > 0$ and $y > 0$ we consider the four $mcstp$ represented by the following figures:
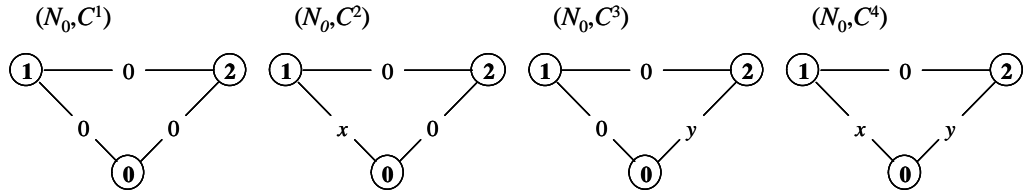
We know that $\psi(N_0, C^1) = (a, -a)$ for some $a \in \mathbb{R}$. Since ψ satisfies IOC, $\psi(N_0, C^2) = \psi(N_0, C^1)$, $\psi(N_0, C^3) = \psi(N_0, C^1)$, $\psi_1(N_0, C^4) = \psi_1(N_0, C^2) = a$, and $\psi_2(N_0, C^4) = \psi_2(N_0, C^3) = -a$. Thus, $\psi(N_0, C^4) = (a, -a)$, which is a contradiction because $m(N_0, C^4) = \min\{x, y\} > 0$.

(b) In Bergantiños and Vidal-Puga (2004b), for any irreducible form (N_0, C^*), we proved that

$$\varphi_i(N_0, C^*) = \frac{1}{n!} \sum_{\pi \in \Pi_N} c^*_{i^\pi i}$$

where $i^\pi \in Pre(i, \pi) \cup \{0\}$ and $c^*_{i^\pi i} = \min\{c^*_{ji} \mid j \in Pre(i, \pi) \cup \{0\}\}$.

Thus, given irreducible $C^*, C'^* \in \mathcal{C}^N$,

$$\varphi_i(N_0, C^*) = \frac{1}{n!} \sum_{\pi \in \Pi_N} c^*_{i^\pi i} = \frac{1}{n!} \sum_{\pi \in \Pi_N} c'^*_{i^\pi i} = \varphi_i(N_0, C'^*).$$

6.6 Proof of Lemma 4.2

Let $k, k' \in N_0$ and $C^\alpha, C^\beta \in \mathcal{C}^N$ such that $c^\alpha_{kk'} = \alpha < \beta = c^\beta_{kk'}$, otherwise $c^\alpha_{ij} = c^\beta_{ij}$. It is enough to prove that $C^{\alpha*} \leq C^{\beta*}$.

Since $C^\alpha \leq C^\beta$ we deduce that $m\left(N_0, C^\alpha\right) \leq m\left(N_0, C^\beta\right)$.

Assume first that there exists some mt t in (N_0, C^α) such that $(k, k') \notin t$. Thus,

$$c\left(N_0, C^\alpha, t\right) = c\left(N_0, C^\beta, t\right) = m\left(N_0, C^\alpha\right).$$

Hence, $m\left(N_0, C^\alpha\right) = m\left(N_0, C^\beta\right)$ and t is an mt in $\left(N_0, C^\beta\right)$. Moreover, $c^\alpha_{ij} = c^\beta_{ij}$ for all $(i, j) \in t$. Under Remark 3.1, this means that $C^{\alpha*} = C^{\beta*}$.

Now assume that for all mt t in (N_0, C^α), $(k, k') \in t$. Let G be the set of trees which do not contain the arc (k, k').

Let $t^G \in G$ such that

$$c\left(N_0, C^\alpha, t^G\right) = \min_{t \in G}\left\{c\left(N_0, C^\alpha, t\right)\right\}.$$

Since $(k, k') \notin t^G$, we deduce that $c\left(N_0, C^\alpha, t^G\right) = c\left(N_0, C^\beta, t^G\right)$.

We define $\gamma = c\left(N_0, C^\alpha, t^G\right) - m\left(N_0, C^\alpha\right)$. Note that, under our hypothesis, $\gamma > 0$. We now distinguish two cases:

Case 1: $\beta - \alpha \leq \gamma$. Let $t = \{(i^0, i)\}_{i \in N}$ be an mt in C^α. We can assume wlog that $k' = k^0$. Since $\beta - \alpha \leq \gamma$, we have that t is also an mt in C^β.

We know that $C^{\alpha*} = (C^\alpha)^t$ and $C^{\beta*} = (C^\beta)^t$. Now, $(C^\alpha)^t \leq (C^\beta)^t$ is a trivial consequence of the definition of minimal network.

Case 2: $\beta - \alpha > \gamma$. Thus, t^G is an mt in $\left(N_0, C^\beta\right)$ but not t.

Let $C'' \in \mathcal{C}^N$ such that $c''_{kk'} = \alpha + \gamma$, otherwise $c''_{ij} = c^\alpha_{ij}$. Note that $C^\alpha \leq C'' \leq C^\beta$. Moreover, C^α and C'' are under the hypothesis of Case 1, and thus $C^{\alpha*} \leq C''^*$.

We know that t^G is an mt in both $\left(N_0, C^\beta\right)$ and (N_0, C''), and $(k, k') \notin t^G$. Thus, $c^\beta_{ij} = c''_{ij}$ for all $(i, j) \in t^G$. This means that $\left(N_0, C^\beta\right)$ and (N_0, C'') are tree-equivalent. Under Remark 3.1, $C''^* = C^{\beta*}$. Thus, $C^{\alpha*} \leq C^{\beta*}$.

6.7 Proof of Theorem 4.1

Independence of Irrelevant Trees (IIT). This is a consequence of Proposition $3.5(a)$.

Solidarity (SOL). Given a $mcstp$ (N_0, C), $S \subset N$, and $i \in S$, under Proposition $3.3(b)$ it is not difficult to deduce that

$$v_{C^*}(S) - v_{C^*}(S \setminus \{i\}) = \min_{j \in S_0 \setminus \{i\}} c_{ij}^*.$$

Let (N_0, C) and (N_0, C') be such that $C \leq C'$. Under Lemma 4.2, $C^* \leq C'^*$. Thus,

$$v_{C^*}(S) - v_{C^*}(S \setminus \{i\}) \leq v_{C'^*}(S) - v_{C'^*}(S \setminus \{i\}).$$

Under the strong monotonicity of the Shapley value (Young, 1985),

$$\varphi(N_0, C) = Sh(N, v_{C^*}) \leq Sh(N, v_{C'^*}) = \varphi(N_0, C').$$

Cost Monotonicity (CM). Since φ satisfies SOL we conclude that φ also satisfies CM.

Population Monotonicity (PM). We must prove that for all $mcstp$ (N_0, C), all $S \subset N$, and all $i \in S$, we have $\varphi_i(N_0, C) \leq \varphi_i(S_0, C)$. It is enough to prove it for $S = N \setminus \{k\}$ for some $k \in N$. We assume wlog that $k = n$.

We first prove the following Claim:

Claim. *Assuming $c_{0n} = \alpha$, $c_{in} = \beta$ for all $i \in N \setminus \{n\}$, and $\beta > \alpha > \max_{i,j \in N_0 \setminus \{n\}} \{c_{ij}\}$, we have*

$$\varphi_i(N_0, C) = \begin{cases} \alpha & \text{if } i = n \\ \varphi_i((N \setminus \{n\})_0, C) & \text{otherwise.} \end{cases}$$

Proof. Let t be an mt in C and let $\pi \in \Pi_N$ be an order in which the agents in N connect to the source via t following Prim's algorithm. Under the hypothesis of the Claim, it is clear that n is the last node to be connected to the source, *i.e.* $n = \pi_n$, and moreover, that $(0, n) \in t$.

On the other hand, it is clear that $t^{-n} = t \setminus \{(0, n)\}$ is also an mt in $((N \setminus \{n\})_0, C)$ and that $\pi_{N \setminus \{n\}}$ is an order in which the agents in $N \setminus \{n\}$ connect to the source via t^{-n} following Prim's algorithm.

Under Proposition 3.2, $\hat{t} = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ is an mt in (N_0, C^*) satisfying $(A1)$ and $(A2)$. Analogously, $\hat{t}^{-n} = \{(\pi_{s-1}, \pi_s)\}_{s=1}^{n-1}$ is an mt in $((N \setminus \{n\})_0, C^*)$ satisfying $(A1)$ and $(A2)$.

Let $(N \setminus \{n\}, v_{-n})$ be the cooperative game associated with $((N \setminus \{n\})_0, C^*)$.

Under Proposition 3.3(a), $v_{C^*}(S) = v_{-n}(S \setminus \{n\}) + \alpha$ if $n \in S$ and $v_{C^*}(S) = v_{-n}(S)$ if $n \notin S$.

Consider the cooperative games (N, w_1) and (N, w_2) where for each $S \subset N$,

$$w_1(S) = \begin{cases} v_{-n}(S \setminus \{n\}) & \text{if } n \in S \\ v_{-n}(S) & \text{if } n \notin S. \end{cases} \quad \text{and}$$

$$w_2(S) = \begin{cases} \alpha & \text{if } n \in S \\ 0 & \text{if } n \notin S. \end{cases}$$

It is not difficult to see that

$$
\begin{aligned}
Sh_i(N, w_1) &= Sh_i(N \setminus \{n\}, v_{-n}) = \varphi_i((N \setminus \{n\})_0, C^*) \text{ if } i \neq n, \\
Sh_n(N, w_1) &= 0, \\
Sh_i(N, w_2) &= 0 \text{ if } i \neq n, \text{ and} \\
Sh_n(N, w_2) &= \alpha.
\end{aligned}
$$

Since $v_{C^*}(S) = w_1(S) + w_2(S)$ for all $S \subset N$, $\varphi(N_0, C) = Sh(N, v_{C^*})$, and the Shapley value is additive, the result is concluded. ∎

Let $\alpha = \max_{i,j \in N_0} \{c_{ij}\} + 1$ and $\beta = \alpha + 1$. Let $C^0 \in \mathcal{C}^N$ be such that $c_{0n}^0 = \alpha$, otherwise $c_{ij}^0 = c_{ij}$. For all $k = 1, ..., n-1$, we define $C^k \in \mathcal{C}^N$ such that $c_{kn}^k = \beta$, otherwise $c_{ij}^k = c_{ij}^{k-1}$.

Take $i \in N \setminus \{n\}$. Since φ satisfies SOL,

$$\varphi_i(N_0, C) \leq \varphi_i(N_0, C^0) \leq \varphi_i(N_0, C^1) \leq .... \leq \varphi_i(N_0, C^{n-1}).$$

Under the Claim,

$$\varphi_i\left(N_0, C^{n-1}\right) = \varphi_i\left((N \setminus \{n\})_0, C^{n-1}\right) = \varphi_i\left((N \setminus \{n\})_0, C\right).$$

Thus, $\varphi_i\left(N_0, C\right) \leq \varphi_i\left((N \setminus \{n\})_0, C\right)$.

Separability (SEP). Since PM implies SEP, φ satisfies SEP.

Core Selection (CS). Since PM implies CS, φ satisfies CS.

Continuity (CON). Clearly, $\varphi\left(N_0, C\right) = (f \circ g \circ h)\left(C\right)$ for all $C \in \mathcal{C}^N$, where $h\left(C\right) = C^*$ and $g\left(C\right) = v_C$ for all $C \in \mathcal{C}^N$, and $f\left(v\right) = Sh\left(N, v\right)$ for all v. Since f, g, and h are continuous functions, φ is also continuous.

Symmetry (SYM). See Bergantiños and Vidal-Puga (2004a).

Equal Share of Extra Costs $(ESEC)$. Let (N_0, C) and (N_0, C') be as in the definition of $ESEC$. It is straightforward to check that both C^* and C'^* also satisfy the conditions in the definition of $ESEC$. Moreover, $c_0^* = c_0$ and $c_0'^* = c_0'$.

It is not difficult to prove that there exists a tree t satisfying $(A1)$ and $(A2)$ in C^* and C'^*. Hence, for any order $\pi \in \Pi_N$, $B_i^\pi\left(N_0, C^*\right) = B_i^\pi\left(N_0, C'^*\right)$ if $i \neq \pi_1$, $B_{\pi_1}^\pi\left(N_0, C^*\right) = c_0$ and $B_{\pi_1}^\pi\left(N_0, C'^*\right) = c_0'$.

Thus, given $i \in N$,

$$
\begin{aligned}
\varphi_i\left(N_0, C'\right) &= \frac{1}{n!} \sum_{\pi \in \Pi_N} B_i^\pi\left(N_0, C'^*\right) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi_N} B_i^\pi\left(N_0, C^*\right) + \frac{(n-1)!\left(c_0' - c_0\right)}{n!} \\
&= \varphi_i\left(N_0, C'\right) + \frac{c_0' - c_0}{n}.
\end{aligned}
$$

Positivity (POS). Given $S \subset N$, under Proposition 3.3(b), $v_{C^*}\left(S\right) - v_{C^*}\left(S \setminus \{i\}\right) \geq 0$ for all $i \in N$. Thus, $\varphi_i\left(N_0, C\right) = Sh_i\left(N, v_{C^*}\right) \geq 0$.

6.8 Proof of Proposition 4.1

Under Lemma 4.1, φ satisfies IOC in irreducible forms. Under Theorem 4.1, φ satisfies SYM.

We now prove uniqueness. Let ψ be a rule satisfying SYM and IOC.

Let (N_0, C^*) be an irreducible form and let $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ be an mt satisfying $(A1)$ and $(A2)$.

Let μ be the number of different costs in C^*, namely

$$\mu = \left| \left\{ c_{ij}^* \mid i, j \in N_0 \right\} \right|.$$

Let $\{x_1, ..., x_\mu\}$ be the μ different costs in C^*. We can assume wlog that $x_1 < ... < x_\mu$. We prove that ψ is unique by induction on μ.

If $\mu = 1$, then $c_{ij}^* = x_1$ for all $i, j \in N_0$. Thus, all agents are symmetric. Under SYM, $\psi_i(N_0, C^*) = \psi_j(N_0, C^*)$ for all $i, j \in N$. Thus, $\psi_i(N_0, C^*) = \frac{m(N_0, C^*)}{n} = \varphi_i(N_0, C^*)$ for all $i \in N$.

Now assume that $\psi = \varphi$ when $\mu < p$. We prove it for $\mu = p$.

Let $N' \subset N$ be the set of agents whose minimal cost is x_1, namely

$$N' = \left\{ i \in N \mid c_{ij}^* = x_1 \text{ for some } j \in N_0 \backslash \{i\} \right\}.$$

We now apply an induction argument to $|N'|$.

If $N' = \{k\}$, we deduce $c_{0k}^* = x_1$ and $c_{ik}^* > x_1$ for all $i \in N \backslash \{k\}$. We define C' as $c_{0k}' = x_2$, otherwise $c_{ij}' = c_{ij}^*$. It is straightforward to check that C' is an irreducible matrix. Moreover, for all $i \in N \backslash \{k\}$ and $j \in N_0$, $c_{ij}' = c_{ij}^*$. Under the induction hypothesis applied to μ, we have that $\psi(N_0, C') = \varphi(N_0, C')$. Under IOC, for all $i \in N \backslash \{k\}$,

$$\psi_i(N_0, C^*) = \psi_i(N_0, C') = \varphi_i(N_0, C') = \varphi_i(N_0, C^*).$$

Thus, $\psi(N_0, C^*) = \varphi(N_0, C^*)$.

Assume $\psi = \varphi$ when $\mu = p$ and $|N'| < n'$. We prove it for $\mu = p$ and $|N'| = n'$. Fix $k \in N'$. We define

$$X_k = \{i \in N \mid c_{ik}^* = x_1\} \cup \{k\}.$$

Clearly, $X_k \subset N'$. We first prove that the agents in X_k are symmetric. Assuming $k = \pi_r$, given $\pi_q \in X_k$ with $r < q$ (the reasoning for $q < r$ is analogous), we have

$$x_1 = c^*_{\pi_r \pi_q} = \max_{s|r < s \le q} \left\{ c^*_{\pi_{s-1} \pi_s} \right\}$$

and thus $\pi_s \in X_k$ for all $s \in \{r, ..., q\}$. From this it is not difficult to deduce that $X_k = \{\pi_s\}_{s=s_0}^{s_1}$ for some s_0, s_1. Moreover, $c^*_{ij} = x_1$ for all $i, j \in X_k$.

Let $\pi_q \in X_k$, and let $\pi_p \in N \backslash X_k$ with $p < s_0$ (the reasoning for $s_1 < p$ is analogous). We have

$$c^*_{\pi_p \pi_q} = \max_{s|p < s \le q} \left\{ c^*_{\pi_{s-1} \pi_s} \right\} = \max_{s|p < s \le r} \left\{ c^*_{\pi_{s-1} \pi_s} \right\} = c^*_{\pi_p \pi_r}$$

and hence π_q and π_r are symmetric.

We define C' as follows: $c'_{ij} = x_2$ if $i, j \in X_k$, otherwise $c'_{ij} = c^*_{ij}$. It is straightforward to check that C' is an irreducible matrix. Moreover, for all $i \in N \backslash X_k$ and $j \in N_0$, $c'_{ij} = c^*_{ij}$. Under the induction hypothesis applied to μ (if $X_k = N'$) or $|N'|$ (if $X_k \subsetneq N'$), we have that $\psi(N_0, C') = \varphi(N_0, C')$. Under IOC, $\psi_i(N_0, C^*) = \varphi_i(N_0, C^*)$ for all $i \in N \backslash X_k$. Since all agents in X_k are symmetric in C^*, for all $i \in X_k$,

$$\psi_i(N_0, C^*) = \frac{m(N_0, C^*) - \sum_{j \in N \backslash X_k} \psi_j(N_0, C^*)}{|X_k|} = \varphi_i(N_0, C^*).$$

6.9   Proof of Theorem 4.2

Under Theorem 4.1, we know that φ satisfies IIT, SEP, and $ESEC$.

We now prove its uniqueness. Let ψ be a rule satisfying these properties. We apply an induction argument over n. If $n = 1$, the result is trivial. Assume the result holds for fewer than n agents. We prove it for n agents.

Since ψ satisfies IIT, under Proposition 3.5(a), we can restrict ourselves to irreducible matrices. Let $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ be an mt in (N_0, C^*) satisfying ($A1$) and ($A2$). Let $\pi_r \in N$ be such that $c^*_{\pi_{r-1} \pi_r} = \max_{s|0 < s \le n} \left\{ c^*_{\pi_{s-1} \pi_s} \right\}$. We have two cases:

1. $r > 1$. Take $S = \{\pi_s\}_{s=1}^{r-1}$. Under Proposition 3.3($a$), we know that $\{(\pi_{s-1}, \pi_s)\}_{s=1}^{r-1}$ is an mt in (S_0, C^*) and $\{(0, \pi_r)\} \cup \{(\pi_{s-1}, \pi_s)\}_{s=r+1}^{n}$ is an mt in $((N \setminus S)_0, C^*)$. Moreover, $c_{0\pi_r}^* = \max\limits_{0 < s \leq r} \{c_{\pi_{s-1}\pi_s}^*\} = c_{\pi_{r-1}\pi_r}^*$. Thus,

$$m(S_0, C^*) + m((N \setminus S)_0, C^*) = \sum_{s=1}^{n} c_{\pi_{s-1}\pi_s}^* = m(N_0, C^*).$$

Under SEP, $\psi_i(N_0, C^*) = \psi_i(S_0, C^*)$ when $i \in S$ and $\psi_i(N_0, C^*) = \psi_i((N \setminus S)_0, C^*)$ when $i \notin S$.

We know that $S \neq \emptyset$ and $N \setminus S \neq \emptyset$ because $r > 1$. Under the induction hypothesis, $\psi(S_0, C^*) = \varphi(S_0, C^*)$ and $\psi((N \setminus S)_0, C^*) = \varphi((N \setminus S)_0, C^*)$.

Since φ satisfies SEP we conclude that $\psi(N_0, C^*) = \varphi(N_0, C^*)$.

2. $r = 1$. Let $p > 1$ be such that $c_{\pi_{p-1}\pi_p}^* = \max\limits_{1 < s \leq n} \{c_{\pi_{s-1}\pi_s}^*\}$. Clearly, $c_{\pi_{p-1}\pi_p}^* \leq c_{0\pi_1}^*$. We define C' as follows: $c_{ij}' = c_{ij}^*$ if $i, j \in N$ and $c_{0i}' = c_{0i}^* - \alpha$ for all $i \in N$, where $\alpha = c_{0\pi_1}^* - c_{\pi_{p-1}\pi_p}^* \geq 0$. Under $ESEC$, for all $i \in N$,

$$\begin{aligned}
\psi_i(N_0, C^*) &= \psi_i(N_0, C') + \frac{\alpha}{n} \text{ and} \\
\varphi_i(N_0, C^*) &= \varphi_i(N_0, C') + \frac{\alpha}{n}.
\end{aligned}$$

It is straightforward to prove that C' is an irreducible matrix satisfying $c_{\pi_{p-1}\pi_p}' = \max\limits_{0 < s \leq n} \{c_{\pi_{s-1}\pi_s}'\}$. Applying Case 1 to C' we conclude that $\psi(N_0, C') = \varphi(N_0, C')$. Hence, $\psi(N_0, C^*) = \varphi(N_0, C^*)$.

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