

# Merge-proofness in minimum cost spanning tree problems\*

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## Abstract

In the context of cost sharing in minimum cost spanning tree problems, we introduce a property called *merge-proofness*. This property says that no group of agents can be better off claiming to be a single node. We show that the sharing rule that assigns to each agent his own connection cost (the Bird rule) satisfies this property. Moreover, we provide a characterization of the Bird rule using merge-proofness.

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## 1 Introduction

Minimum cost spanning tree problems (*mcstp*) are used to model situations where a group of agents, located at different geographical points, want a particular service which is provided by a common supplier, or *source*. The agents will be served through connections which entail some cost. However, agents do not care whether they are connected directly or indirectly to the source. Many economic situations can be modeled in this way, for instance, some houses in a village may want to be connected to a common water source or to a power plant. Other examples include communication networks such as Internet, cable television or telephone.

The optimal network is a *minimum cost spanning tree* (*mt*). An algorithm for building an *mt* is provided by Prim (1957). But building an *mt* is only a part of the problem. Another important issue is how to divide the cost associated with an *mt* among the agents. A *rule* is a mapping that determines, for each problem, a division of the amount to be paid by the agents involved in the problem.

Bird (1976) was the first to associate a coalitional game with any *mcstp*. Moreover, in case the *mcstp* has a unique *mt*, he showed that the rule that assigns to each agent his own connection satisfies core selection, *i.e.* the assignment (known as the *Bird rule*) always belongs to the core of the coalitional game. Other rules derived from solutions in coalitional games, such as the core itself and the nucleolus were further studied by Granot and Huberman (1981, 1984). See Sharkey (1995) for a survey. More recently, additional rules have been studied: Kar (2002) characterized the Shapley value in the context of *mcstp*; Dutta and Kar (2004) proposed and characterized a new rule, and furthermore they characterized the Bird rule using a property of "restricted consistency"; and Bergantiños and Vidal-Puga (2007a) defined

and characterized another rule,  $\varphi$ . On the other hand, Bergantiños and Lorenzo-Freire (2008) showed that  $\varphi$  coincides with a rule previously defined by Feltkamp et al. (1994) and also studied in Bergantiños and Vidal-Puga (2004). This rule has also been characterized in Branzei et al. (2004) and Bergantiños and Vidal-Puga (2007b).

Different rules are usually associated with alternative sets of desired axioms. The aim of the axiomatic characterization is to identify a well-defined set of properties for each rule. It allows to obtain different insights on the principles underlying the rule, and on the type of problems for which it might be suitable.

In this paper, we focus on a new property called *merge-proofness*. The idea behind this property is that there exists a planner who wants to construct a network to connect all the agents to a source. In this kind of situations some agents may have incentives to join in advance in order to be treated as a single agent. A rule satisfies merge-proofness if the agents have no incentives to do so.

This property is related to others that have been studied in other contexts. For instance, a bankruptcy problem describes a situation in which an arbitrator has to allocate a given amount among a group of agents who have claims on it, those claims adding up to more than is available. A property known as *no-advantageous-merging* (O'Neill<sup>1</sup>, 1982) has been studied in this framework. This property means that no group of creditors have incentives to pool their claims and to present themselves as a single creditor. We can find a different type of manipulation in Social Choice, where *group-strategyproofness* ensures that no subset of agents can gain by reporting false preferences.

We show that merge-proofness is a very strong property as no symmetric rule satisfies it. We hence have to restrict ourselves to a smaller class of *mcstp*. On the domain of problems in which all the connecting costs in any *mt* are different or zero, we show that the Bird rule is the only rule satisfying

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<sup>1</sup>O'Neill uses the name *strategy-proofness*.

*merge-proofness*, *no-subsidy*, and *independence of extreme null points*. No-subsidy states that no coalition of agents should subsidize the other agents, paying more than the cost of connecting themselves to the source<sup>2</sup>.

Merge-proofness has been independently studied by Özsoy (2006) under the name *overt-merge-proofness*. On the class of *mcstp* admitting a unique *mt* with all its costs different, Özsoy shows that the Bird rule is the only rule satisfying *merge-proofness*, *no-subsidy*<sup>3</sup> and *tree invariance*. Hence, the critical difference between Özsoy's results and ours is this last property. *Tree invariance* states that if the cost of the connections increases, but there is an *mt* of the original problem that remains the same and its cost does not increase, then each agent should pay the same in the new problem. On the other hand, *independence of extreme null points* states that if an agent is a leaf (extreme point) that connect at null cost to any *mt*, then the remaining agents pay the same if we remove it from the problem<sup>4</sup>.

The paper is organized as follows. In Section 2 we introduce the model and present the properties used in the characterization. In Section 3 we show by an example that merge-proofness is a very strong property in the general framework. We hence have to restrict ourselves to a smaller class of *mcstp*. In Section 4 we present the domain restriction and prove that the Bird rule is well-defined in this domain. In Section 5 we prove that the Bird rule satisfies the properties and we also present the characterization result. In Section 6 we prove that the properties are independent.

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<sup>2</sup>This property is equivalent to the property of core selection in the associated coalitional game. See Bird (1976).

<sup>3</sup>Özsoy calls it *core selection*.

<sup>4</sup>*Tree invariance* and *independence of extreme null points* are different properties. For example, the Dutta-Kar rule satisfies *tree invariance* but not *independence of extreme null points*. On the other hand, consider the rule:  $\varphi_i(N_0, C) = 0$  if  $i$  is an extreme null point, and  $\varphi_i(N_0, C) = Sh_i((N \setminus S)_0, C)$  if  $i$  is not an extreme null point, where  $S$  is the set of extreme null points and  $Sh$  is the Shapley value of  $(N_0, C)$ , as studied by Kar (2002).  $\varphi$  satisfies *independence of extreme null points* but not *tree invariance*.

## 2 The model

Let  $N$  be a finite set of *agents* who want to be connected to a source. Usually, we denote the set of agents as  $N = \{1, 2, \dots, n\}$ . Let  $N_0 = N \cup \{0\}$ , where 0 is the *source*.

A *cost matrix* on  $N_0$ ,  $C = (c_{ij})_{i,j \in N_0}$  gives the cost of direct link between any pair of nodes. We assume that  $c_{ij} = c_{ji} \geq 0$  for each  $i, j \in N_0$ , and  $c_{ii} = 0$  for each  $i \in N_0$ .

We denote the set of all cost matrices on  $N$  as  $\mathcal{C}^N$ . Given two matrices  $C, C' \in \mathcal{C}^N$ , we say  $C \leq C'$  if  $c_{ij} \leq c'_{ij}$  for all  $i, j \in N_0$ .

A *minimum cost spanning tree problem*, briefly *mcstp*, is a pair  $(N_0, C)$  where  $N$  is the set of agents, 0 is the source, and  $C \in \mathcal{C}^N$ .

A *network*  $g$  over  $N_0$  is a subset of  $\{(i, j) : i, j \in N_0\}$ . The elements of  $g$  are called *arcs*. We assume that the arcs are undirected, i.e.  $(i, j)$  and  $(j, i)$  represent the same arc.

Given a network  $g$  and a pair of nodes  $i$  and  $j$ , a *path* from  $i$  to  $j$  in  $g$  is a sequence of distinct arcs  $\{(i_{h-1}, i_h)\}_{h=1}^l$  satisfying  $(i_{h-1}, i_h) \in g$  for all  $h \in \{1, 2, \dots, l\}$ ,  $i = i_0$  and  $j = i_l$ .

A *tree* over  $S \subset N_0$  is a network such that for all  $i, j \in S$  there exists a unique path from  $i$  to  $j$ .

Given a network  $g$ , we say that two nodes  $i, j$  are *connected* in  $g$  if there exists a path from  $i$  to  $j$  in  $g$ .

Let  $\mathcal{G}^N$  denote the set of all networks over  $N_0$ . Let  $\mathcal{G}_0^N$  denote the set of all networks over  $N_0$  such that every node in  $N$  is connected to the source. Let  $\mathcal{T}_0^N$  denote the set of all trees over  $N_0$ . Clearly,  $\mathcal{T}_0^N \subset \mathcal{G}_0^N \subset \mathcal{G}^N$ .

Given  $g \in \mathcal{G}^N$ , we define the *cost* associated with  $g$  in  $(N_0, C)$  as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write  $c(g)$  or  $c(C, g)$  instead of  $c(N_0, C, g)$ .

A *minimum cost spanning tree* for  $(N_0, C)$ , briefly an *mt*, is a tree  $t \in \mathcal{T}_0^N$  such that  $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$ . Given an *mcstp*  $(N_0, C)$ , an *mt* always exists

but it may not be unique. We denote the cost associated with any  $mt$  on  $(N_0, C)$  as  $m(N_0, C)$ .

Given  $S \subset N_0$ , let  $C_S$  denote the matrix  $C$  restricted to  $S$ . We denote the restriction to  $S$  of the *mcstp*  $(N_0, C)$  as  $(S, C_S)$ , and the cost associated with any  $mt$  on  $(S, C_S)$  as  $m(S, C_S)$ , that is,  $m(S, C_S)$  is the cost of connection of the agents in  $S$  among themselves. Note that when  $0 \notin S$ ,  $m(S, C_S)$  does not include the cost of connection to the source.

Given a tree  $t$ , we define the *predecessor set* of a node  $i$  in  $t$  as  $P(i, t) = \{j \in N_0 : j \text{ is in the unique path from } i \text{ to the source}\}$ . We assume that  $i \notin P(i, t)$  and  $0 \in P(i, t)$  when  $i \neq 0$ . For notational convenience,  $P(0, t) = \emptyset$ . The *distance* from node  $i$  to the source in  $t$  is the cardinality of  $P(i, t)$ . The *immediate predecessor* of agent  $i$  in  $t$ , denoted as  $i^0$ , is the node that comes immediately before  $i$ , that is,  $i^0 \in P(i, t)$  and  $k \in P(i, t)$  implies either  $k = i^0$  or  $k \in P(i^0, t)$ . Note that  $P(i^0, t) \subset P(i, t)$  and  $P(i, t) \setminus P(i^0, t) = \{i^0\}$ . The *follower set* of an agent  $i$  in  $t$  is the set  $F(i, t) = \{j \in N : i \in P(j, t)\}$ . The *immediate followers* of agent  $i$  in  $t$ , denoted as  $IF(i, t)$ , is the set of agents that come immediately after agent  $i$ , that is,  $IF(i, t) = \{j \in F(i, t) : j^0 = i\}$ .

Let  $\Pi_N$  denote the set of all orders in  $N$ . Given  $\pi \in \Pi_N$ , let  $P_i^\pi$  denote the set of elements in  $N$  which come before  $i$  in the order given by  $\pi$ , *i.e.*

$$P_i^\pi := \{j \in N : \pi(j) < \pi(i)\}.$$

Notice that, in this case,  $0 \notin P_i^\pi$  since  $0$  is not an agent in  $N$ .

Given  $S \subset N_0$ , we say that  $i, j \in S$ ,  $i \neq j$  are  $(C, S)$ -*connected* (Norde et al., 2004) if there exists a path  $g$  from  $i$  to  $j$  satisfying that  $c_{kl} = 0$  for all  $(k, l) \in g$ .

We say that  $S \subset N_0$  is a  $C$ -*component* if two conditions hold: First, for all  $i, j \in S$ ,  $i$  and  $j$  are  $(C, S)$ -*connected*. Second,  $S$  is maximal, *i.e.* if  $S \subsetneq T$ , there exist  $i, j \in T$ ,  $i \neq j$  such that  $i$  and  $j$  are not  $(C, T)$ -*connected*.

The set of  $C$ -*components* constitutes a partition of  $N_0$ .

There are several algorithms in the literature to construct an *mt*. Prim (1957) provides one. Sequentially, the agents connect, either directly or in-

directly to the source. At each stage, one of the cheapest arcs between the connected and the unconnected agents is added.

**Prim's algorithm**

Let  $S_g^0 = \{0\}$  and  $g^0 = \emptyset$ .

Stage 1: Take an arc  $(0, i)$  such that  $c_{0i} = \min_{j \in N} \{c_{0j}\}$ . Now,  $S_g^1 = \{0, i\}$  and  $g^1 = \{(0, i)\}$ .

Stage  $p$ : Assume we have defined  $S_g^{p-1} \subset N_0$  and  $g^{p-1} \in \mathcal{G}^N$ . We now define  $S_g^p$  and  $g^p$ . Take an arc  $(i, j)$ ,  $i \in S_g^{p-1}$ ,  $j \in N_0 \setminus S_g^{p-1}$ , such that  $c_{ij} = \min_{k \in S_g^{p-1}, l \in N_0 \setminus S_g^{p-1}} \{c_{kl}\}$ . Now  $S_g^p = S_g^{p-1} \cup \{i\}$  and  $g^p = g^{p-1} \cup \{(i, j)\}$ .

This process terminates in  $n$  stages. We say that  $g^n$  is a tree obtained via Prim's algorithm.

This algorithm provides a *mt*. An *mt* may not be unique (when the minimizer arc is not unique), but each *mt* can be obtained via Prim's algorithm.

A (*cost allocation*) *rule* is a function  $\phi$  that assigns to each *mcstp*  $(N_0, C)$  a vector  $\phi(N_0, C) \in \mathbb{R}^N$  such that  $\sum_{i \in N} \phi_i(N_0, C) = m(N_0, C)$ , where  $\phi_i(N_0, C)$  represents the cost assigned to agent  $i$ .

Notice that we implicitly assume that the agents build an *mt*.

Bird (1976) introduced a rule which is defined through Prim's algorithm. He assumed that there is a unique *mt*.

**Definition 2.1** (*Bird, 1976*) *Given an mcstp  $(N_0, C)$  and an mt  $t = \{(i^0, i)\}_{i \in N}$  in  $(N_0, C)$ , the Bird rule (B) is defined as:*

$$B_i^t(N_0, C) = c_{i^0 i}$$

for each  $i \in N$ .

The idea of the Bird rule is quite simple: The agents connect to the source following Prim's algorithm and each agent pays the cost of the adjacent arc in the path to the source.

We now introduce different properties of the rules.

**Definition 2.2** A rule  $\phi$  satisfies no-subsidy if for all mcstp  $(N_0, C)$  and all  $S \subset N$ ,  $\sum_{i \in S} \phi_i(N_0, C) \leq m(S_0, C_{S_0})$ .

This property says that no group of agents can be better off constructing their own network instead of paying what the rule  $\phi$  proposes to them.

Before moving on to the next property, we introduce the concept of an *extreme null point*.

**Definition 2.3** Given an mcstp  $(N_0, C)$  and an mt  $t$  in  $(N_0, C)$ , we say that  $i \in N$  is an extreme point in  $t$  if  $F(i, t) = \emptyset$ .

**Definition 2.4** Given an mcstp  $(N_0, C)$  we say that  $i \in N$ , in all the mt in  $(N_0, C)$ , is an extreme null point if it is an extreme point and moreover  $c_{i0_i} = 0$ .

Dutta and Kar (2004) defined the concept of extreme point. They argued that since node  $i$  is an extreme point, this node is of no use to the rest of the network since no node is connected to the source through node  $i$ .

We argue the same for every extreme null point. Moreover, since  $i$  connects to the source with null cost, it does not increase the total cost of the network. Hence, we can consider that node  $i$  is not beneficial for the rest of agents but neither is a problem for them. So, it seems reasonable that the allocation of the rest of the agents does not change if he connects to the source.

The property that we define states that if agent  $i$  is an extreme null point, no agent  $j$  will pay a different cost in order to include agent  $i$  in the network. Formally:

**Definition 2.5** A rule  $\phi$  satisfies independence of extreme null points if for all mcstp  $(N_0, C)$  and all extreme null point  $i \in N$ ,

$$\phi_j(N_0 \setminus \{i\}, C_{N_0 \setminus \{i\}}) = \phi_j(N_0, C)$$

for all  $j \in N \setminus \{i\}$ .



This property implies that each extreme null point pays zero.

**Remark 2.1** *This property is similar to one defined by Derks and Haller (1999) for TU coalitional games, called “Null Player Out”, which requires that a null player (that is, an individual whose contribution to any coalition is zero) does not influence the utility allocation within the rest of the society<sup>5</sup>.*

We next consider the possibility that a group of agents  $S \subset N$  joins in advance to be treated as a single node  $\alpha \in S$ .

The result is a new problem, called *reduced problem*, where the cost of connection between some node  $i$  in  $N_0 \setminus S$  and  $\alpha$  is the minimal connection cost between node  $i$  and any of the agents in  $S$ . The other costs remain as in the initial problem. Formally,

**Definition 2.6** *Given an mcstp  $(N_0, C)$  and  $\alpha \in S \subset N$ , the reduced problem  $(N_0^{S\alpha}, C^{S\alpha})$  is defined as  $N^{S\alpha} = (N \setminus S) \cup \{\alpha\}$ ,  $c_{ij}^{S\alpha} = c_{ij}$  for all  $i, j \in N_0 \setminus S$ , and  $c_{i\alpha}^{S\alpha} = \min_{j \in S} \{c_{ij}\}$  for all  $i \in N_0 \setminus S$ .*

We introduce a new property in *mcstp*:

**Definition 2.7** *A rule  $\phi$  satisfies merge-proofness if for all mcstp  $(N_0, C)$ ,*

$$\sum_{i \in S} \phi_i(N_0, C) \leq \phi_\alpha(N_0^{S\alpha}, C^{S\alpha}) + m(S, C_S)$$

*for all  $\alpha \in S \subset N$ .*

This property asserts that no group of agents have any incentive to join in advance, paying the cost, to be treated as a single agent.

### 3 Two examples

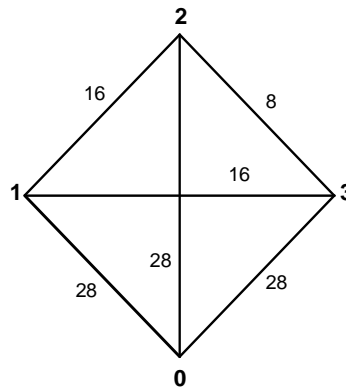
In this section we introduce two examples. The first one illustrates *merge-proofness* and shows that this property is incompatible with *symmetry*. The second example shows that some of the rules in the literature are not *merge-proof*.

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<sup>5</sup>Hamiache (2006) uses the term “Independence of Irrelevant Players”.

**Example 3.1** *There are three agents, 1, 2 and 3. The connection cost between each agent and the source is 28. The connection cost between agents 1 and 2 and between agents 1 and 3 is 16. The connection cost between agents 2 and 3 is 8.*

*This problem is represented in the following figure:*

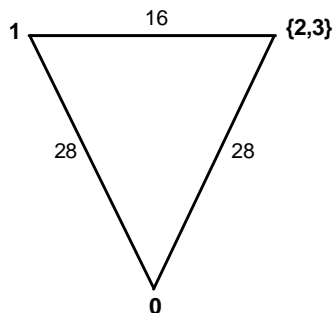


*where 0 is the source.*

*The minimum connection cost is 52 (there exist more than one minimum cost spanning tree).*

*Let  $x_i$  be the assignment that a rule proposes to each agent  $i$ .*

*Suppose now that the agents in  $\{2, 3\}$  join and act as a single one. The resulting problem can be represented as follows<sup>6</sup>:*




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<sup>6</sup>Formally speaking, agent  $\{2, 3\}$  should be replaced by either agent 2 or agent 3.

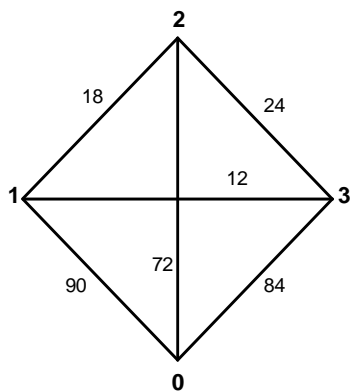
By a symmetry argument, the allocation in this problem should be  $(22, 22)$ . Moreover, since the cost of connecting agents 2 and 3 is 2, under merge-proofness the rule should assign them no more than 30, i.e.  $x_2 + x_3 \leq 22 + 8$ .

Proceeding in the same way with coalitions  $\{1, 2\}$  and  $\{1, 3\}$ , we obtain  $x_1 + x_2 \leq 18 + 16$  and  $x_1 + x_3 \leq 18 + 16$ . However, the minimum connection cost is 52, hence we have an incompatibility.

**Remark 3.1** Even without symmetry, Özsoy (2006, Proposition 2) shows that there exists no merge-proofness rule on the general domain.

It may be argued that this is a special example, because there are more than one possible minimum cost spanning tree. In the literature of *mcstp* it is usual to assume a unique minimum cost spanning tree, or even that there are not two arcs with the same cost (see, for instance, Bird (1976) and Dutta and Kar (2004)). We study this situation next.

**Example 3.2** Consider the problem represented in the following figure:



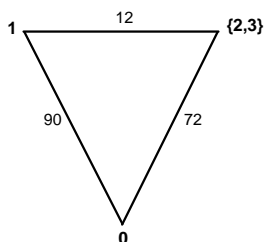
where 0 is the source.

We study the assignment proposed by several rules in the literature. These are given in the following table. The rules with an asterisk are the rules that

satisfy no-subsidy.

	1	2	3
<i>Shapley value (Kar, 2002)</i>	37	28	37
<i>Bird* (1976)</i>	18	72	12
<i>Dutta and Kar* (2004)</i>	12	18	72
<i>Nucleolus* (Granot and Huberman, 1984)</i>	32	32	38
<i>Bergantiños and Vidal-Puga* (2007a)</i>	33	36	33

Assume now that the agents in  $\{2, 3\}$  join and act as one. The resulting problem can be represented as follows:



In this case, the proposal given by each of the previous rules is:

	1	$\{2, 3\}$
<i>Shapley value (Kar, 2002)</i>	51	33
<i>Bird (1976)</i>	12	72
<i>Dutta and Kar (2004)</i>	72	12
<i>Nucleolus (Granot and Huberman, 1984)</i>	51	33
<i>Bergantiños and Vidal-Puga (2007a)</i>	42	42

The question is: Do these rules satisfy merge-proofness? If we compare

the costs that agents 2 and 3 have to pay in both situations, we have

<i>Shapley value (Kar, 2002)</i>	$28 + 37 > 33 + 24$
<b><i>Bird (1976)</i></b>	<b><math>72 + 12 &lt; 72 + 24</math></b>
<i>Dutta and Kar (2004)</i>	$18 + 72 > 12 + 24$
<i>Nucleolus (Granot and Huberman, 1984)</i>	$32 + 38 > 33 + 24$
<i>Bergantiños and Vidal-Puga (2007a)</i>	$36 + 33 > 42 + 24$

Hence, in this example, only the rule defined by Bird (1976) satisfies merge-proofness. We will prove that this result holds in general.

## 4 The domain restriction

In this Section we define the domain restriction on the set of permissible cost matrices that will be used. This restriction is necessary because of the incompatibility presented in Section 3. Additionally, we required the domain to be closed under the operation of merging, *i.e.* if a coalition  $S \ni \alpha$  merges, the resulting problem  $(N_0^{S\alpha}, C^{S\alpha})$  should also belong to the domain.

**Definition 4.1** For each  $C \in \mathcal{C}^N$ , let  $g^*(C)$  be the set of arcs that belong to at least one *mt* in  $C$ . We define  $\mathcal{D} := \{C \in \mathcal{C}^N : \text{no two arcs in } g^*(C) \text{ with positive cost have the same value}\}$ .

**Remark 4.1** Dutta and Kar (2004) define two different domain restrictions. These definitions are the following:

$$\mathcal{C}^1 := \{C \in \mathcal{C}^N : C \text{ induces a unique } mt\},$$

$$\mathcal{C}^2 := \{C \in \mathcal{C}^1 : \text{no two edges of the unique } mt \text{ have the same cost}\}.$$

It is clear that  $\mathcal{C}^2$  is contained in  $\mathcal{D}$ . The domain  $\mathcal{C}^1$  is not contained in  $\mathcal{D}$ . In fact, any domain that contains  $\mathcal{C}^1$  is not closed under the operation of merging. Take  $N = \{1, 2, 3, 4, 5, 6\}$ , and  $C \in \mathcal{C}^1$  defined by  $c_{01} = c_{02} = c_{03} = 28$ ;  $c_{14} = 8$ ;  $c_{25} = c_{36} = 16$  and  $c_{ij} = 30$  otherwise. It is clear that there exists a unique *mt*. After players  $\{2, 4\}$ ,  $\{3, 5\}$  and  $\{1, 6\}$  merge, we have a

problem that is not in  $\mathcal{C}^1$  and, moreover, it coincides with the one presented in Example 3.1 to show the non existence of symmetric, merge-proof rules<sup>7</sup>.

In the next Proposition we state that in the reduced problem the matrix  $C^{S\alpha}$  belongs to  $D$  when  $C$  also belongs to  $D$ .

**Proposition 4.1** *If  $C \in \mathcal{D}$ , then  $C^{S\alpha} \in \mathcal{D}$  for all  $\alpha \in S \subset N$ .*

The proof is straightforward and we omit it.

Recall that Bird (1976) defines the *Bird rule* when there is a unique *mt*. Over  $\mathcal{D}$ , there might exist several *mt*. The next Proposition states that even though this is true, the *Bird* assignment is the same for all of them.

**Proposition 4.2** *Let  $C \in \mathcal{D}$  and let  $t, t'$  be two different *mt* on  $(N_0, C)$ . Then,  $B_i^t(N_0, C) = B_i^{t'}(N_0, C)$  for all  $i \in N$ .*

**Proof.** We will construct  $t$  and  $t'$  following Prim's algorithm.

Since  $C \in \mathcal{D}$ , both trees have the same arcs until one agent connects to the source with null cost. In that case, more than one arc with the same cost may exist. Assume this happens in stage  $p$ . Hence,  $t^{p-1} = t'^{p-1}$  and  $S_i^{p-1} = S_{i'}^{p-1}$ . By definition of the Bird rule,  $B_i^t(N_0, C) = B_i^{t'}(N_0, C)$  for all  $i \in S_i^{p-1} = S_{i'}^{p-1}$ .

Since we have found an arc with null cost, we have a non-trivial *C-component*, say  $S \subset N_0$ . Since all the agents in  $S$ , but the first one, connect with null cost, whatever the order of connection of the agents from the *C-component*, each of them should pay zero under  $t$  and  $t'$ . Hence  $B_i^t(N_0, C) = B_i^{t'}(N_0, C)$  for all  $i \in S$ .

When all the agents in  $S$  are connected to the source, the following arc that connects to the source, if any, has positive cost. Hence, the domain  $\mathcal{D}$  requires that the arcs formed in both trees,  $t$  and  $t'$  be the same again until a new *C-component* appears.

---

<sup>7</sup>Despite the domain in the characterization result cannot contain  $\mathcal{C}^1$ , it is still possible to prove that the Bird rule satisfies merge-proofness in  $\mathcal{C}^1$ . See Özsoy (2006, Proposition 3).

The procedure for the remaining  $C$ -components is analogous because all the agents in the  $C$ -component (but the first one) connect at zero cost, and they pay zero. ■

## 5 The main result

In this section we present a characterization of the Bird rule on  $\mathcal{D}$ . It is the only rule satisfying *no-subsidy*, *independence of extreme null points* and *merge-proofness*. First, in Proposition 5.1, we prove that the *Bird rule* satisfies the mentioned properties.

**Proposition 5.1** *Over the domain  $\mathcal{D}$ , the Bird rule satisfies no-subsidy, independence of extreme null points, and merge-proofness.*

**Proof.**  $B$  satisfies *no-subsidy*. See Bird (1976).

$B$  satisfies *independence of extreme null points*. It is straightforward.

$B$  satisfies *merge-proofness*:

Let  $t = \{(i^0, i)\}_{i \in N}$  be an  $mt$  in the problem  $(N_0, C)$ . Let  $\alpha \in S \subset N$ .

Consider now  $(S, C_S)$  and let  $\tau$  be an  $mt$  in  $(S, C_S)$ , *i.e.*  $c(\tau) = m(S, C_S)$ .

Let  $i^*$  be the first agent in  $S$  that connects to the source following Prim's algorithm in  $t$  (*i.e.*  $S_t^p \cap S = \{i^*\}$  and  $S_t^{p-1} \cap S = \emptyset$ ). Note that  $i^{*0} \in N_0 \setminus S$ .

Given  $t$ , we construct a new network in the problem  $(N_0, C)$ . Let

$$t^* = (t \setminus \{(i^0, i)\}_{i \in S}) \cup \{(i^{*0}, i^*)\} \cup \tau.$$

It is straightforward to check that  $t^*$  is a tree in  $(N_0, C)$ .

Since  $t$  is an  $mt$  in this problem,  $c(t) \leq c(t^*)$ . Hence,

$$c(t) \leq c(t) - \sum_{i \in S} c_{i^0 i} + c_{i^{*0} i^*} + c(\tau)$$

that is,

$$\sum_{i \in S} c_{i^0 i} \leq c_{i^{*0} i^*} + c(\tau) = c_{i^{*0} i^*} + m(S, C_S).$$

Since  $i^*$  is the first agent in  $S$  to be connected,  $c_{i^*0i^*} = \min_{j \in S} \{c_{i^*0j}\} = c_{i^*0\alpha}^{S\alpha}$ . Hence, we can rewrite the above expression as:

$$\sum_{i \in S} c_{i^*0i} \leq c_{i^*0\alpha}^{S\alpha} + m(S, C_S).$$

It is clear that, following Prim's algorithm, we can construct an  $mt$   $t'$  in  $(N_0^{S\alpha}, C^{S\alpha})$  such that  $(i^*0, \alpha) \in t'$ . Hence, by definition of the Bird rule,

$$\sum_{i \in S} B_i(N_0, C) \leq B_\alpha(N_0^{S\alpha}, C^{S\alpha}) + m(S, C_S).$$

■

Now we present a characterization of the Bird rule.

**Theorem 5.1** *On the domain  $\mathcal{D}$ , a rule  $\phi$  satisfies merge-proofness, no-subsidy, and independence of extreme null arcs if and only if  $\phi = B$ .*

**Proof.** We know by Proposition 5.1 that  $B$  satisfies these properties. Let  $t = \{(i^0, i)\}_{i \in N}$  be an  $mt$  in the problem  $(N_0, C)$ . Let  $\phi$  be a rule satisfying *merge-proofness, no-subsidy* and *independence of extreme null points*.

Since  $t$  is an  $mt$ , we have  $\sum_{i \in N} \phi_i(N_0, C) = \sum_{i \in N} c_{i^0i}$ . Hence, it is enough to prove that, for each  $i \in N$ ,  $c_{i^0i} \leq \phi_i(N_0, C)$ .

Let  $F_i = F(i, t)$  and let  $\eta = |IF(i, t)|$ . This means that there are  $\eta$  agents connected directly to agent  $i$  in the  $mt$ , maybe  $\eta = 0$ .

Consider a new problem  $(N_0^\epsilon, C^\epsilon)$  similar to  $(N_0, C)$ , but adding an "imperfect substitute" for agent  $i$ . Formally,  $N_0^\epsilon = N_0 \cup \{\alpha\}$  with  $\epsilon > 0$  sufficiently small,  $c_{jj'}^\epsilon = c_{jj'}$  for all  $j, j' \in N_0$ ,  $c_{i\alpha}^\epsilon = 0$ ,  $c_{j\alpha}^\epsilon = c_{ji} + \epsilon_j$ ,  $\epsilon_j \leq \epsilon$  for all  $j \in IF(i, t)$ , and  $c_{j\alpha}^\epsilon$  large enough for all  $j \in N_0 \setminus (IF(i, t) \cup \{i\})$ . Under these conditions,  $t^\epsilon := t \cup \{(i, \alpha)\}$  is an  $mt$  for  $(N_0^\epsilon, C^\epsilon)$ ,  $c(t^\epsilon) = c(t)$ , and  $\alpha$  is an extreme null point in  $(N_0^\epsilon, C^\epsilon)$ .

Assume the agents in  $F_i \cup \{\alpha\}$  join to be treated as a single node  $\alpha$ . That is, consider the reduced problem<sup>8</sup>  $(N_0^{\epsilon S\alpha}, C^{\epsilon S\alpha})$  with  $S = F_i \cup \{\alpha\}$ .

By definition,  $c_{i\alpha}^{\epsilon S\alpha} = \min_{j \in S} \{c_{ij}^\epsilon\}$ , hence  $c_{i\alpha}^{\epsilon S\alpha} \leq c_{i\alpha}^\epsilon$ . Since  $c_{i\alpha}^\epsilon = 0$ ,  $c_{i\alpha}^{\epsilon S\alpha} = 0$ .

<sup>8</sup>We write  $N_0^{\epsilon S\alpha}$  instead of  $(N_0^\epsilon)^{S\alpha}$  and  $C^{\epsilon S\alpha}$  instead of  $(C^\epsilon)^{S\alpha}$ .



It is straightforward to check that  $t' = \{(j^0, j)\}_{j \in N \setminus F_i} \cup \{(i, \alpha)\}$  is an  $mt$  in  $(N_0^{\epsilon S \alpha}, C^{\epsilon S \alpha})$  and, moreover,  $(i, \alpha)$  is an extreme null point for  $(N_0^{\epsilon S \alpha}, C^{\epsilon S \alpha})$ .

Since  $\phi$  satisfies *merge-proofness*,

$$\sum_{j \in S} \phi_j(N_0^\epsilon, C^\epsilon) \leq \phi_\alpha(N_0^{\epsilon S \alpha}, C^{\epsilon S \alpha}) + m(S, C_S^\epsilon).$$

Under *independence of extreme null points*,  $\phi_\alpha(N_0^{\epsilon S \alpha}, C^{\epsilon S \alpha}) = 0$ . Hence,

$$\sum_{j \in S} \phi_j(N_0^\epsilon, C^\epsilon) \leq m(S, C_S^\epsilon). \quad (1)$$

We study both terms.

Let  $K = N \setminus F_i$ . Note that  $i \in K$  and  $K = N^\epsilon \setminus S$ .

**Claim I**  $\sum_{j \in S} \phi_j(N_0^\epsilon, C^\epsilon) = m(N_0, C) - \sum_{j \in K \setminus \{i\}} \phi_j(N_0^\epsilon, C^\epsilon) - \phi_i(N_0^\epsilon, C^\epsilon)$ .

In the problem  $(N_0^\epsilon, C^\epsilon)$ ,

$$m(N_0^\epsilon, C^\epsilon) = \sum_{j \in K \setminus \{i\}} \phi_j(N_0^\epsilon, C^\epsilon) + \phi_i(N_0^\epsilon, C^\epsilon) + \sum_{j \in S} \phi_j(N_0^\epsilon, C^\epsilon).$$

Clearly,  $m(N_0^\epsilon, C^\epsilon) = m(N_0, C)$  and hence the result.

**Claim II**  $m(S, C_S^\epsilon) \leq m(N_0, C) - m(K_0 \setminus \{i\}, C_{K_0 \setminus \{i\}}^\epsilon) - c_{i^0 i} + \eta \epsilon$ .

Let  $IF_i = IF(i, t)$ . By definition of  $C^\epsilon$ , for an  $\epsilon$  sufficiently small, we can construct an  $mt$   $\tau$  on  $(S, C_S^\epsilon)$  such that the immediate followers of agent  $i$  in  $t$  connect to  $\alpha$  and the rest of agents in  $S$  connect to the same nodes as in  $t$ , *i.e.*  $\tau = \{(j^0, j)\}_{j \in F_i \setminus IF_i} \cup \{(\alpha, j)\}_{j \in IF_i}$  is an  $mt$  on  $(S, C_S^\epsilon)$ .

Hence,  $m(S, C_S^\epsilon) = \sum_{j \in F_i \setminus IF_i} c_{j^0 j}^\epsilon + \sum_{j \in IF_i} c_{\alpha j}^\epsilon$ .

By definition,  $c_{j^0 j}^\epsilon = c_{j^0 j}$  for all  $j \in F_i \setminus IF_i$  and  $c_{\alpha j}^\epsilon \leq c_{ij} + \epsilon$  for all  $j \in IF_i$ . Since  $\eta = |IF_i|$ ,

$$m(S, C_S^\epsilon) \leq \sum_{j \in F_i} c_{j^0 j} + \eta \epsilon. \quad (2)$$

Consider now  $(F_i \cup \{i\}, C_{F_i \cup \{i\}})$ . It is straightforward to check that we

can construct an  $mt$   $t^*$  on  $(F_i \cup \{i\}, C_{F_i \cup \{i\}})$  such that each agent in  $F_i$  connects to the same nodes as in  $t$ , *i.e.*  $t^* = \{(j^0, j)\}_{j \in F_i}$  is an  $mt$  on  $(F_i \cup \{i\}, C_{F_i \cup \{i\}})$ .

Hence,  $m(F_i \cup \{i\}, C_{F_i \cup \{i\}}) = \sum_{j \in F_i} c_{j^0 j}$ .

Replacing this expression in (2),

$$m(S, C_S^\epsilon) \leq m(F_i \cup \{i\}, C_{F_i \cup \{i\}}) + \eta\epsilon.$$

On the other hand, since no agent in  $K$  connects to the source through agent  $i$  in  $t$ ,

$$m(N_0, C) = m(K_0 \setminus \{i\}, C_{K_0 \setminus \{i\}}) + c_{i^0 i} + m(F_i \cup \{i\}, C_{F_i \cup \{i\}}).$$

Combining the last two expressions:

$$\begin{aligned} m(S, C_S^\epsilon) &\leq m(N_0, C) - m(K_0 \setminus \{i\}, C_{K_0 \setminus \{i\}}) - c_{i^0 i} + \eta\epsilon \\ &= m(N_0, C) - m(K_0 \setminus \{i\}, C_{K_0 \setminus \{i\}}^\epsilon) - c_{i^0 i} + \eta\epsilon \end{aligned}$$

and hence Claim II is proved.

Applying Claim I, Claim II and equation (1),

$$m(K_0 \setminus \{i\}, C_{K_0 \setminus \{i\}}^\epsilon) - \sum_{j \in K \setminus \{i\}} \phi_j(N_0^\epsilon, C^\epsilon) + c_{i^0 i} - \eta\epsilon \leq \phi_i(N_0^\epsilon, C^\epsilon).$$

Since  $\phi$  satisfies *no-subsidy*,

$$c_{i^0 i} - \eta\epsilon \leq \phi_i(N_0^\epsilon, C^\epsilon).$$

Under *independence of extreme null points*,  $\phi_i(N_0^\epsilon, C^\epsilon) = \phi_i(N_0, C)$ . Thus,  $c_{i^0 i} - \eta\epsilon \leq \phi_i(N_0, C)$ . But  $\phi_i(N_0, C)$  does not depend on  $\epsilon$ . Hence,  $c_{i^0 i} \leq \phi_i(N_0, C)$ . ■

## 6 Independence of the axioms

In this section we show that the three axioms used in Theorem 5.1 are independent.

The following rule assigns to each agent half of the cost of his adjacent arcs in the  $mt$ . Moreover, the agents that connect directly to the source pay the entire connection cost with the source. Formally,

$$\phi_i^1(N_0, C) := \begin{cases} c_{0i} + \frac{1}{2} \sum_{j \in IF(i,t)} c_{ij} & \text{if } i^0 = 0 \\ \frac{1}{2} c_{i^0 i} + \frac{1}{2} \sum_{j \in IF(i,t)} c_{ij} & \text{if } i^0 \neq 0 \end{cases}$$

for all  $C \in \mathcal{D}$  and  $i \in N$ .

This rule satisfies a stronger property than *merge-proofness*, which states that if a group of agents  $S$  join in advance in order to be treated as a single node, no agent in  $N \setminus S$  gets worse off in the reduced problem. Formally,

**Definition 6.1** *Let  $C \in \mathcal{D}$  and  $\alpha \in S \subset N$ . A rule  $\phi$  satisfies strong merge-proofness if*

$$\phi_i(N_0^{S\alpha}, C^{S\alpha}) \leq \phi_i(N_0, C)$$

for all  $i \in N \setminus S$ .

**Remark 6.1** *It is not difficult to check that strong merge-proofness implies merge-proofness. The opposite is not true. For example, the Bird rule does not satisfy strong merge-proofness: Consider the  $mstp(N_0, C)$  with  $N = \{1, 2, 3, 4\}$ ,  $c_{01} = 5$ ,  $c_{12} = 10$ ,  $c_{23} = 8$ ,  $c_{34} = 9$  and  $c_{ij} > 10$  for the rest of the arcs. The unique  $mt$  is  $t = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$ . In this case,  $B_3^t(N_0, C) = 8$ .*

*Assume now that the agents in  $\{1, 4\}$  join. The resulting problem will be:  $N^{S\alpha} = \{1, 2, 3\}$ ,  $c_{01}^{S\alpha} = 5$ ,  $c_{13}^{S\alpha} = 9$ ,  $c_{23}^{S\alpha} = 8$ , and  $c_{ij}^{S\alpha} > 10$  for the rest of the arcs. Now the unique  $mt$  is  $t^{S\alpha} = \{(0, 1), (1, 3), (3, 2)\}$ , and  $B_3^{t^{S\alpha}}(N_0, C^{S\alpha}) = 9$ .*

Even though *strong merge-proofness* is defined for any  $S \subset N$ , we can restrict ourselves to the case  $|S| = 2$ . The reason is that any rule that satisfies

strong merge-proofness for  $|S| = 2$  will also satisfy strong merge-proofness for every  $|S| > 2$ . To see why, assume the agents in  $S$  merge one by one. Each time an agent joins the merging coalition, strong merge-proofness for two-agent coalitions implies that no agent in  $N \setminus S$  is worse off.

Hence, it is useful to study the reduced problem when  $S = \{\alpha, \beta\}$ . Given an  $mt$   $t$  in  $(N_0, C)$ , we can construct an  $mt$  in  $(N_0^{S\alpha}, C^{S\alpha})$  by simply deleting the most expensive arc in the path that joins  $\alpha$  and  $\beta$ , as shown in Figure 1.

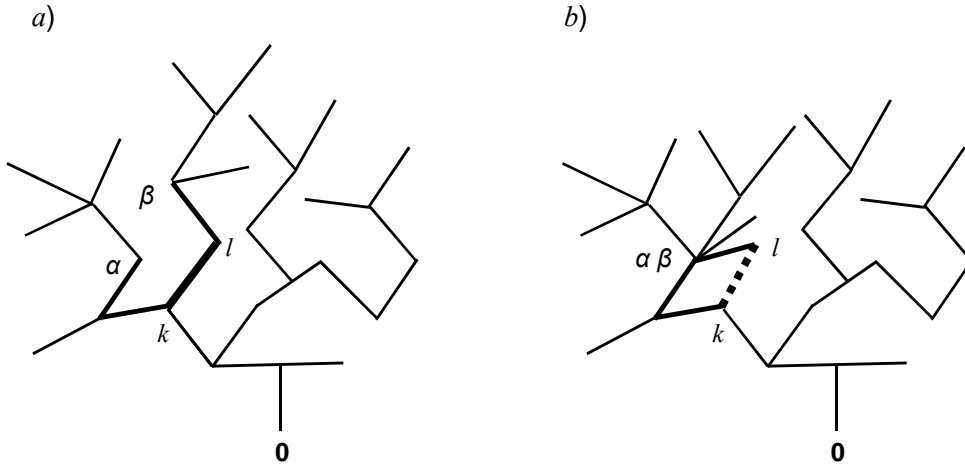


Figure 1: Figure 1a) represents an  $mt$  in  $(N_0, C)$ . The most expensive arc in the path that connects  $\alpha$  and  $\beta$  is  $(k, l)$ . In Figure 1b) nodes  $\alpha$  and  $\beta$  join and the most expensive arc is removed. The resulting tree is an  $mt$  in the reduced problem.

This result is formally stated in the next lemma:

**Lemma 6.1** *Let  $(N_0, C)$  be an  $mcstp$  and let  $t$  be an  $mt$  in  $(N_0, C)$ . Given  $S = \{\alpha, \beta\} \subset N$ , let  $\tau_{\alpha\beta}$  be the path that connects  $\alpha$  and  $\beta$  in  $t$ . Let  $\hat{t} := t \setminus \{(k, l)\}$  for some  $(k, l) \in \arg \max_{(i,j) \in \tau_{\alpha\beta}} c_{ij}$ .*

*The network*

$$t^{S\alpha} := \left( \hat{t} \setminus \{(\beta, i)\}_{(\beta, i) \in \hat{t}} \right) \cup \{(\alpha, i)\}_{(\beta, i) \in \hat{t}}$$

*is an  $mt$  on  $(N_0^{S\alpha}, C^{S\alpha})$ .*

**Proof.** Consider the *mcstp*  $(N_0, C')$  defined as  $c'_{ij} = c_{ij}$  for all  $\{i, j\} \neq \{\alpha, \beta\}$  and  $c'_{\alpha\beta} = c_{kl}$ . It is straightforward to check that  $t$  is also an *mt* on  $(N_0, C')$  (see for example the proof of Proposition 2.2iii in Aarts and Driessen (1993)).

Since the cost of the arc  $(\alpha, \beta)$  does not affect the definition of  $(N^{S\alpha}, C^{S\alpha})$ ,  $C^{S\alpha}$  and  $C'^{S\alpha}$  coincide. Hence, it is enough to prove that  $t^{S\alpha}$  is an *mt* on  $(N^{S\alpha}, C'^{S\alpha})$ . We proceed by a contradiction argument. Assume there exists a tree  $t^*$  on  $(N^{S\alpha}, C'^{S\alpha})$  such that  $c(t^*, C'^{S\alpha}) < c(t^{S\alpha}, C'^{S\alpha})$ . The counterpart of  $t^*$  in  $(N_0, C')$  is defined as follows. Let

$$O_\beta := \{(\alpha, i) \in t^* : c'_{\alpha i}{}^{S\alpha} = c'_{\beta i}\}$$

be the set of arcs in  $t^*$  that would be adjacent to  $\beta$  (the rest of the arcs  $(\alpha, i) \in t^*$  satisfy  $c'_{\alpha i}{}^{A\alpha} = c'_{\alpha i}$ ). We define the following tree in  $(N_0, C')$ :

$$t' := (t^* \setminus O_\beta) \cup \{(\alpha, \beta)\} \cup \{(\beta, i)\}_{(\alpha, i) \in O_\beta}.$$

To see that  $t'$  is indeed a tree in  $(N_0, C')$ , notice that it has exactly  $n$  arcs ( $n - 1$  arcs from  $t^*$  plus  $(\alpha, \beta)$ ) and all of the nodes in  $N$  are connected to the source: those that connect to the source through  $O_\beta$  in  $t^*$  will now connect first to  $\beta$  and then to  $\alpha$  through  $(\alpha, \beta)$ .

We will prove that  $c(t', C') < c(t, C')$ , which is a contradiction because  $t$  is an *mt* on  $(N_0, C')$ . Notice that  $c'_{ij}{}^{S\alpha} = c'_{ij}$  for all  $(i, j) \in t^* \setminus O_\beta$ . Thus,

$$c(t^* \setminus O_\beta, C') = c(t^* \setminus O_\beta, C'^{S\alpha}).$$

Hence,

$$\begin{aligned} c(t', C') &= c(t^* \setminus O_\beta, C') + c'_{\alpha\beta} + \sum_{(\alpha, i) \in O_\beta} c'_{\beta i} \\ &= c(t^* \setminus O_\beta, C'^{S\alpha}) + c'_{\alpha\beta} + \sum_{(\alpha, i) \in O_\beta} c'_{\alpha i}{}^{S\alpha} \\ &= c(t^*, C'^{S\alpha}) + c'_{\alpha\beta} < c(t^{S\alpha}, C'^{S\alpha}) + c'_{\alpha\beta} \leq c(t, C') \end{aligned}$$

where the last inequality comes from

$$\begin{aligned}
c(t^{S\alpha}, C'^{S\alpha}) &= c(\hat{t} \setminus \{(\beta, i)\}_{(\beta, i) \in \hat{t}}, C'^{S\alpha}) + \sum_{(\beta, i) \in \hat{t}} c'_{\alpha i}{}^{S\alpha} \\
&\leq c(\hat{t} \setminus \{(\beta, i)\}_{(\beta, i) \in \hat{t}}, C') + \sum_{(\beta, i) \in \hat{t}} c'_{\beta i} \\
&= c(\hat{t}, C') = c(t, C') - c'_{kl} = c(t, C') - c'_{\alpha\beta}.
\end{aligned}$$

■

**Proposition 6.1** *On the domain  $\mathcal{D}$ ,  $\phi^1$  satisfies strong merge-proofness and independence of extreme null points.*

**Proof.** Let  $C \in \mathcal{D}$  and let  $t$  be an  $mt$  in  $(N_0, C)$ .

It is straightforward to check that  $\phi^1$  satisfies *independence of extreme null points*.

We prove that  $\phi^1$  satisfies *strong merge-proofness*.

We can assume that  $|S| = 2$ . Let  $S = \{\alpha, \beta\} \subset N$ .

Under Lemma 6.1,  $t^{S\alpha}$  is an  $mt$  in  $(N_0^{S\alpha}, C'^{S\alpha})$ .

We prove that no agent  $i \in N^{S\alpha} \setminus \{\alpha\}$  is worse off in the reduced problem than in  $(N_0, C)$ .

Let  $(i, j) \in t^{S\alpha}$ ,  $i \neq \alpha$ . We have two cases:

- If  $j = \alpha$ , by definition of  $t^{S\alpha}$ ,  $(i, \alpha) \in t$  or  $(i, \beta) \in t$ . By definition,  $c'_{i\alpha}{}^{S\alpha} = \min\{c_{i\alpha}, c_{i\beta}\}$ . Hence, whatever agent  $i$  pays in the reduced problem for  $(i, \alpha)$  is not more than what he pays in  $(N_0, C)$  for  $(i, \alpha)$  or  $(i, \beta)$ .
- If  $j \neq \alpha$ , by definition of  $t^{S\alpha}$ ,  $(i, j) \in t$ . Moreover,  $c'_{ij}{}^{S\alpha} = c_{ij}$ . Hence, whatever agent  $i$  pays in the reduced problem for  $(i, j)$  is the same as what he pays in  $(N_0, C)$  for  $(i, j)$ .

■

This rule violates *no-subsidy*. Consider the *mcstp*  $(N_0, C)$  with  $N = \{1, 2\}$ ,  $c_{01} = 10$ ,  $c_{02} = 15$ , and  $c_{12} = 6$ . The unique  $mt$  is  $t = \{(0, 1), (1, 2)\}$ . In this case,  $\phi_1^1(N_0, C) = 13 > 10 = m(\{0, 1\}, C_{\{0, 1\}})$ .

Now we present a rule that satisfies *no-subsidy* and *strong merge-proofness* (and hence *merge-proofness*) but does not satisfy *independence of extreme null points*. We describe the rule in four steps.

**Step 1:** Order the agents following their cost of direct link to the source. Since the order does not have to be unique, let  $\Pi'_N$  denote the set of such orders.

**Step 2:** Given an order  $\pi \in \Pi'_N$ , let  $t^\pi$  be an *mt* obtained from Prim's algorithm such that the indifferences are solved in favor of the agent that is before in the order  $\pi$ . For each  $i \in N$ , select the highest cost in the unique path in  $t^\pi$  between agent  $i$  and each  $j \in P_i^\pi \cup \{0\}$ .

**Step 3:** The rule  $\phi^{2\pi}$  assigns to agent  $i$  the minimum of such numbers.

**Step 4:** The rule  $\phi^2$  assigns to the agents the average of these  $\phi^{2\pi}$  over all the orders in  $\Pi'_N$ .

We define the rule formally as follows:

Let  $\Pi'_N := \{\pi \in \Pi_N : \pi(i) < \pi(j) \Rightarrow c_{0i} \leq c_{0j}\}$ . Given  $\pi \in \Pi'_N$ , let  $t^\pi$  be the unique *mt* obtained from Prim's algorithm solving indifferences using the preference order  $\pi$ . Given  $i, j \in N$ , let  $\tau_{ij}^\pi$  denote the path that connects  $i$  and  $j$  in  $t^\pi$ . The rule  $\phi^2$  is defined as follows: For each  $N$ , each  $C \in \mathcal{D}$  and  $i \in N$ ,

$$\phi_i^{2\pi}(N_0, C) := \min_{j \in P_i^\pi \cup \{0\}} \left\{ \max_{(k,l) \in \tau_{ij}^\pi} \{c_{kl}\} \right\}$$

and

$$\phi_i^2(N_0, C) := \frac{1}{|\Pi'_N|} \sum_{\pi \in \Pi'_N} \phi_i^{2\pi}(N_0, C).$$

This rule appears in Bergantiños and Vidal-Puga (2007a, after Corollary 4.1) with a different formulation.

Consider the following example:

**Example 6.1** Let  $(N_0, C)$  such that  $N = \{1, 2, 3, 4\}$ ,  $c_{01} = 10$ ,  $c_{03} = 15$ ,  $c_{04} = 15$ ,  $c_{12} = 2$ ,  $c_{14} = 6$ ,  $c_{34} = 0$ , and  $c_{ij} = 20$  otherwise.

We compute  $\phi^2(N_0, C)$ . There are two orders in  $\Pi'_N$ :  $\pi = (1, 2, 3, 4)$  and  $\pi' = (1, 2, 4, 3)$ . The respective mt are  $t^\pi = t^{\pi'} = \{(0, 1), (1, 2), (1, 4), (4, 3)\}$ . We compute  $\phi^{2\pi}(N_0, C)$ .

**Agent 1:** We have  $P_1^\pi \cup \{0\} = \{0\}$  and  $\tau_{10}^\pi = \{(1, 0)\}$ . Hence,

$$\phi_1^{2\pi}(N_0, C) = c_{10} = 10.$$

**Agent 2:** We have  $P_2^\pi \cup \{0\} = \{0, 1\}$ ,  $\tau_{20}^\pi = \{(2, 1), (1, 0)\}$  and  $\tau_{21}^\pi = \{(2, 1)\}$ . Hence,

$$\begin{aligned} \phi_2^{2\pi}(N_0, C) &= \min \{ \max \{c_{21}, c_{10}\}, c_{21} \} \\ &= \min \{ \max \{2, 10\}, 2 \} = 2. \end{aligned}$$

**Agent 3:** We have  $P_3^\pi \cup \{0\} = \{0, 1, 2\}$ ,  $\tau_{30}^\pi = \{(3, 4), (4, 1), (1, 0)\}$ ,  $\tau_{31}^\pi = \{(3, 4), (4, 1)\}$  and  $\tau_{32}^\pi = \{(3, 4), (4, 1), (1, 2)\}$ . Hence,

$$\begin{aligned} \phi_3^{2\pi}(N_0, C) &= \min \{ \max \{c_{34}, c_{41}, c_{10}\}, \max \{c_{34}, c_{41}\}, \max \{c_{34}, c_{41}, c_{12}\} \} \\ &= \min \{ \max \{0, 6, 10\}, \max \{0, 6\}, \max \{0, 6, 2\} \} = 6. \end{aligned}$$

**Agent 4:** We have  $P_4^\pi \cup \{0\} = \{0, 1, 2, 3\}$ ,  $\tau_{40}^\pi = \{(4, 1), (1, 0)\}$ ,  $\tau_{41}^\pi = \{(4, 1)\}$ ,  $\tau_{42}^\pi = \{(4, 1), (1, 2)\}$ , and  $\tau_{43}^\pi = \{(4, 3)\}$ . Hence,

$$\begin{aligned} \phi_4^{2\pi}(N_0, C) &= \min \{ \max \{c_{41}, c_{10}\}, c_{41}, \max \{c_{41}, c_{12}\}, c_{43} \} \\ &= \min \{ \max \{6, 10\}, 6, \max \{6, 2\}, 0 \} = 0. \end{aligned}$$

Hence,  $\phi^{2\pi}(N_0, C) = (10, 2, 6, 0)$ . By an analogous reasoning, we get  $\phi^{2\pi'}(N_0, C) = (10, 2, 0, 6)$ .

Thus, the rule assigns  $\phi^2(N_0, C) = (10, 2, 3, 3)$  which is different from assignment proposed by the Bird rule  $B(N_0, C) = (10, 2, 0, 6)$ .

As  $\phi^1$  does,  $\phi^2$  satisfies strong merge-proofness and hence merge-proofness. We prove this result in the next proposition.

**Proposition 6.2** *Over the domain  $\mathcal{D}$ ,  $\phi^2$  satisfies strong merge-proofness and no-subsidy.*



**Proof.** First we prove that  $\phi^2$  satisfies *no-subsidy*:

Bergantiños and Vidal-Puga (2007a, after Corollary 4.1) prove that this rule satisfies Population Monotonicity (*PM*), and moreover *PM* implies no-subsidy<sup>9</sup>.

We now prove in a intuitive way that  $\phi^2$  satisfies *strong merge-proofness*:  
Let  $C \in \mathcal{D}$  and let  $t$  be an *mt* on  $(N_0, C)$ .

We can assume that  $|S| = 2$ . Let  $S = \{\alpha, \beta\} \subset N$ .

Under Lemma 6.1,  $t^{S\alpha}$  is an *mt* in  $(N_0^{S\alpha}, C^{S\alpha})$ .

We prove that every agent  $i \in N \setminus S$  is not worse off in the reduced problem than in  $(N_0, C)$ .

Let  $\Omega_1^C := \arg \min_{i \in N} c_{0i}$  be the set of agents in  $N$  with the lowest cost to the source. Let  $\Omega_2^C := \arg \min_{i \in N \setminus \Omega_1^C} c_{0i}$  be the set of agents in  $N$  with the second-lowest cost the source, and so on. Let  $\gamma$  be the number of such groups. Hence, an order  $\pi$  belongs to  $\Pi'_N$  if and only if the agents in  $\Omega_1^C$  come first, then the agents in  $\Omega_2^C$ , and so on. In particular,  $|\Pi'_N| = |\Omega_1^C|! \dots |\Omega_\gamma^C|!$ .

We also define  $\Omega_l^{C^{S\alpha}}$  in an analogous way. In particular,  $|\Pi'_{N^{S\alpha}}| = |\Omega_1^{C^{S\alpha}}|! \dots |\Omega_\eta^{C^{S\alpha}}|!$ .

Assume  $\alpha \in \Omega_k^C$  and  $\beta \in \Omega_{k'}^C$ . We can assume wlog  $k \leq k'$ . We have two cases:

**Case 1:**  $k < k'$ . In this case, it is straightforward to check that  $\Omega_l^{C^{S\alpha}} = \Omega_l^C$  for all  $l \neq k'$ , whereas  $\Omega_{k'}^{C^{S\alpha}} = \Omega_{k'}^C \setminus \{\beta\}$ . Notice that we implicitly assume  $\Omega_{k'}^{C^{S\alpha}} = \emptyset$  in the particular case  $\Omega_{k'}^C = \{\beta\}$ , so that  $\gamma = \eta$ .

**Case 2:**  $k = k'$ . In this case,  $\Omega_l^{C^{S\alpha}} = \Omega_l^{C^{S\alpha}}$  for all  $l \neq k$ , whereas  $\Omega_k^{C^{S\alpha}} = \Omega_k^C \setminus \{\beta\}$ .

In both cases, it is straightforward to check that  $|\Pi'_{N^{S\alpha}}| = |\Pi'_N| / |\Omega_k^C|$ . Moreover, each order in  $\Pi'_{N^{S\alpha}}$  is induced by exactly  $|\Omega_{k'}^C|$  different orders in  $\Pi'_N$ . Hence, given  $i \in N \setminus S$ , it is enough to prove that  $\phi_i^{2\pi^{S\alpha}}(N_0^{S\alpha}, C^{S\alpha}) \leq \phi_i^{2\pi}(N_0, C)$  for any  $\pi \in \Pi'_N$  and  $\pi^{S\alpha} \in \Pi'_{N^{S\alpha}}$  such that  $\pi^{S\alpha}$  is induced by  $\pi$ .

Fix  $i \in N \setminus S$ ,  $\pi \in \Pi'_N$ , and  $\pi^{S\alpha} \in \Pi'_{N^{S\alpha}}$  such that  $\pi^{S\alpha}$  is induced by  $\pi$ . It is straightforward to check that when  $t = t^\pi$ , we have that, essentially,

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<sup>9</sup>*No-subsidy* is called *Core Selection* in Bergantiños and Vidal-Puga (2007a).

$t^{S^\alpha} = t^{\pi^{S^\alpha}}$  (they may change in the identity of the arcs in a  $C$ -component, which is irrelevant in the posterior analysis). Hence, we omit the superscript.

Let  $j \in P_i^\pi \cup \{0\}$ . It is enough to prove that the maximum cost of the path between  $i$  and  $j$  in  $t^{S^\alpha}$  (we take  $\alpha$  when  $j = \beta$ ) is not more than the maximum cost of the path between  $i$  and  $j$  in  $t$ . Let  $(k, l)$  be the most expensive arc in the path from  $\alpha$  to  $\beta$  in  $t$ . We distinguish two cases:

- If  $(k, l)$  does not belong to the path from  $i$  to  $j$  in  $t$ , the path from  $i$  to  $j$  in  $t^{S^\alpha}$  is the same. Hence, the maximum cost of the path from  $i$  to  $j$  is the same in  $t$  as in  $t^{S^\alpha}$ .
- If  $(k, l)$  belongs to the path from  $i$  to  $j$  in  $t$ , the cost from  $i$  to  $j$  in  $t^{S^\alpha}$  is not more because we have removed the most expensive arc of the path from  $\alpha$  to  $\beta$ .

■

This rule violates *independence of extreme null points*. Consider the *mcstp* given in Example 6.1. Agent 3 is an extreme null point. If removed,  $\phi_4^2(N_0 \setminus \{3\}, C_{N_0 \setminus \{3\}}) = 6 \neq 3 = \phi_4^2(N_0, C)$ .

Finally, we define a rule,  $\phi^3$ , that satisfies *no-subsidy* and *independence of extreme null points* but does not satisfy *merge-proofness*.

This rule is similar to the Dutta-Kar rule (Dutta and Kar, 2004). However,  $\phi^3$  assigns zero cost to the extreme null points and the assignment of the rest of the agents does not depend on the extreme null points.

Formally:

Let  $\Omega := \{i \in N : i \text{ is an extreme null point in } (N_0, C)\}$ .

Consider the following algorithm:

Let  $S^0 = \{0\}$ ,  $t^0 = \emptyset$ ,  $p^0 = 0$ . Let  $S_c^0 = N_0 \setminus S^0$ .

*Step 1:* Choose an ordered pair  $(a_1^0, a_1)$  such that  $(a_1^0, a_1) \in \arg \min_{\substack{(i,j) \in S^0 \times S_c^0 \\ j \notin \Omega}} c_{ij}$ .

Define  $p^1 = \max(p^0, c_{a_1^0 a_1})$ ,  $S^1 = S^0 \cup \{a_1\}$ ,  $t^1 = t^0 \cup \{(a_1^0, a_1)\}$ ,  $S_c^1 = N_0 \setminus S^1$ .

*Step k:* Assume we have defined  $p^{k-1}$ ,  $S^k$ ,  $t^{k-1}$  and  $S_c^k$ . Take an ordered pair  $(a_k^0, a_k) \in \arg \min_{\substack{(i,j) \in S^{k-1} \times S_c^{k-1} \\ j \notin \Omega}} c_{ij}$ . Now,  $S^k = S^{k-1} \cup \{a_k\}$ ,  $t^k = t^{k-1} \cup \{(a_k^0, a_k)\}$ ,  $p^k = \max(p^{k-1}, c_{a_k^0 a_k})$  and  $S_c^k = N_0 \setminus S^k$ .

Also,

$$\phi_{a_{k-1}}^3(N_0, C) = \min(p^{k-1}, c_{a_k^0 a_k}). \quad (3)$$

The algorithm terminates at step  $m = |N| \setminus |\Omega|$ . We define:

$$\phi_{a_m}^3(N_0, C) = p^m \quad (4)$$

and

$$\phi_i^3(N_0, C) = 0 \text{ for all } i \in \Omega. \quad (5)$$

The rule  $\phi^3$  is described by equations (3),(4) and (5).

If the set of extreme null points is empty,  $\phi^3$  coincides with the rule proposed by Dutta and Kar (2004).

To show that  $\phi^3$  does not satisfy *merge-proofness*, we use Example 3.2. For this example, the assignment proposed by  $\phi^3$  coincides with the assignment proposed by the Dutta-Kar rule because there are no extreme null points.

**Proposition 6.3** *Over the domain  $\mathcal{D}$ ,  $\phi^3$  satisfies no-subsidy and independence of extreme null points.*

**Proof.** *It is clear that  $\phi^3$  satisfies independence of extreme null points. On the other hand, the proof that  $\phi^3$  satisfies no-subsidy is similar to the proof that the Dutta-Kar rule satisfies no-subsidy<sup>10</sup> (Dutta and Kar, 2004, Theorem 1) and we omit it. ■*

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<sup>10</sup>No-subsidy is called *core selection* in Dutta and Kar (2004).

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