Characterization of monotonic rules in minimum cost spanning tree problems*

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Abstract

We provide, in minimum cost spanning tree problems, a general framework to identify the family of rules satisfying monotonicity over cost and population. We also prove that the set of allocations induced by the family coincides with the so-called irreducible core, that results from decreasing the cost of the arcs as much as possible, without reducing the minimal cost.

Keywords: Cost sharing, minimum cost spanning tree problems, monotonicity, irreducible core.

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*, for short). A group of agents (denoted by N), located at different geographical places, want a particular service which can only be provided by a common supplier, called the source (denoted by 0). Agents will be served through connections which involve some cost. However, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix C, where c_{ij} denotes the connection costs between i and j ($i, j \in N \cup \{0\}$). Many real situations can be modeled in this way. For instance communication networks, such as telephone, Internet, wireless telecommunication, or cable television.

The first issue addressed in *mcstp* is the definition of polynomial algorithms for constructing minimum cost spanning trees (*mcst*). We can mention, for instance, Kruskal (1956). But constructing a *mcst* is only the first step. Another important issue is how to divide the cost associated with the *mcst* among the agents. Some authors study the problem from

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a non-cooperative perspective trying to provide a decentralized mechanism for dividing the cost among the agents. Some papers following this approach are Bergantiños and Lorenzo (2004), Bergantiños and Vidal-Puga (2010), Hougaard and Tvede (2012), and Hernández *et al* (2013). Other authors study the problem from a cooperative perspective providing a centralized mechanism. It is assumed that a planner should decide how to divide the cost among the agents following some fairness criteria. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often help the planner to compare different rules and to decide which rule is preferred in a particular situation. Some examples of papers following the axiomatic approach are Bird (1976), Kar (2002), Branzei *et al* (2004), Bergantiños and Vidal-Puga (2007, 2009), Bogomolnaia and Moulin (2010), and Trudeau (2012a, 2012b).

In this paper we follow the axiomatic approach and we focus on two monotonicity properties. *Population monotonicity* claims that if new agents join a "society" no agent from the "initial society" can be worse off; and *cost monotonicity* claims that if connection costs weakly increase, no agent can be better off. Population monotonicity in *mcstp* has been studied, among others, by Bergantiños and Gómez-Rúa (2010), Bergantiños and Vidal-Puga (2007, 2009), Bogomolnaia and Moulin (2010), Lorenzo and Lorenzo-Freire (2009), and Norde *et al* (2004). Cost monotonicity in *mcstp* has been studied in by Bergantiños and Gómez-Rúa (2010), Bergantiños *et al* (2010, 2011), Bergantiños and Vidal-Puga (2007, 2009), Bogomolnaia and Moulin (2010), and Trudeau (2012a). In the literature there exist two families of rules satisfying both properties. The optimistic weighted Shapley rules studied by Bergantiños and Lorenzo-Freire (2008a, 2008b) and the obligation rules studied by Tijs *et al* (2006), Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010).

The main objective of this paper is to study the set of rules satisfying population monotonicity and cost monotonicity. We focus on two aspects: to characterize the set of rules satisfying both properties and to characterize the set of allocations induced by these rules.

Given a *mcstp* C, Bird (1976) considers the irreducible problem C^*, which is obtained from C by reducing the cost of the edges as much as possible, but without reducing the cost of the *mcst*. Bird (1976) associates to each *mcstp* C a cooperative cost game with transferable utility (N, c_C). We prove that the set of allocations induced by rules satisfying population monotonicity and cost monotonicity coincides with the core of the game (N, c_{C^*}), the so called *irreducible core*.

A weaker version of population monotonicity is *separability*, which claims that if two groups of agents can connect to the source independently of each other, then we can compute their payments separately. A weaker version of cost monotonicity is *reductionism*, which claims that the rule must depend only on the irreducible problem. We identify a necessary and sufficient condition for a family of rules to cover all the ones satisfying separability and reductionism. In order to describe this condition, we need to define the so-called, neighborhoods and extra-costs functions. A *neighborhood* is a group of agents that are "closer" to each other than to any of the other agents or to the source. Namely, the largest connection cost among agents of the neighborhood is smaller than the smallest connection cost between an agent in the neighborhood and a node outside the neighborhood. An *extra-costs function* is a way of dividing the savings obtained by the agents of a neighborhood when they connect each other through a *mcst*. The intuition behind such rules is the following: Initially each agent is connected to the source in the irreducible problem. Now, agents inside neighborhoods are connected among them. For each neighborhood, the savings are divided between the agents in the neighborhood following the extra-costs function.

We characterize the set of rules satisfying population monotonicity and cost monotonicity, which is a subset of the previous set. We need to select the extra-costs functions satisfying the so called *aggregate neighborhood monotonicity*, which says that for each agent, the aggregate sum of the savings given by the extra-costs function should not decrease when the connection cost between two consecutive neighborhoods is increased.

In particular, we show how optimistic weighted Shapley rules and obligation rules can be defined using the extra-costs functions. Hence, we provide a general framework to identify the monotonic rules. Besides, our results can be applied to identify new classes of rules satisfying both monotonicity properties. We do it by introducing a class of rules that generalize the obligation rules.

The paper is organized as follows. In Section 2 we introduce the model and the notation. In Section 3 we characterize the set of allocations induced by the rules satisfying population and cost monotonicity. In Section 4 we characterize the set of rules satisfying separability and reductionism. In Section 5 we characterize the set of rules satisfying population monotonicity and cost monotonicity and we apply these results to some known rules in the literature. Some concluding remarks appear in Section 6. The proofs are presented in the Appendix.

2 Minimum cost spanning tree problems

We first introduce minimum cost spanning tree problems and some notation used throughout the paper.

Let $U = \{1, 2, ... |U|\}$ be the finite set of possible agents, and let 0 be a special node called the *source*.

A *graph* is a pair (V, E) where $\varnothing \neq V \subset U \cup \{0\}$ and $E = \{\{i, j\} : i, j \in V, i \neq j\}$. The elements of V are called *nodes* (or *vertices*) and the elements of E are called *edges*. Given $g \subset E$, a *path* in g between i and j is a sequence of nodes $i = i_1, ..., i_k = j$ such that $\{i_s, i_{s+1}\} \in g$ for each $s = 1, ..., k-1$. A path is *simple* if $i_s \neq i_{s'}$ for each $s, s' \in \{1, ..., k\}$ with $s \neq s'$. We say that $g \subset E$ is a *spanning tree in* V if for each pair of nodes $i, j \in V$ there exists exactly one simple path in t between i and j. Let $\mathbb{T}(V)$ (or simply $\mathbb{T}$) denote the set of all spanning trees in V.

Given $x, y \in \mathbb{R}^V$ we say that $x \leq y$ if $x_i \leq y_i$ for all $i \in V$. As usual, $\mathbb{R}_+$ denotes the set of non-negative real numbers. Let $\Delta(V) = \left\{ (x_i)_{i \in V} \in \mathbb{R}^V_+ : \sum\limits_{i \in V} x_i = 1 \right\}$ be the simplex in $\mathbb{R}^V$.

Let Π_V denote the set of all orders in V. Given $\pi \in \Pi_V$, let $Pre(i, \pi)$ denote the set of elements of V which come before i in the order given by π, *i.e.*, $Pre(i, \pi) = \{j \in V \mid \pi(j) < \pi(i)\}$. Given $\pi \in \Pi_V$ and $S \subset V$, π_S denotes the order induced by π among nodes in S.

A *cost matrix on* V is a matrix $C = (c_{ij})_{i,j \in V}$ such that $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N_0$. As usual, c_{ij} denotes the cost of constructing the edge $\{i, j\}$ connecting agents i and j.

Given $g \subset E$, the *cost* of g in (V, C) is defined as $c(g, C) = \sum_{\{i,j\} \in g} c_{ij}$. A *minimum cost spanning tree* (*mcst*) in (V, C) is a spanning tree t in V with minimum cost, namely $c(t, C) = \min_{t' \in \mathbb{T}(V)} c(t', C)$. A *mcst* is not necessarily unique. However, all *mcst* in (V, C) have the same cost, that we denote as $m(V, C)$.

Given a pair (V, C) we denote $\max(V, C) := \max_{i,j \in V} c_{ij}$. Given $\varnothing \neq S \subset V$, we denote as (S, C_S) the restriction of (V, C) to S. Given $i, j \in V$ and $\alpha \in \mathbb{R}_+$, we denote as αI_{ij} the matrix C given by $c_{kl} = 0$ for all $\{k, l\} \neq \{i, j\}$ and $c_{ij} = \alpha$.

Given a pair (V, C) and a *mcst* t, following Bird (1976) we define the *minimal network* (V, C^t) associated with t as follows:

$$c_{ij}^t = \max_{\{i_l, i_{l+1}\} \subset \tau_{ij}} c_{i_l i_{l+1}} \tag{1}$$

where $\tau_{ij} = \{i_1, ..., i_k\}$ denotes the unique simple path in t from i to j. Bird (1976) used this minimal network to study a subset of the core of a *mcstp*.

Even though this definition is dependent on the choice of *mt* t, it is independent of the chosen t. Proof of this can be found, for instance, in Aarts and Driessen (1993).

We define the *irreducible form* of (V, C) as the minimal network (V, C^*) associated with a particular *mcst* t. We say that (V, C) is an *irreducible problem* if $C = C^*$. We denote as $\mathcal{C}_0^*$ the set of all irreducible problems such that $0 \in V$ and as $\mathcal{C}^*$ the set of all irreducible problems such that $0 \notin V$.

It is well known that if (V, C) is an irreducible problem and we reduce any of the costs of C, then the total cost of connecting all agents with the source is also reduced.

A *minimum cost spanning tree problem* (*mcstp*) is a pair (V, C) where $0 \in V$ and $\varnothing \neq V \backslash \{0\} \subset U$. Usually we denote it as (N_0, C) where $N = V \backslash \{0\}$ and $N_0 = N \cup \{0\}$. For simplicity, when N is clear, we write C instead of (N_0, C). Let $\mathcal{C}_0$ be the set of all *mcstp*. Besides, let $\mathcal{C}$ be the set of all pairs (V, C) such that $0 \notin V$.

A *rule* is a function f that assigns to each *mcstp* $(N_0, C) \in \mathcal{C}$ a vector $f(N_0, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$. As usual, $f_i(N_0, C)$ represents the payoff assigned to agent $i \in N$.

We now introduce some properties of rules, which we will use in this paper.

Population Monotonicity (*PM*) For all *mcstp* (N_0, C), $S \subset N$, and $i \in S$, we have

$$f_i(N_0, C) \leq f_i(S_0, C_{S_0}).$$

This property says that if new agents join a network, no agent from the initial network can be worse off.

Cost Monotonicity (CM) For all *mcstp* (N_0, C) and (N_0, C') such that $C \leq C'$, we have

$$f(N_0, C) \leq f(N_0, C').$$

This property says that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off. This property is also called solidarity or strong cost monotonicity in some papers such as Bergantiños and Vidal-Puga (2007) and Bergantiños and Kar (2010).

Separability (SEP) For all *mcstp* (N_0, C) and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C_{S_0}) + m\left((N \backslash S)_0, C_{(N \backslash S)_0}\right)$, we have

$$f_i(N_0, C) = \begin{cases} f_i(S_0, C_{S_0}) & \text{if } i \in S \\ f_i\left((N \backslash S)_0, C_{(N \backslash S)_0}\right) & \text{if } i \in N \backslash S. \end{cases}$$

Two subsets of agents, S and $N \backslash S$, can be connected to the source either separately or jointly. If there are no savings when they are jointly connected to the source, this property says that the agents will pay the same in both circumstances. This property is also called decomposition in some papers such as Megiddo (1978) and Granot and Huberman (1981).

Reductionism (RED) For all *mcstp* (N_0, C),

$$f(N_0, C) = f(N_0, C^*).$$

If a rule satisfies this property, then it only depends on irreducible matrices. RED appears in Bogomolnaia and Moulin (2010) and it is introduced in Bergantiños and Vidal-Puga (2007) where it is called independence of irrelevant trees.

Bergantiños and Vidal-Puga (2007) study the relationships between these properties. They prove that PM implies SEP, SEP does not imply PM, CM implies RED, and RED does not imply SEP.

3 The irreducible core

Bird (1976) introduces the irreducible core of a *mcstp* (N_0, C). We define the *set of monotonic allocations* as the set of allocations induced by rules satisfying CM and PM. In this section we prove that the set of monotonic allocations coincides with the irreducible core.

A *cost game with transferable utility*, briefly a *cost game*, is a pair (N, c) where $\varnothing \neq N \subset V$ and $c : 2^N \to \mathbb{R}$ satisfies $c(\varnothing) = 0$.

The *core* of a cost game (N, c) is defined as

$$core\,(N,c) = \left\{ (x_i)_{i\in N} : \sum_{i\in N} x_i = c\,(N) \text{ and } \sum_{i\in S} x_i \leq c\,(S)\,\forall S \subset N \right\}.$$

Bird (1976) associates with each *mcstp* (N_0, C) the cost game (N, c_C). For each coalition $S \subset N$, $c_C\,(S) = m\,(S_0, C_S)$.

The *irreducible core* of a *mcstp* (N_0, C), denoted as $IC\,(N_0, C)$, is the core of the cost game (N, c_{C^*}) where (N_0, C^*) is the irreducible problem associated with (N_0, C).

Even though the core of a cost game could be empty, there are some class of situations where it is always non empty. For instance, Bird (1976) proved that the core of any *mcstp* (N_0, C) is non empty. Besides he realizes that the core contains "many" allocations. Thus, he introduced the irreducible core, which is a non-empty set of the core. One advantage of the irreducible core is its nice structure, it is the convex hull of the vector of marginal contributions of (N, c_{C^*}).

Given a *mcstp* (N_0, C), let $AM\,(N_0, C)$ denote the set of allocations induced by the rules satisfying CM and PM. Namely, $x \in AM\,(N_0, C)$ if and only if there exists a rule f satisfying CM and PM such that $x = f\,(N_0, C)$.

In the next theorem we prove that any rule f satisfying CM and PM gives, for any *mcstp* (N_0, C), an element $f\,(N_0, C)$ in the irreducible core of (N_0, C).

Theorem 3.1 *For each mcstp* (N_0, C), $AM\,(N_0, C) = IC\,(N_0, C)$.

Nevertheless, given a rule f such that, for each *mcstp* (N_0, C), $f\,(N_0, C) \in IC\,(N_0, C)$, it could be the case that f does not satisfy both monotonicity properties. We now provide a rule f that does not satisfy PM (a similar example could be provided for CM). Let $\pi^{id} \in \Pi_U$ as in the proof of Theorem 3.1. We define $\pi' \in \Pi_U$ such that $\pi'\,(i) = |U| - i$ for all $i \in U$. Let $f^{\pi^{id}}$ and $f^{\pi'}$ be as in the proof of Theorem 3.1. We define the rule f as follows:

$$f\,(N_0, C) = \begin{cases} f^{\pi^{id}}\,(N_0, C) & \text{if } |N| \text{ is even} \\ f^{\pi'}\,(N_0, C) & \text{if } |N| \text{ is odd.} \end{cases}$$

Let (N_0, C) be a *mcstp* where $|N|$ is even (the case where $|N|$ is odd is similar and we omit it). Then, $f\,(N_0, C) = f^{\pi^{id}}\,(N_0, C)$. Since $f^{\pi^{id}}$ satisfies PM and CM (we have proved it in the proof of Theorem 3.1), $f^{\pi^{id}}\,(N_0, C) \in IC\,(N_0, C)$. Nevertheless, f does not satisfy PM. Let $N = \{1, 2, 3\}$, $c_{01} = c_{02} = c_{03} = 9$, and $c_{12} = c_{13} = c_{23} = 5$. Then, $f_3\left(\{1, 3\}_0, C_{\{13\}}\right) = 5$ but $f_3\,(N_0, C) = 9$.

In Theorem 3.1 above we have stated that the set of allocations induced by monotonic rules coincide with the irreducible core. A natural question that arises is if we can obtain all allocations in the irreducible core with a smaller set of rules. The answer is affirmative. From the proof of Theorem 3.1 we can deduce that any allocation in the irreducible core could be obtained as an allocation induced by the convex hull of the "order induced" rules $\{f^\pi\}_{\pi \in \Pi_U}$.

4 The set of rules satisfying separability and reductionism

In this section we characterize the set of rules satisfying SEP and RED. For doing it we need some new definitions. A *neighborhood* is a group of agents that are “closer” to each other than to any of the other agents or to the source. An *extra-costs function* is a way of dividing the savings obtained by the agents of a neighborhood when they connect among themselves. The rules satisfying both properties could be described as follows. Initially each agent is connected to the source in the irreducible matrix. Now, agents inside neighborhoods are connected among them. For each neighborhood, the savings are divided among the agents in the neighborhood following the extra-costs function.

We first introduce the concepts which will be crucial in our results.

Given a *mcstp* (N_0, C) and $S \subset N$, we define

$$\delta_S = \begin{cases} \min\limits_{j \in N_0 \setminus \{i\}} c_{ij} & \text{if } S = \{i\} \\ \min\limits_{i \in S, j \in N_0 \setminus S} c_{ij} - \max\limits_{\{i,j\} \in \tau(S)} c_{ij} & \text{if } |S| > 1 \end{cases}$$

where $\tau(S)$ is a *mcst* in (S, C_S) connecting all the agents in S.

Even though $\tau(S)$ is not necessarily unique, as in the case of the irreducible problem $\max\limits_{\{i,j\} \in \tau(S)} c_{ij}$ does not depend on the particular $\tau(S)$ and hence δ_S is well defined.

Roughly speaking, δ_S may be interpreted, when positive, as some kind of "distance" between S and $N_0 \setminus S$.

Definition 4.1 *Let (N_0, C) be a mcstp. We say that $S \subset N$, $|S| > 1$, is a* neighborhood *in (N_0, C) if $\delta_S > 0$. We denote the set of all neighborhoods in (N_0, C) as $Ne(N_0, C)$. Sometimes we will write $Ne(C)$ instead of $Ne(N_0, C)$.*

Example 4.1 *Let $N = \{1,2,3,4,5,6\}$ and C be such that $c_{01} = 50$, $c_{12} = 20$, $c_{13} = 40$, $c_{34} = 10$, $c_{15} = 60$, $c_{36} = 70$, and $c_{ij} > 70$ otherwise. There are exactly two neighborhoods containing node 1: $\{1,2\}$ because $\delta_{\{1,2\}} = c_{13} - c_{12} = 20$, and $\{1,2,3,4\}$ because $\delta_{\{1,2,3,4\}} = c_{01} - c_{13} = 50 - 40 = 10$. Notice that $\{1,2,3\}$ is not a neighborhood because $\delta_{\{1,2,3\}} = c_{34} - c_{13} = -30$.*

Some comments about neighborhoods.

1. From the definition of the irreducible problem (N_0, C^*) we deduce that neighborhoods of (N_0, C) and (N_0, C^*) coincide.

2. In general, $(C^*)_S \neq (C_S)^*$. Take for example $N = \{1,2,3\}$, $c_{12} = c_{13} = 1$, $c_{23} = 2$ and $S = \{2,3\}$. Then, $c^*_{23} = 1$ and hence $C' = (C^*)_S$ satisfies $c'_{23} = 1$ whereas $C'' = (C_S)^*$ satisfies $c''_{23} = 2$. Later on (Proposition 1.1) we will prove that the equality holds when S is a neighborhood.

The next proposition gives some results about neighborhoods.

Proposition 4.1 *1. $S \subset N$ is a neighborhood in (N_0, C) if and only if S is a neighborhood in (N_0, C^*). Besides, $(C_S)^* = (C^*)_S$ and*

$$\delta_S = \min_{i \in S, j \in N_0 \setminus S} c^*_{ij} - \max_{i,j \in S} c^*_{ij}.$$

2. If S is a neighborhood in (N_0, C) and $i \in S$, then

$$S = \left\{ j \in N : c^*_{ij} < \min_{k \in S, l \in N_0 \setminus S} c^*_{kl} \right\}.$$

3. If S, S' are two neighborhoods in (N_0, C), (N_0, C) is an irreducible problem, and $S \cap S' \neq \emptyset$, then either $S \subset S'$ or $S' \subset S$.

4. For each $i \in N$, there exists a unique family of subsets of N, $S_1, S_2, ..., S_q$ with $q \geq 0$[1] *such that $\{S_1, ..., S_q\}$ is the set of neighborhoods in (N_0, C) that contain i, and $S_1 \subset S_2 \subset ... \subset S_q$.*

5. There exist no neighborhood in (N_0, C) if and only if $\{\{0, i\}\}_{i \in N}$ is a mcst in (N_0, C).

Under Proposition 1.1, for each neighborhood $S \subset N$, we have $(C^*)_S = (C_S)^*$. We denote this matrix as C^*_S.

We now introduce the family of extra-costs functions, which will be used in the definition of the rules we characterize.

Definition 4.2 *An* extra-costs function *is a function $e : \mathcal{C}^* \times \mathbb{R}_+ \to \mathbb{R}^U_+$ satisfying:*

(E1) $e_i((N, C), x) = 0$ *for all* $(N, C) \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, *and* $i \in U \setminus N$.

(E2) $\sum_{i \in N} e_i((N, C), x) = x$ *for all* $(N, C) \in \mathcal{C}^*$ *and* $x \in \mathbb{R}_+$.

Definition 4.3 *For each extra-costs function e we define the rule f^e as follows. Given a mcstp (N_0, C) and $i \in N$,*

$$f^e_i(N_0, C) := c^*_{0i} - \sum_{S \in Ne(N_0, C), i \in S} \left(\delta_S - e_i\left(\left(S, C^*_S\right), \delta_S\right)\right).$$

When no confusion arises we write $e_i(C^*_S, \delta_S)$ instead of $e_i((S, C^*_S), \delta_S)$

The intuition behind such rules is the following. Initially each agent i pays c^*_{0i}. Now, agents inside neighborhoods are connected among them. For each neighborhood S, the savings are divided among the agents in S following e. The larger is $e_i(C^*_S, \delta_S)$, the smaller is the saving $(\delta_S - e_i(C^*_S, \delta_S))$ corresponding to agent i in neighborhood S.

We compute f^e in two examples.

[1] Case $q = 0$ covers the situation in which agent i has no neighborhoods.

Example 4.2 *Let* $N = \{1,2\}$, $c_{01} = 10$, $c_{02} = 15$, *and* $c_{12} = 2$. *Then,* $c^*_{10} = c^*_{20} = 10$ *and* $c^*_{12} = 2$. *Let* e *be such that for each irreducible problem* (N, C') *and each* $x \in \mathbb{R}_+$, $e_1(C', x) = \frac{3x}{4}$ *and* $e_2(C', x) = \frac{x}{4}$. *There is a unique neighborhood* $S = N$ *with* $\delta_N = 10 - 2 = 8$. *Now,*

$$
\begin{aligned}
f^e_1(C_0) &= c^*_{01} - (\delta_N - e_1(C^*, 8)) = 10 - \left(8 - \frac{3}{4}8\right) = 8 \text{ and} \\
f^e_1(C_0) &= c^*_{02} - (\delta_N - e_2(C^*, 8)) = 10 - \left(8 - \frac{1}{4}8\right) = 4.
\end{aligned}
$$

Example 4.3 ***(continuation of Example 4.1)*** *Let* e *be defined as* $e_j(C', x) = \frac{x}{|N'|}$ *for all* $(N', C') \in \mathcal{C}$ *and* $j \in N$*'. We compute* $f^e_1(C)$. *There are two neighborhoods containing agent 1:* $S_1 = \{1,2\}$ *and* $S_2 = \{1,2,3,4\}$. *Besides* $c^*_{01} = 50$, $\delta_{S_1} = 20$ *and* $\delta_{S_2} = 10$.

Then,

$$
\begin{aligned}
f^e_1(C_0) &= 50 - \left(\delta_{S_2} - e_2\left(C^*_{S_2}, 10\right)\right) - \left(\delta_{S_1} - e_1\left(C^*_{S_1}, 20\right)\right) \\
&= 50 - (10 - 2.5) - (20 - 10) = 32.5.
\end{aligned}
$$

Under $(E2)$, for each $i \in N$, f^e_i can be computed as

$$
f^e_i(N_0, C) = c^*_{0i} - \sum_{S \in Ne(N_0, C), i \in S} \left(\sum_{j \in S \setminus \{i\}} e_j\left(C^*_S, \delta_S\right) \right).
$$

In Proposition 4.2 below we prove that each f^e is a rule, namely, $\sum_{i \in N} f^e_i(N_0, C) = m(N_0, C)$.

Proposition 4.2 *For each extra-costs function* e, f^e *is a rule.*

In Theorem 4.1 below we characterize this family of rules.

Theorem 4.1 *A rule* f *satisfies separability and reductionism if and only if* $f = f^e$ *for some extra-costs function* e.

We give the idea of the proof of Theorem 2. See Appendix for a formal proof. It is easy to see that each f^e satisfies separability and reductionism. Given a rule f satisfying both properties, we need to find e such that $f = f^e$. Given an irreducible problem (N, C) and $x \in \mathbb{R}_+$ we define the *mcstp* (N_0, C') and (N_0, C'') where

$$
\begin{aligned}
c'_{ij} &= \begin{cases} c_{ij} & \text{if } 0 \notin \{i,j\} \\ \max(N, C) & \text{if } 0 \in \{i,j\} \end{cases} \quad \text{and} \\
c''_{ij} &= \begin{cases} c_{ij} & \text{if } 0 \notin \{i,j\} \\ \max(N, C) + x & \text{if } 0 \in \{i,j\}. \end{cases}
\end{aligned}
$$

Now, for each $i \in N$ we define $e_i(N, C) = f_i(N_0, C'') - f_i(N_0, C')$. We then prove that e is an extra-costs function. Finally, using an induction argument over the number of neighborhoods, we prove that $f = f^e$.

5 The set of rules satisfying population monotonicity and cost monotonicity

In this section we characterize the set of rules satisfying both monotonicity properties. Since PM implies SEP and CM implies RED, this set of rules will be a subset of the set characterized in the previous section. This subset is proper, since not all the rules characterized in the previous section satisfy PM or CM. Take for example the extra-cost function given as

$$e_i\left(\left(N,C^*\right),x\right)=\begin{cases}\frac{x}{|N|} & \text{if } x<1 \text{ or } |N|=1\\ 0 & \text{if } x\geq 1,\ |N|>1 \text{ and } i=\min_{j\in N} j\\ \frac{x}{|N|-1} & \text{if } x\geq 1,\ |N|>1 \text{ and } i\neq \min_{j\in N} j\end{cases}$$

for all $(N,C^*)\in\mathcal{C}^*$, $x\in\mathbb{R}_+$ and $i\in N$. The resulting rule f^e proposes an egalitarian share as long as the cost is small (less than 1). Otherwise, the agent with the lowest index should pay nothing (unless $|N|=1$). This rule does not satisfy PM nor CM.

We will prove that the set of rules satisfying PM and CM coincides with the set of rules induced by extra-costs functions satisfying a neighborhood monotonicity property.

We first introduce the concepts we will use. Given $(N^1,C^1),(N^2,C^2)\in\mathcal{C}$, $N^1\cap N^2=\emptyset$, and $a\in\mathbb{R}_+$, we define

$$\left(N^1\cup N^2, C^1\oplus_a C^2\right)$$

as the pair $(N^1\cup N^2, C)$ where $c_{ij}=c^{\alpha}_{ij}$ if $i,j\in N^{\alpha}$ for some $\alpha\in\{1,2\}$, and $c_{ij}=a+\max(N^1,C^1)$ for all $i\in N^1$, $j\in N^2$.

For convenience, we write $C^1\oplus_a C^2\oplus_b C^3$ instead of $(C^1\oplus_a C^2)\oplus_b C^3$, and so on.

Given $a=(a_1,...,a_\Gamma)\in\mathbb{R}^\Gamma_+$, $\left(C^1,...,C^\Gamma\right)\in\mathcal{C}^\Gamma$, and $\gamma\leq\Gamma$ we denote

$$C^\gamma(a)=C^1\oplus_{a_1}C^2\oplus_{a_2}...\oplus_{a_{\gamma-1}}C^\gamma.$$

Notice that, given $\gamma>1$,

$$C^\gamma(a)=C^{\gamma-1}(a)\oplus_{a_{\gamma-1}}C^\gamma. \qquad (2)$$

Definition 5.1 *We say that an extra-costs function* e *satisfies the* Aggregated Neighborhood Monotonicity *(ANM) property if for all disjoint sequences* $\{(N^\gamma,C^\gamma)\}^\Gamma_{\gamma=1}\subset\mathcal{C}^*$, $\Gamma\geq 1$, $i\in N^{\gamma_i}$ *with* $\gamma_i\neq 2$, $a\in\mathbb{R}^\Gamma_+$ *with* $a_\gamma\geq\max(N^{\gamma+1},C^{\gamma+1})-\max(N^\gamma,C^\gamma)$ *for all* $\gamma=1,...,\Gamma-1$, *and* $y\in[0,a_2]$ *(*$y\geq 0$ *when* $\Gamma=1$*), we have*

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^\gamma(a'),a'_\gamma\right)\geq\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^\gamma(a),a_\gamma\right)$$

where $a'=(a_1+y,a_2-y,a_3,...,a_\Gamma)$ *(*$a'=(a_1+y)$ *when* $\Gamma=1$*).*

Let us clarify the implications of this property in the following example.

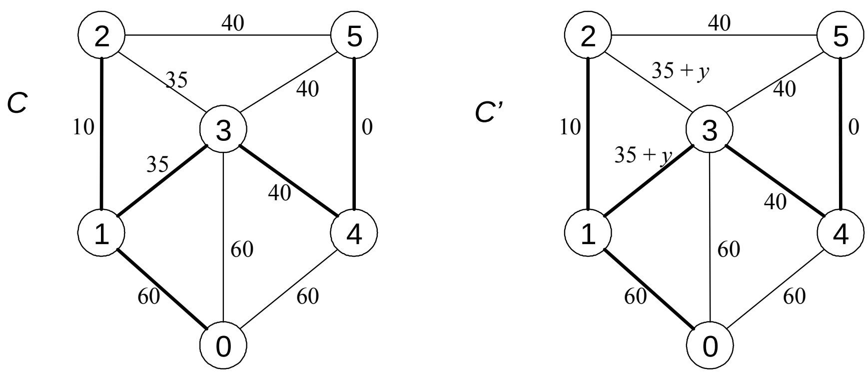

Figure 1: Irreducible cost matrices C and C' with $y \in [0,1]$. The remaining costs can be derived from (1) taking any *mcst*. For example, $c_{02} = \max\{c_{01}, c_{12}\} = 60$ when $t = \{\{0,1\},\{1,2\},\{1,3\},\{3,4\},\{4,5\}\}$.

Example 5.1 *Let* (N_0, C) *and* (N_0, C') *be such that* $N = \{1,2,3,4,5\}$ *and* C *and* C' *are described in Figure 1.For both mcstp we can find a sequence* $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma}$ *as in the definition of* ANM. *Let* $\Gamma = 3$, $N^1 = \{1,2\}, c^1_{12} = 10, N^2 = \{3\}, N^3 = \{4,5\}$, $c^3_{45} = 0$, $a = (25,5,20)$, $a' = (25+y, 5-y, 20)$, *and* $y \in [0,5]$. *We are under the conditions imposed on the definition of* ANM *because* $a_1 = 25 \geq 0 - 10 = \max(N^2, C^2) - \max(N^1, C^1)$, $a_2 = 5 \geq 0 - 0 = \max(N^3, C^3) - \max(N^2, C^2)$.

$C^\gamma(a)$ *and* $C^\gamma(a')$ *are described in Figure 2.*

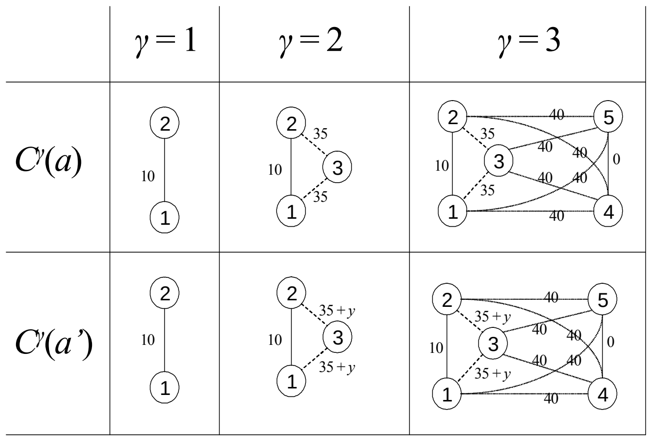

Figure 2: $C^\gamma(a)$ and $C^\gamma(a')$ for $\gamma = 1,2,3$.

Given $i \in N^1$, ANM *says that*

$$\begin{aligned} & e_i\left(C^1(a'), 25+y\right) + e_i\left(C^2(a'), 5-y\right) + e_i\left(C^3(a'), 20\right) \\ \geq\ & e_i\left(C^1(a), 25\right) + e_i\left(C^2(a), 5\right) + e_i\left(C^3(a), 20\right). \end{aligned}$$

Given $i \in N^2$, ANM *says nothing (since we assume* $\gamma_i \neq 2$*).*

Given $i \in N^3$, ANM says that

$$e_i\left(C^3\left(a'\right),20\right) \geq e_i\left(C^3\left(a\right),20\right).$$

We now explain the intuition behind this technical property in the previous example (the same ideas could be applied to the general case). Consider the sequence of neighborhoods $\{1,2\}, \{1,2,3\}$, and $\{1,2,3,4,5\}$ in (N_0, C) and (N_0, C') as in Proposition 1.4. Notice that the costs associated with neighborhood $\{1,2,3\}$ are larger in C' than in C. The extra-costs function e tell us how to divide the savings induced by the neighborhoods. If we want a rule (associated with an extra-costs function) to satisfy PM and CM, some conditions on this sequence of neighborhoods should be imposed. There are three kind of agents in the sequence.

- Agents in $\bigcup_{\gamma=3}^{\Gamma} N^{\gamma}$. In this example this set is $N^3 = \{4,5\}$. Agent 4 (similar arguments could be applied to 5) has only a saving derived from neighborhood $\{1,2,3,4,5\}$. The total saving associated with $\{1,2,3,4,5\}$ is 20 in both *mcstp*. The connection costs of agent 4 are the same in both *mcstp* ($c_{4i} = c'_{4i}$ for all $i \in N_0$). Some agents (5 in this example) also have the same connection costs in both *mcstp*. Some other agents have larger connection costs in C' than in C. In this example each agent $k \in \{1,2,3\}$ satisfies that $c_{ki} \leq c'_{ki}$ for all $i \in N_0$ and $c_{ki'} < c'_{ki'}$ for some $i' \in N_0$. Nevertheless no agent decreases her costs (no agent k satisfies $c_{ki} \geq c'_{ki}$ for all $i \in N_0$ and $c_{ki'} > c'_{ki'}$ for some $i' \in N_0$). Thus, the relative position of agent 4 has improved from C to C' (because others agents are worse off). ANM says that the saving of agent 4 should be not larger in C than in C'.

- Agents in $N^1 = \{1,2\}$. Agent 1 (similar arguments could be applied to 2) has a saving derived from the three neighborhoods of the sequence.

  In $\{1,2\}$ her relative position is the same because the connection costs of agents 1 and agent 2 are the same in C' as in C. The total saving is larger in C' than in C ($25+y$ instead of 25). Thus, the saving of agent 1 associated with neighborhood $\{1,2\}$ should be not larger in C than in C'.

  In $\{1,2,3\}$ her relative position is similar because the connection costs of agents 1, 2, and 3 are larger in C' than in C. The total saving is smaller in C' than in C ($5-y$ instead of 5). Thus, the saving of agent 1 associated with neighborhood $\{1,2,3\}$ should be not larger in C' than in C.

  In $\{1,2,3,4,5\}$ her relative position is worse because the connection costs of agents 1, 2, and 3 are larger in C' than in C and the increment (y) is the same for all agents. Besides, the connection costs of agents 4 and 5 are the same in C' than in C. The total saving is the same in C' than in C (20 in both). Thus, the saving of agent 1 associated with neighborhood $\{1,2,3,4,5\}$ should be not larger in C' than in C.

What should happen with the aggregate saving of agent 1? Since in $\{1,2\}$ there are y units more to be divided between two agents, whereas in $\{1,2,3\}$ there are y units less to be divided between three agents, the aggregated saving of agent 1 associated with neighborhoods $\{1,2\}$ and $\{1,2,3\}$ should be not larger in C than in C'. In $\{1,2,3,4,5\}$ the total amount to be divided is the same. Thus, if we give more importance to changes in the total saving to be divided than in the relative position of each agent (when the total saving do not change), then the total saving of agent 1 should not be larger in C than in C'. That is the way in which ANM works.

- Agents in N^2. ANM says nothing.

We now present the characterization.

Theorem 5.1 *A rule f satisfies PM and CM if and only if $f = f^e$ for some extra-costs function e satisfying the ANM property.*

We give the idea of the proof of Theorem 3. See Appendix for a formal proof. The most elaborate part is the proof that f^e satisfies CM. We do it by considering several cases depending on the structure of the neighborhoods containing each agent. To prove that f^e satisfies PM is relatively easy and we use that f^e satisfies CM. Let f be a rule satisfying PM and CM. By Theorem 2 we can find an extra-costs function e such that $f = f^e$. Making some computations we prove that such e satisfies AGM.

In the literature some authors studied families of rules satisfying both monotonicity properties. The *folk rule* was originally introduced by Feltkamp *et al* (1994) and later studied in Branzei *et al* (2004) and Bergantiños and Vidal-Puga (2007), among others. The *optimistic weighted Shapley rules* are a family of rules defined by Bergantiños and Lorenzo-Freire (2008a, 2008b). *Obligation rules* were introduced by Tijs *et al* (2006) and studied later in Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010). The folk rule is an optimistic weighted Shapley rule. Besides, optimistic weighted Shapley rules are a subset of obligation rules.

We now show how these rules can be included in our family.

Proposition 5.1

1. *Obligation rules are the rules f^e where for each (N, C^*, x) and each $i \in N$,*

$$e_i(C^*, x) = o_i(N)\, x$$

where o is a function that assigns to each N a vector $o(N) \in \Delta(N)$ such that $o_i(S) \geq o_i(N)$ for all $i \in S \subset N$.

2. *Optimistic weighted Shapley rules are the rules f^e such that for each (N, C^*, x) and each $i \in N$,*

$$e_i(C^*, x) = \frac{\omega_i}{\sum_{i \in N} \omega_i} x.$$

where $\omega \in \mathbb{R}^U_+$.

3. The folk rule is the rule f^e where for each (N, C^*, x) and each $i \in N$

$$e_i (C^*, x) = \frac{1}{|N|} x.$$

From Proposition 3, it is clear that the folk rule is a particular case of an optimistic weighted Shapley rule, and those are also obligation rules. Hence, our paper provides a unified framework for all these rules.

Theorem 3 can also be used for identifying classes of rules satisfying PM and CM different from the class of rules studied in Proposition 3. We do it in the following. Let $\{o^x\}_{x \in \mathbb{R}_+}$ be a parametric family of obligation functions, i.e. for each $x \in \mathbb{R}_+$, $o^x (N) \in \Delta (N)$ and $o_i^x (S) \geq o_i^x (N)$ for all $i \in S \subsetneq N$. We assume $o_i^x (N)$ is an integrable function of x for all $i \in N$ and

$$\int_a^{a+c} o_i^x (S)\, dx \geq \int_b^{b+c} o_i^x (N)\, dx \tag{3}$$

for all $i \in S \subsetneq N$ and $a, b, c \in \mathbb{R}_+$.

Proposition 5.2 *Let $\{o^x\}_{x \in \mathbb{R}_+}$ be defined as before. The rule f^e with e defined as*

$$e_i (C^*, x) = \int_0^x o_i^t (N)\, dt$$

for all (C^*, x) and $i \in N$, satisfies CM and PM.

This family contains the obligation rules (simply take $o^x = o$ for all x). Besides, not all the obligation rules can be defined in this way. Take for example $\hat{o} = \{\hat{o}^x\}_{x \in \mathbb{R}_+}$ defined as follows:

$$\hat{o}_i^x (N) := \begin{cases} \frac{1}{|N|} & \text{if } |N| \neq 2 \\ \frac{1+1_{x \leq 1}}{3} & \text{if } |N| = 2 \text{ and } i = \min_{j \in N} j \\ \frac{2-1_{x \leq 1}}{3} & \text{if } |N| = 2 \text{ and } i = \max_{j \in N} j \end{cases}$$

for all $i \in N$, where $1_{x \leq 1} = 1$ if $x \leq 1$ and $1_{x \leq 1} = 0$ otherwise. The resulting rule $f^{\hat{o}}$ is not a obligation rule.

6 Concluding remarks

In this section we summarize the main findings of the paper. Our main objective is to study in *mcstp* the rules satisfying PM and CM.

Given a *mcstp*, its irreducible problem is obtained by reducing the cost of the edges as much as possible, but without changing the total cost associated with any *mcst*. The irreducible core is the core of the irreducible problem and it is a non-empty subset of the core. Our first result says that the set of allocations induced by the rules satisfying PM and CM coincides with the irreducible core.

We introduce the concept of neighborhood. We say that a group of agents S are in a neighborhood if any connection cost between any agent of the neighborhood and any agent outside the neighborhood is larger than any connection cost between any pair of agents in the neighborhood. We define δ_S as the difference between the previous amounts. Initially each agent is connected to the source in the irreducible matrix. Now, agents inside neighborhoods are connected among them. For each neighborhood S, the savings (given by δ_S) are divided among the agents in the neighborhood S following the extra-costs function.

Our second result says that the set of rules satisfying SEP and RED coincides with the set of rules induced by extra-costs functions.

Our third result says the set of rules satisfying PM and CM coincides with the set of rules induced by extra-costs functions satisfying the ANM property.

We also explain how some rules of the literature satisfying PM and CM can be expressed in terms of extra-costs functions. Besides, with the help of our result, we identify a new class of rules satisfying PM and CM.

7 Appendix

We prove the results of the paper.

7.1 Proof of Theorem 3.1

Let (N_0, C) be a *mcstp*. We first prove that $IC(N_0, C) \subset AM(N_0, C)$.

Given $\pi \in \Pi_U$ we define the rule f^π such that for each *mcstp* (N_0', C') and each $i \in N'$,

$$f_i^\pi (N_0', C') = c_{C'^*}(Pre(i, \pi_{N'}) \cup \{i\}) - c_{C'^*}(Pre(i, \pi_{N'})).$$

This rule f^π is well defined because

$$\sum_{i \in N'} f_i^\pi (N_0', C') = c_{C'^*}(N') = m(N_0', C'^*) = m(N_0', C').$$

We now prove that for each $\pi \in \Pi_U$, f^π satisfies PM and CM.

Given a *mcstp* (N_0', C') and $S \subset N'$ the next expression appears, for instance, in Lemma 0 (b) of Bergantiños and Gómez-Rúa (2010),

$$c_{C'^*}(S \cup \{i\}) - c_{C'^*}(S) = \min_{k \in S \cup \{0\}} \{c_{ik}'^*\}.$$

Then,

$$f_i^\pi (N_0', C') = \min_{k \in Pre(i, \pi_{N'}) \cup \{0\}} \{c_{ik}'^*\}$$

Let $S \subset N'$, and $i \in S$. Since $Pre(i, \pi_S) \subset Pre(i, \pi_{N'})$,

$$\min_{k \in Pre(i, \pi_{N'}) \cup \{0\}} \{c_{ik}'^*\} \le \min_{k \in Pre(i, \pi_S) \cup \{0\}} \{c_{ik}'^*\}.$$

Under (1), $C_S'^* \le (C_S')^*$. Then,

$$\min_{k \in Pre(i, \pi_S) \cup \{0\}} \{c_{ik}'^*\} \le \min_{k \in Pre(i, \pi_S) \cup \{0\}} \{(C_S')_{ik}^*\} = f_i^\pi (S_0, C_S').$$

Hence, f^{π} satisfies PM.

Let (N_0', C') and (N_0', C'') be such that $C' \leq C''$. Bergantiños and Vidal-Puga (2007) prove in Lemma 4.2 that $C'^{*} \leq C''^{*}$. Then, for each $i \in N'$,

$$f_i^{\pi}(N_0', C') = \min_{k \in Pre(i, \pi_{N'}) \cup \{0\}} \{c_{ik}'^{*}\} \leq \min_{k \in Pre(i, \pi_{N'}) \cup \{0\}} \{c_{ik}''^{*}\} = f_i^{\pi}(N_0, C'').$$

Hence, f^{π} satisfies CM.

It is well known that (N, c_{C^*}) is a concave game. See, for instance Proposition 3.3 (c) in Bergantiños and Vidal-Puga (2007). Then, $core\,(N, c_{C^*})$ is the convex hull of the family of vector of marginal contributions. Namely, given $x = (x_i)_{i \in N} \in IC\,(N_0, C)$, there exists $w = (w_{\pi})_{\pi \in \Pi_N} \in \Delta\,(\Pi_N)$ such that for each $i \in N$,

$$x_i = \sum_{\pi \in \Pi_N} w_{\pi} \left[c_{C^*}(Pre\,(i, \pi) \cup \{i\}) - c_{C^*}(Pre\,(i, \pi)) \right].$$

Given $\pi \in \Pi_N$ we define $\pi^U \in \Pi_U$ such that we first order agents in N as in π and later agents in $U \backslash N$ as in the identical order. Formally, let $\pi^{id} \in \Pi_U$ be such that $\pi^{id}(i) = i$ for all $i \in U$. We now take $\pi^U = \left(\pi, \pi^{id}_{U \backslash N}\right)$.

We define the rule $f^w = \sum_{\pi \in \Pi_N} w_{\pi} f^{\pi^U}$. For each $i \in N$

$$\begin{aligned} f_i^w(N_0, C) &= \sum_{\pi \in \Pi_N} w_{\pi} f_i^{\pi^U}(N_0, C) \\ &= \sum_{\pi \in \Pi_N} w_{\pi} \left[c_{C^*}\left(Pre\left(i, \pi_N^U\right) \cup \{i\}\right) - c_{C^*}\left(Pre\left(i, \pi_N^U\right)\right) \right] \\ &= \sum_{\pi \in \Pi_N} w_{\pi} \left[c_{C^*}(Pre\,(i, \pi) \cup \{i\}) - c_{C^*}(Pre\,(i, \pi)) \right] \\ &= x_i. \end{aligned}$$

It only remains to prove that f^w satisfies PM and CM. Let (N_0', C') and (N_0', C'') be such that $C' \leq C''$. Since for each $\pi \in \Pi_N$, f^{π^U} satisfies CM, we have that $f^{\pi^U}(N_0', C') \leq f^{\pi^U}(N_0', C'')$. Now

$$f^w(N_0', C') = \sum_{\pi \in \Pi_N} w_{\pi} f^{\pi^U}(N_0', C') \leq \sum_{\pi \in \Pi_N} w_{\pi} f^{\pi^U}(N_0', C'') = f^w(N_0', C'').$$

Similarly, we can prove that f^w satisfies PM. Then, $IC\,(N_0, C) \subset AM\,(N_0, C)$.

We now prove that $AM\,(N_0, C) \subset IC\,(N_0, C)$. Let $x \in AM\,(N_0, C)$. There exists a rule f satisfying CM and PM such that $x = f\,(N_0, C)$. Since CM implies RED

$$f\,(N_0, C) = f\,(N_0, C^*).$$

Bergantiños and Vidal-Puga (2007) prove that if f satisfies PM, then it satisfies core selection, namely $f\,(N_0, C) \in core\,(N, c_C)$. Therefore,

$$f\,(N_0, C^*) \in core\,(N, c_{C^*}) = IC\,(N_0, C). \blacksquare$$

7.2 Proof of Proposition 4.1

(1) Assume that S is a neighborhood in (N_0, C). Because of the definition of the irreducible problem (N_0, C^*), we have that $\min_{i\in S, j\in N_0\backslash S} c_{ij} = \min_{i\in S, j\in N_0\backslash S} c^*_{ij}$. Let $C^1 = (C_S)^*$ and let $C^2 = (C^*)_S$. Let $\tau(S)$ be a *mcst* in (S, C_S). Since S is a neighborhood in (N_0, C), $\tau(S)$ is also a *mcst* in (S, C^1) and (S, C^2). Given $i, j \in S$, let τ_{ij} be the unique simple path in $\tau(S)$ from i to j. Then,

$$c^1_{ij} = \max_{\{i_l, i_{l+1}\}\subset\tau_{ij}} c_{i_l i_{l+1}} = c^2_{ij}.$$

Because of the definition of C^* we have that $\max_{(i,j)\in\tau(S)} c_{ij} = \max_{(i,j)\in\tau(S)} c^*_{ij} = \max_{(i,j)\in S} c^*_{ij}$. Now,

$$\begin{aligned}\delta^*_S &= \min_{i\in S, j\in N_0\backslash S} c^*_{ij} - \max_{\{i,j\}\in\tau(S)} c^*_{ij} \\ &= \min_{i\in S, j\in N_0\backslash S} c_{ij} - \max_{\{i,j\}\in\tau(S)} c_{ij} = \delta_S\end{aligned}$$

which means that S is a neighborhood in (N_0, C^*).

Similarly we can prove that if S is a neighborhood in (N_0, C^*), then S is a neighborhood in (N_0, C).

(2) "$\supset$" Let $j \in N$ be such that $c^*_{ij} < \min_{k\in S, l\in N_0\backslash S} c^*_{kl}$. If $j \notin S$, then $c^*_{ij} \geq \min_{k\in S, l\in N_0\backslash S} c^*_{kl}$, which is a contradiction. Hence, $j \in S$.

"$\subset$": Let $j \in N$ be such that $c^*_{ij} \geq \min_{k\in S, l\in N_0\backslash S} c^*_{kl}$. If $j \in S$, then

$$\delta_S = \min_{k\in S, l\in N_0\backslash S} c^*_{kl} - \max_{k,l\in S} c^*_{kl} \leq c^*_{ij} - c^*_{ij} = 0$$

which cannot be true because S is a neighborhood. Hence, $j \notin S$.

(3) Let $i \in S \cap S'$. If $\min_{k\in S, l\in N_0\backslash S} c^*_{kl} \leq \min_{k\in S', l\in N_0\backslash S'} c^*_{kl}$ then it follows from Proposition 1.2 that $S \subset S'$. If $\min_{k\in S', l\in N_0\backslash S'} c^*_{kl} \leq \min_{k\in S, l\in N_0\backslash S} c^*_{kl}$ then it follows from Proposition 1.2 that $S' \subset S$.

(4) It follows from Proposition 1.3.

(5) Assume that $\{(0, i)\}_{i\in N}$ is not a *mcst*. Let $\{k, l\} \subset N$ be such that $c_{kl} = \min_{i,j\in N} c_{ij}$. Thus, $c_{kl} < \min_{i\in N} c_{0i}$. Then, $S = \{k\} \cup \left\{ i \in N : \max_{\{i_l i_{l+1}\}\subset\tau_{ik}} c_{i_l i_{l+1}} \leq c_{kl} \right\}$ is a neighborhood in (N_0, C).

Assume $\{(0, i)\}_{i\in N}$ is a *mcst*. Then, given any $S \subset N$, we have $\min_{i\in S, j\in N_0\backslash S} c_{ij} = \min_{i\in S} c_{0i}$ and $\max_{\{i,j\}\in\tau(S)} c_{ij} \geq \min_{i\in S} c_{0i}$. Hence

$$\delta_S = \min_{i\in S, j\in N_0\backslash S} c_{ij} - \max_{\{i,j\}\in\tau(S)} c_{ij} \leq 0$$

and S is not a neighborhood. ■

7.3 Proof of Proposition 4.2

Let (N_0, C) be a *mcstp*. Then,

$$\begin{aligned}
\sum_{i \in N} f_i^e(N_0, C) &= \sum_{i \in N} c_{0i}^* - \sum_{i \in N} \sum_{S \in Ne(N_0,C), i \in S} (\delta_S - e_i(C_S^*, \delta_S)) \\
&= \sum_{i \in N} c_{0i}^* - \sum_{S \in Ne(N_0,C)} \left(\sum_{i \in S} (\delta_S - e_i(C_S^*, \delta_S)) \right) \\
&= \sum_{i \in N} c_{0i}^* - \sum_{S \in Ne(N_0,C)} (|S| - 1)\, \delta_S.
\end{aligned}$$

Thus, it is enough to prove that for each *mcstp* (N_0, C),

$$m(N_0, C) + \sum_{S \in Ne(N_0,C)} (|S| - 1)\, \delta_S = \sum_{i \in N} c_{0i}^*.$$

Assume first there exists no neighborhood. Under Proposition 1.5, $\{\{0, i\}\}_{i \in N}$ is a *mcst* in (N_0, C). Hence, $\{\{0, i\}\}_{i \in N}$ is also a *mcst* in (N_0, C^*) and the result is easily checked.

Assume now that there are exactly $k > 0$ neighborhoods and the result is true when there exists less than k neighborhoods. Let S' be a minimal neighborhood (there is no neighborhood S such that $S \subsetneq S'$). Let $\tau(S')$ denote a *mcst* in S'. Since S' is minimal, there exists $\alpha \geq 0$ such that $c_{ij} = \alpha$ for all $(i, j) \in \tau(S')$.

Let t be a *mcst* in (N_0, C). We define C' as $c'_{ij} = \alpha + \delta_{S'}$ if $\{i, j\} \subset S'$ and $c'_{ij} = c_{ij}$ otherwise. Thus,

- t is also a *mcst* in (N_0, C');
- $c'^*_{0i} = c^*_{0i}$ for all $i \in N$;
- $m(N_0, C') = m(N_0, C) + (|S'| - 1)\, \delta_{S'}$; and
- $\{S : S \text{ is a neighborhood in } (N_0, C')\}$ coincides with $\{S : S \text{ is a neighborhood in } (N_0, C)\} \setminus \{S'$

Now, applying the induction hypothesis, we have

$$\begin{aligned}
& m(N_0, C) + \sum_{S \in Ne(N_0,C)} (|S| - 1)\, \delta_S \\
=\ & m(N_0, C') - (|S'| - 1)\, \delta_{S'} + \sum_{S \in Ne(N_0,C)} (|S| - 1)\, \delta_S \\
=\ & m(N_0, C') + \sum_{S \in Ne(N_0,C')} (|S| - 1)\, \delta_S \\
=\ & \sum_{i \in N} c'^*_{0i} = \sum_{i \in N} c^*_{0i}. \ \blacksquare
\end{aligned}$$

7.4 Proof of Theorem 4.1

Let e be any extra-costs function and f^e be the associated rule. It is obvious that f^e satisfies RED.

In order to prove that f^e also satisfies SEP, let $S \subset N$ such that $m(N_0, C) = m(S_0, C_{S_0}) + m\left((N \backslash S)_0, C_{(N \backslash S)_0}\right)$. Then,

$$Ne(N_0, C) = Ne(S_0, C_{S_0}) \cup Ne\left((N \backslash S)_0, C_{(N \backslash S)_0}\right).$$

Hence, $f_i^e(N_0, C) = f_i^e(S_0, C_{S_0})$ and this proves that f satisfies SEP.

We now prove that if f satisfies SEP and RED, then $f = f^e$ for some extra-costs function e. Let f be such a rule.

Given $(N, C^*) \in \mathcal{C}^*$ and $a \in \mathbb{R}_+$, we define $\left(N_0, C^{*(a)}\right)$ as the *mcstp* given by $c_{ij}^{*(a)} = c_{ij}^*$ for all $i, j \in N$ and $c_{0i}^{*(a)} = a$ for all $i \in N$. Notice that $\left(N, C^{*(a)}\right)$ is an irreducible problem when $a \geq \max(N, C^*)$.

For all $(N, C^*) \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \in N$ we define

$$e_i(C^*, x) = f_i\left(N_0, C^{*(\max(N,C^*)+x)}\right) - f_i\left(N_0, C^{*(\max(N,C^*))}\right).$$

Given $i \in U \backslash N$ we define $e_i(C^*, x) = 0$.

We first prove that e is an extra-costs function. By definition, $e_i(C^*, x) = 0$ for all $(N, C^*) \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, $i \in U \backslash N$. Besides,

$$\begin{aligned}\sum_{i \in N} e_i(C^*, x) &= m\left(N_0, C^{*(\max(N,C^*)+x)}\right) - m\left(N_0, C^{*(\max(N,C^*))}\right) \\ &= m(N, C^*) + \max(N, C^*) + x - m(N, C^*) - \max(N, C^*) \\ &= x.\end{aligned}$$

Hence, e is an extra-costs function.

We need to prove that $f = f^e$. It is obvious that for any *mcstp* (N_0, C), $f^e(N_0, C) = f^e(N_0, C^*)$. Since f satisfies RED, $f(N_0, C) = f(N_0, C^*)$. Thus, it is enough to prove that $f^e(N_0, C^*) = f(N_0, C^*)$.

We proceed by induction on the number of neighborhoods $Ne(N_0, C)$. Assume $|Ne(N_0, C)| = 0$.

Under Proposition 1.5, $\{(0, i)\}_{i \in N}$ is a *mcst* in (N_0, C). Since f satisfies SEP, $f_i(N_0, C) = f_i\left(\{i\}_0, C_{\{i\}_0}\right) = c_{0i}$. Besides, since $\{(0, i)\}_{i \in N}$ is a *mcst* in (N_0, C), we have $c_{0i} = c_{0i}^*$ for all $i \in N$ and hence $f^e(N_0, C) = f(N_0, C)$.

Assume now the result is true for *mcstp* with less than $|Ne(N_0, C)|$ neighborhoods.

Assume first that $\max(N_0, C^*) \geq \max_{i \in N} c_{0i}^*$. By Proposition 3.1 in Bergantiños and Vidal-Puga (2007) we can find a *mcstp* t in (N_0, C^*) and $(i, j) \in t$ such that $c_{ij}^* = \max(N_0, C^*)$ and i is in the unique simple path in t from j to 0. Let S be the set of agents k satisfying that j is in the unique simple path in t from k to 0. Then, $S \neq \varnothing$, $S \neq N$ and $m(N_0, C^*) = m\left(S_0, C_{S_0}^*\right) + m\left((N \backslash S)_0, C_{(N \backslash S)_0}\right)$. Under SEP, $f_i(N_0, C^*) = f_i\left(S_0, C_{S_0}^*\right)$ for all $i \in S$ and

$f_i(N_0, C^*) = f_i\left((N\backslash S)_0, C_{(N\backslash S)_0}\right)$ for all $i \in N\backslash S$. Repeating this argument we can find a partition $\{S_1, ..., S_p\}$ of N satisfying that for each $k = 1, ...p$ $\max\left(S_k, C^*_{S_k}\right) < \max_{i\in S_k} c^*_{0i}$ and $f_i(N_0, C^*) = f_i\left((S_k)_0, C^*_{(S_k)_0}\right)$ for each $i \in S_k$.

Hence, we can assume that $\max(N_0, C^*) < \max_{i\in N} c^*_{0i}$. Since (N_0, C^*) is irreducible, $\max_{i\in N} c^*_{0i} = c^*_{0i}$ for all $i \in N$. Hence, N is a neighborhood in (N_0, C^*) and $\delta_N = \max_{i\in N} c^*_{0i} - \max(N_0, C^*)$. Now, for each $i \in N$,

$$\begin{aligned} f_i(N_0, C^*) &= f_i\left(N_0, C^{*(\max(N_0,C^*)+\delta_N)}\right) \\ &= e_i(C^*, \delta_N) + f_i\left(N_0, C^{*(\max C^*)}\right). \end{aligned}$$

We define $C' = C^{*(\max(N_0,C^*))}$. Then (N_0, C') is an irreducible problem satisfying $Ne(N_0, C^*) = Ne(N_0, C') \cup \{N\}$. For each $S \in Ne(N_0, C')$, $\delta_S = \delta'_S$, and $c'^*_{0i} = c^*_{0i} - \delta_N$. Hence, applying the induction hypothesis, for each $i \in N$,

$$\begin{aligned} f_i(N_0, C^*) &= e_i(C^*, \delta_N) + f_i(N_0, C') \\ &= e_i(C^*, \delta_N) + c'^*_{0i} + \sum_{S\in Ne(N_0,C')} \left(e_i\left(C^*_S, \delta_S\right) - \delta_S\right) \\ &= e_i(C^*, \delta_N) + c^*_{0i} - \delta_N + \sum_{S\in Ne(N_0,C')} \left(e_i\left(C^*_S, \delta_S\right) - \delta_S\right) \\ &= c^*_{0i} + \sum_{S\in Ne(N_0,C^*)} \left(e_i\left(C^*_S, \delta_S\right) - \delta_S\right) \\ &= f^e_i(N_0, C^*). \blacksquare \end{aligned}$$

7.5 Proof of Theorem 5.1

We start the proof with the following Lemma.

Lemma 1. (i) Given $(N', C'), (N'', C'') \in \mathcal{C}^*$ and $a \in \mathbb{R}_+$ with $N' \cap N'' = \emptyset$ and $a \geq \max(N'', C'') - \max(N', C')$, then $C' \oplus_a C'' \in \mathcal{C}^*$.

(ii) Given a disjoint sequence $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset \mathcal{C}^*$, $\Gamma > 1$, $a \in \mathbb{R}^\Gamma_+$ with $a_\gamma \geq \max(N^{\gamma+1}, C^{\gamma+1}) - \max(N^\gamma, C^\gamma)$ for all $\gamma = 1, ..., \Gamma - 1$, and $y \in [0, a_2]$, then $C^\gamma(a) \in \mathcal{C}^*$ and $C^\gamma(a') \in \mathcal{C}^*$ for all $\gamma = 1, ..., \Gamma$, where $a' = (a_1 + y, a_2 - y, a_3, ..., a_\Gamma)$.

Proof of Lemma 1. (i) Let $C = C' \oplus_a C''$. Then, $a + \max C' = \max C$. Hence, we can find a *mcst* t in $(N' \cup N'', C)$ and $(N' \cup N'', C^*)$ such that $t = t^1 \cup t^2 \cup \{(k^1, k^2)\}$ where t^1 is a *mcst* in (N', C'), t^2 is a *mcst* in (N'', C''), $k^1 \in N^1$ and $k^2 \in N^2$. Since $c_{k^1k^2} = \max C \geq c_{ij}$ for all $(i, j) \in t^1 \cup t^2$ we can deduce, using the definition of irreducible matrix, that $C = C^*$.

(ii) We assume $\gamma > 1$, since the case $\gamma = 1$ is trivial. We proceed by induction on Γ. For $\Gamma = 2$, the result follows from (i) because $a'_1 = a_1 + y \geq a_1 \geq \max C^2 - \max C^1$. Assume the result is true for sequences with less than Γ *mcstp*'s, $\Gamma \geq 3$. Under the induction hypothesis, we have $C^\gamma(b)$, $C^\gamma(b') \in \mathcal{C}^*$ where $\gamma = 1, ..., \Gamma - 1$, $b = (a_1, ..., a_{\Gamma-1})$ and $b' = (a_1 + y, a_2 - y, a_3, ..., a_{\Gamma-1})$. Now, it is clear that $C^\gamma(a) = C^\gamma(b)$ and $C^\gamma(a') = C^\gamma(b')$

for all $\gamma = 1, ..., \Gamma - 1$. Hence, the result holds for any $\gamma < \Gamma$. Assume now $\gamma = \Gamma$. We have

$$C^{\Gamma}(a) \overset{(2)}{=} C^{\Gamma-1}(a) \oplus_{a_{\Gamma-1}} C^{\Gamma}(a) \overset{(i)}{\in} \mathcal{C}^*$$

and

$$C^{\Gamma}(a') \overset{(2)}{=} C^{\Gamma-1}(a') \oplus_{a'_{\Gamma-1}} C^{\Gamma}(a').$$

In order to apply (i) to this last expression (so that $C^{\Gamma}(a') \in \mathcal{C}^*$) we have to prove that

$$a'_{\Gamma-1} \geq \max C^{\Gamma}(a') - \max C^{\Gamma-1}(a'). \tag{4}$$

By definition, $\max C^{\gamma}(a') = \max C^{\gamma}(a)$ for all $\gamma \neq 2$, whereas $\max C^2(a') = \max C^2(a) + y$. Hence, for $\Gamma > 3$,

$$\max C^{\Gamma}(a') - \max C^{\Gamma-1}(a') = \max C^{\Gamma}(a) - \max C^{\Gamma-1}(a) \leq a_{\Gamma-1} = a'_{\Gamma-1}$$

and for $\Gamma = 3$,

$$\max C^3(a') - \max C^2(a') = \max C^3(a) - \max C^2(a) - y \leq a_2 - y = a'_2. \blacksquare$$

We now prove that if $f = f^e$ with e satisfying ANM, then f satisfies CM and PM.

Following Norde *et al* (2004), for each N_0 we define the set Σ_{N_0} of linear orders on the edges as the set of all bijections $\sigma : \left\{1, ..., \binom{n+1}{2}\right\} \to \{\{i,j\} : i,j \in N_0\}$. For each *mcstp* (N_0, C), there exists at least one linear order $\sigma \in \Sigma_{N_0}$ such that $c_{\sigma(1)} \leq c_{\sigma(2)} \leq ... \leq c_{\sigma\left(\binom{n+1}{2}\right)}$. For any $\sigma \in \Sigma_{N_0}$, we define the set

$$K^{\sigma} = \left\{(N_0, C) : c_{\sigma(k)} \leq c_{\sigma(k+1)} \text{ for all } k = 1, 2, ... \binom{n+1}{2}\right\},$$

which we call the *Kruskal cone* with respect to σ. One can easily see that $\bigcup_{\sigma \in \Sigma_{N_0}} K^{\sigma}$ coincides with the set of all *mcstp* where the set of agents is N.

We say that a non-empty set $S \subset N$ is a *quasi-neighborhood* in (N_0, C) if $\delta_S \geq 0$. Let $qNe(N_0, C) = \{S \subset N, S \neq \emptyset : \delta_S \geq 0\}$ denote the set of quasi-neighborhoods in (N_0, C). Clearly, $Ne(N_0, C) \subset qNe(N_0, C)$.

We now prove that f satisfies CM. It is enough to prove that $f(N_0, C) \leq f(N_0, C')$ when there exists $\{k, l\} \subset N_0$ such that $c'_{kl} > c_{kl}$ and $c'_{ij} = c_{ij}$ otherwise. Let (k, l), (N_0, C) and (N_0, C') be defined in this way.

For any $x \in [0,1]$, the *mcstp* (N_0, C^x) defined as $c^x_{ij} = (1-x)c_{ij} + xc'_{ij}$ satisfies $c'_{kl} \geq c^x_{kl} \geq c_{kl}$ and $c^x_{ij} = c_{ij}$ otherwise. Since Σ_{N_0} is a finite set, there exist a sequence $\{x^1, x^2, ...x^p\} \subset [0,1]$ with $x^1 = 0$ and $x^p = 1$ such that, for all r, we have $x^r < x^{r+1}$ and $\left(N_0, C^{x^r}\right)$ and $\left(N_0, C^{x^{r+1}}\right)$ belong to the same Kruskal cone.

Hence, it is enough to prove that $f(N_0, C) \leq f(N_0, C')$ when both (N_0, C) and (N_0, C') belong to the same Kruskal cone. An immediate consequence is that there exists a common *mcst* t in both (N_0, C) and (N_0, C').

Since f satisfies RED, $f(N_0, C) = f(N_0, C^*)$. If $\{k, l\} \notin t$, then $C^* = C'^*$. Thus

$$f(N_0, C) = f(N_0, C^*) = f(N_0, C'^*) = f(N_0, C').$$

Hence, we assume $\{k, l\} \in t$. This implies $c_{kl} = c^*_{kl}$ and $c'_{kl} = c'^*_{kl}$. Let $\alpha = c'^*_{kl} - c^*_{kl} > 0$.

Another consequence of (N_0, C), (N_0, C') being in the same Kruskal cone is that, for any $S \subset N$, $|S| > 1$, there exist a $mcst$ $\tau(S)$ in (N_0, C) and (N_0, C'), $i^1, i^2, j^2 \in S$, $j^1 \in N_0 \backslash S$ with $\{i^2, j^2\} \in \tau(S)$ such that

$$\begin{aligned} \delta_S &= \min_{i' \in S, j' \in N_0 \backslash S} c_{i'j'} - \max_{\{i',j'\} \in \tau(S)} c_{i'j'} = c_{i^1 j^1} - c_{i^2 j^2} \text{ and} \\ \delta'_S &= \min_{i' \in S, j' \in N_0 \backslash S} c'_{i'j'} - \max_{\{i',j'\} \in \tau(S)} c'_{i'j'} = c'_{i^1 j^1} - c'_{i^2 j^2}. \end{aligned}$$

Thus δ_S and δ'_S cannot have opposite sign. Namely, $\delta_S > 0$ implies $\delta'_S \geq 0$. Thus, $Ne(N_0, C) \subset qNe(N_0, C')$ and $Ne(N_0, C') \subset qNe(N_0, C)$.

Given any $X \subset 2^N$ with $Ne(N_0, C) \subset X \subset qNe(N_0, C)$ and $i \in N$ we have

$$f_i(N_0, C) = c^*_{0i} - \sum_{i \in S \in X} (\delta_S - e_i(C^*_S, \delta_S)). \tag{5}$$

The reason is that for any $S \in qNe(N_0, C) \backslash Ne(N_0, C)$, $\delta_S = 0$ and hence $\delta_S - e_i(C^*_S, \delta_S) = 0 - e_i(C^*_S, 0) = 0$.

We define $X = Ne(N_0, C) \cup Ne(N_0, C')$. Clearly, $Ne(N_0, C) \subset X \subset qNe(N_0, C)$ and $Ne(N_0, C') \subset X \subset qNe(N_0, C')$.

Fix $i \in N$. We need to prove that $f_i(N_0, C) \leq f_i(N_0, C')$. Under (5), we have

$$\begin{aligned} f_i(N_0, C) &= c^*_{0i} - \sum_{i \in S \in X} (\delta_S - e_i(C^*_S, \delta_S)) \\ f_i(N_0, C') &= c'^*_{0i} - \sum_{i \in S \in X} (\delta'_S - e_i(C'^*_S, \delta'_S)). \end{aligned}$$

We have seen above that

$$\delta_S = c_{i^1 j^1} - c_{i^2 j^2} \text{ and } \delta'_S = c'_{i^1 j^1} - c'_{i^2 j^2}$$

for some $i^1, i^2, j^2 \in S$, $j^1 \in N_0 \backslash S$ with $\{i^2, j^2\} \in t_S$.

By hypothesis, $c_{jj'} = c'_{jj'}$ for all $\{j, j'\} \neq \{k, l\}$. Hence, $\delta_S = \delta'_S$ unless $\{i^1, j^1\} = \{k, l\}$ or $\{i^2, j^2\} = \{k, l\}$.

Given $S \in X$ and $\delta_S \neq \delta'_S$ we study both cases:

1. If $\{i^1, j^1\} = \{k, l\}$, then $\delta'_S = \delta_S + \alpha$. Besides, there can be at most two such S. One of them contains node k (if any) and the other contains node l (if any). Assume, on the contrary, that there exist two $S' \in X, S \neq S'$ with $k \in S \cap S'$ (the case for $l \in S$ is analogous). Hence,

$$c'_{kl} = c'^*_{kl} = \min_{i' \in S, j' \in N_0 \backslash S} c'^*_{i'j'} = \min_{i' \in S', j' \in N_0 \backslash S'} c'^*_{i'j'}.$$

Since $k \in S \cap S'$, under Proposition 1.4, $S \subsetneq S'$ or $S' \subsetneq S$. Assume w.l.o.g. $S \subsetneq S'$. Then,

$$\begin{aligned} c'^*_{kl} &= \min_{i' \in S, j' \in N_0 \setminus S} c'^*_{i'j'} \leq \min_{i' \in S, j' \in S' \setminus S} c'^*_{i'j'} \\ &\leq \max_{i',j' \in S'} c'^*_{i'j'} \leq \min_{i' \in S', j' \in N_0 \setminus S'} c'^*_{i'j'} = c'^*_{kl} \end{aligned}$$

which implies that no inequality is strict. In particular, $\max_{i',j' \in S'} c'^*_{i'j'} = c'^*_{kl}$. Since $\{k,l\} \nsubseteq S'$, $\max_{i',j' \in S'} c'^*_{i'j'} = \max_{i',j' \in S'} c^*_{i'j'}$ and hence

$$\delta_{S'} = \min_{i' \in S', j' \in N_0 \setminus S'} c^*_{i'j'} - \max_{i',j' \in S'} c^*_{i'j'} = c^*_{kl} - c'^*_{kl} = -\alpha < 0,$$

which is a contradiction.

2. If $\{i^2, j^2\} = \{k, l\}$, then $\delta'_S = \delta_S - \alpha$. Besides, there can be at most one such S. Assume, on the contrary, that there exists $S' \in X$, $S \neq S'$, $k, l \in S \cap S'$, and

$$c_{kl} = c^*_{kl} = \max_{i',j' \in S} c^*_{i'j'} = \max_{i',j' \in S'} c^*_{i'j'}.$$

Since $k \in S \cap S'$, under Proposition 1.4, $S \subsetneq S'$ or $S' \subsetneq S$. Assume w.l.o.g. $S \subsetneq S'$. Then,

$$c^*_{kl} = \max_{i',j' \in S} c^*_{i'j'} \leq \min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} \leq \min_{i' \in S, j' \in S' \setminus S} c^*_{i'j'} \leq \max_{i',j' \in S'} c^*_{i'j'} = c^*_{kl}$$

which implies that no inequality is strict. Thus, $\min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} = c^*_{kl}$ and hence

$$\delta_S = \min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} - \max_{i',j' \in S} c^*_{i'j'} = c^*_{kl} - c^*_{kl} = 0,$$

which implies $\delta'_S = \delta_S - \alpha = -\alpha < 0$, which is a contradiction.

Let $S^k = \{j \in N_0 : c'^*_{kj} < c'^*_{kl}\}$ and let $S^l = \{j \in N_0 : c'^*_{kj} < c'^*_{kl}\}$. Both S^k and S^l are non-empty (because $k \in S^k$ and $l \in S^l$) and disjoint (it follows from $\{k,l\} \in t$). Since they are disjoint, we can assume w.l.o.g. $0 \notin S^k$. Let $S_1 = S^k$. If $|S_1| > 1$, then

$$\begin{aligned} l &\notin S_1, \\ c'^*_{kl} &= \min_{i' \in S_1, j' \in N_0 \setminus S_1} c'^*_{i'j'}, \\ \delta'_{S_1} &= c'^*_{kl} - \max_{i',j' \in S} c'^*_{i'j'} > 0 \end{aligned}$$

and hence either $S_1 \in Ne(N_0, C')$ or $S_1 = \{k\}$.

Assume that $S_1 \in Ne(N_0, C')$. Since (N_0, C) and (N_0, C') are in the same Kruskal cone, $\delta_{S_1} = c^*_{i^1 j^1} - c^*_{i^2 j^2}$ and $\delta'_{S_1} = c'^*_{i^1 j^1} - c'^*_{i^2 j^2}$. Since $\delta'_{S_1} > 0$ we deduce that $\delta_{S_1} \geq 0$. Hence $S_1 \in qNe(N_0, C)$. Now, satisfies condition 1. Then $\delta'_{S_1} = \delta_{S_1} + \alpha$ when $|S_1| > 1$.

Let $S_2 = \left\{ j \in N_0 : c^*_{kj} \leq c^*_{kl} \right\}$. Clearly, $\{k, l\} \subset S_2$. Notice that if $0 \in S_2$ then $S_2 \notin X$. Now, if $0 \notin S_2$ then $S_2 \in X$. Besides $S_1 \subsetneq S_2$ and there is no $S \in X$, $S \neq S_1$, such that $S_1 \subsetneq S \subsetneq S_2$.

In case $0 \notin S_2$, we have that S_2 satisfies condition 2. Hence $\delta'_{S_2} = \delta_{S_2} - \alpha$.

Let $F = \{S \in Ne(N_0, C) : S_1 \subset S, \delta_S = \delta'_S\}$ and let $F' = \{S \in Ne(N_0, C') : S_1 \subset S, \delta_S = \delta'_S\}$. Then, $F = F'$ ($F = F' = \emptyset$ is also possible) and $S_1, S_2 \notin F$. By Proposition 1.3 we can assume that $F = \{S_3, S_4, ..., S_\Gamma\}$ for some $\Gamma \geq 2$ ($\Gamma = 2$ when $F = \emptyset$) and $S_\gamma \subsetneq S_{\gamma+1}$ for all $\gamma = 3, ..., \Gamma - 1$.

Let $G = \{S \in X : S_1 \subset S\}$. Clearly, either $G = \{S_1, ..., S_\Gamma\}$ (when $S_1 \in Ne(N_0, C')$) or $G = \{S_2, ..., S_\Gamma\}$ (when $S_1 = \{k\}$). Besides, $S_\gamma \subsetneq S_{\gamma+1}$ for all $\gamma = 1, 2, ..., \Gamma - 1$.

If $i \notin S_\Gamma$, then $f_i(N_0, C) = f_i(N_0, C')$. We assume $i \in S_\gamma$ for some $\gamma \in \{1, ..., \Gamma\}$. Let γ_i be the minimum of these γ's. We have two cases:

<u>Case 1</u>: $\Gamma = 1$. This means $S_2 \notin X$. Since $\delta_{S_2} \geq 0$, we have $0 \in S_2$, which implies $c^*_{0k} \leq c^*_{kl}$ and also $c'^*_{0k} \leq c'^*_{kl}$.

<u>Subcase 1.1</u>: $S_1 = \{k\} = \{i\}$. This implies $X = \emptyset$ and hence

$$f_i(N_0, C') - f_i(N_0, C) = c'^*_{0i} - c^*_{0i} \geq 0.$$

<u>Subcase 1.2</u>: $S_1 \in X$. This implies $c'^*_{0k} \geq c'^*_{kl}$ and hence $c'^*_{0k} = c'^*_{kl}$. Thus $c'^*_{0i} - c^*_{0i} = \alpha$ and $C^*_{S_1} = C'^*_{S_1}$. Hence,

$$\begin{aligned}
& f_i(N_0, C') - f_i(N_0, C) \\
= \;& c'^*_{0i} - \left(\delta'_{S_1} - e_i\left(C'^*_{S_1}, \delta'_{S_1}\right)\right) - c^*_{0i} + \left(\delta_{S_1} - e_i\left(C^*_{S_1}, \delta_{S_1}\right)\right) \\
= \;& c'^*_{0i} - c^*_{0i} - \left(\delta_{S_1} + \alpha - e_i\left(C^*_{S_1}, \delta_{S_1} + \alpha\right)\right) + \left(\delta_{S_1} - e_i\left(C^*_{S_1}, \delta_{S_1}\right)\right) \\
= \;& e_i\left(C^*_{S_1}, \delta_{S_1} + \alpha\right) - e_i\left(C^*_{S_1}, \delta_{S_1}\right) \geq 0
\end{aligned}$$

where the last inequality comes from applying ANM to $\left\{\left(S_1, C^*_{S_1}\right)\right\}$ with $\Gamma = 1$, $a_1 = \delta_{S_1}$ and $y = \alpha$.

<u>Case 2</u>: $\Gamma > 1$. This means that $S_2 \in X$ and hence $0 \notin S^l$. Thus we can take $S_1 = S^k$ or $S_1 = S^l$. We have that $S_2 = S^k \cup S^l$. If $i \in S_2$ we choose S_1 such that $i \in S_1$. Thus, $\gamma_i \neq 2$ which implies $c'^*_{0i} = c^*_{0i}$.

In this case,

$$\begin{aligned}
& f_i(N_0, C') - f_i(N_0, C) \\
= \;& c'^*_{0i} - c^*_{0i} - \sum_{i \in S \in X} \left(\delta'_S - \delta_S - e_i\left(C'^*_S, \delta'_S\right) + e_i\left(C^*_S, \delta_S\right)\right).
\end{aligned}$$

For any $S \in X \backslash G$ with $i \in S$, we have $C^*_S = C'^*_S$, which implies $\delta_S = \delta'_S$. Hence,

$$\begin{aligned}
& f_i(N_0, C') - f_i(N_0, C) \\
= \;& \sum_{\gamma=\gamma_i}^{\Gamma} \left(-\delta'_{S_\gamma} + \delta_{S_\gamma} + e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right) - e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right)\right) \\
= \;& \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} \left(\delta'_{s_\gamma} - \delta_{s_\gamma}\right).
\end{aligned}$$

The last term is zero, because $\delta'_{S_1} = \delta_{S_1} + \alpha$, $\delta'_{S_2} = \delta_{S_2} - \alpha$, and $\delta'_{S_\gamma} = \delta_{S_\gamma}$ otherwise (remember that $\gamma_i \neq 2$). Hence,

$$f_i(N_0, C') - f_i(N_0, C) = \sum_{\gamma=\gamma_i}^{\Gamma} \left(e_i \left(C'^*_{S_\gamma}, \delta'_{S_\gamma} \right) \right) - \sum_{\gamma=\gamma_i}^{\Gamma} \left(e_i \left(C^*_{S_\gamma}, \delta_{S_\gamma} \right) \right).$$

We now define $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma}$, $a \in \mathbb{R}^{\Gamma}_{+}$ and $y \in [0, a_2]$ so that $e_i \left(C'^*_{S_\gamma}, \delta'_{S_\gamma} \right) = e_i \left(C^\gamma(a'), a'_\gamma \right)$ and $e_i \left(C^*_{S_\gamma}, \delta_{S_\gamma} \right) = e_i \left(C^\gamma(a), a_\gamma \right)$ for all γ. Under ANM, this will prove that the above expression is non-negative.

Let $N^1 = S_1$, $C^1 = C^*_{N^1}$, and $a_1 = \delta_{S_1}$. In general, for any $\gamma = 2, ..., \Gamma$, $N^\gamma = S_\gamma \backslash S_{\gamma-1}$, $C^\gamma = (C^*)_{N^\gamma}$, and $a_\gamma = \delta_{S_\gamma}$. We also define $y = \alpha$. Since $c'^*_{kl} = c^*_{kl} + \alpha$, we have that $\alpha \leq a_2$ and hence $y \in [0, a_2]$.

Clearly, $C'^*_{S_1} = C^1$. Now, we prove that $C'^*_{S_2} = C^1 \oplus_{a_1+\alpha} C^2 = C^2(a')$. Let $C^\alpha = C'^*_{S_2}$ and $C^\beta = C^1 \oplus_{a_1+\alpha} C^2$. We have that $C^\alpha = (C_{S_2} + \alpha I_{kl})^*$. Then, $c^\alpha_{ij} = c^\beta_{ij}$ for all $i, j \in N^1$ and all $i, j \in N^2$.

Let $k^1 \in N^1$ and $k^2 \in N^2$. Then,

$$\begin{aligned} c^\beta_{k^1k^2} &= \max C^1 + a_1 + \alpha = \max C^1 + \delta_{S_1} + \alpha = \min_{\substack{i \in N^1 \\ j \in N_0 \backslash N^1}} c_{ij} + \alpha \\ &= c_{kl} + \alpha = c^\alpha_{k^1k^2}. \end{aligned}$$

Analogously, $C'^*_{S_3} = (C_{S_3} + \alpha I_{kl})^* = (C^1 \oplus_{a_1+\alpha} C^2) \oplus_{a_2-\alpha} C^3 = C^3(a')$. In general, $C'^*_{S_\gamma} = \left(C_{S_\gamma} + \alpha I_{kl} \right)^* = C^1 \oplus_{a_1+\alpha} C^2 \oplus_{a_1-\alpha} C^3 \oplus_{a_3} ... \oplus_{a_{\gamma-1}} C^\gamma = C^\gamma(a')$ for all $\gamma = 3, ..., \Gamma$.

Similarly, we can prove that $C^*_{S_\gamma} = C^\gamma(a)$ for all $\gamma = 1, ..., \Gamma$.

Hence, by applying ANM, we have

$$f_i(N_0, C') - f_i(N_0, C) \geq 0.$$

We now prove that f satisfies PM. Under Theorem 2, we know that f satisfies SEP. It is enough to prove that for each $mcstp$ (N_0, C) and $j \in N$, $f_i(N_0, C) \leq f_i\left(N_0 \backslash \{j\}, C_{N_0 \backslash \{j\}}\right)$ for all $i \in N \backslash \{j\}$. Let (N_0, C') be defined as $c'_{ii'} = c_{ii'}$ for all $i, i' \in N \backslash \{j\}$ and $c'_{ij} = \max C_{N_0 \backslash \{j\}}$ for all $i \in N_0 \backslash \{j\}$. Clearly, $m(N_0, C') = m\left(N_0 \backslash \{j\}, C'_{N_0 \backslash \{j\}}\right) + m\left(\{j\}_0, C'_{\{j\}_0}\right)$. Under SEP, $f_i(N_0, C') = f_i\left(N_0 \backslash \{j\}, C'_{N_0 \backslash \{j\}}\right)$ for all $i \in N \backslash \{j\}$. Given $i \in N \backslash \{j\}$, under CM,

$$f_i(N_0, C) \leq f_i(N_0, C') = f_i\left(N_0 \backslash \{j\}, C'_{N_0 \backslash \{j\}}\right) = f_i\left(N_0 \backslash \{j\}, C_{N_0 \backslash \{j\}}\right).$$

We now prove that if f satisfies CM and PM, then $f = f^e$ for some e satisfying ANM.

We define e as in the proof of Theorem 2. Namely, for all $C^* \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \in N$,

$$e_i(C^*, x) = f_i\left(N_0, C^{*(\max C^* + x)}\right) - f_i\left(N_0, C^{*(\max C^*)}\right)$$

and $e_i(C^*, x) = 0$ for all $i \notin N$. We already proved (proof of Theorem 2) that e is an extra-costs function and $f = f^e$.

Hence, we only need to check that e satisfies ANM. Let $\{(N^{\gamma}, C^{\gamma})\}_{\gamma=1}^{\Gamma} \subset C^*$ be a disjoint sequence with $\Gamma \geq 1$, $i \in N^{\gamma_i}$ with $\gamma_i \neq 2$, $a \in \mathbb{R}_+^{\Gamma}$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^{\gamma}$ for all $\gamma = 1, ..., \Gamma - 1$ and $y \in [0, a_2]$ (or simply $y \geq 0$, when $\Gamma = 1$).

Assume first that $\Gamma = 1$. We need to prove

$$e_i \left(C^1, a_1 + y\right) - e_i \left(C^1, a_1\right) \geq 0.$$

Let $C = C^1$. By definition,

$$\begin{aligned} & e_i (C, a_1 + y) - e_i (C, a_1) \\ = \; & f_i \left(N_0, C^{*(\max C^* + a_1 + y)}\right) - f_i \left(N_0, C^{*(\max C^*)}\right) - f_i \left(N_0, C^{*(\max C^* + a_1)}\right) + f_i \left(N_0, C^{*(\max C^*)}\right) \\ = \; & f_i \left(N_0, C^{*(\max C^* + a_1 + y)}\right) - f_i \left(N_0, C^{*(\max C^* + a_1)}\right) \geq 0 \end{aligned}$$

where the last inequality comes from the fact that $C^{*(\max C^* + a_1 + y)} \geq C^{*(\max C^* + a_1)}$ and f satisfy CM.

Assume now that $\Gamma > 1$. We need to prove

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i \left(C^{\gamma} (a'), a'_{\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i \left(C^{\gamma} (a), a_{\gamma}\right) \geq 0$$

where $a' = (a_1 + y, a_2 - y, a_3, ..., a_\Gamma)$ and $C^{\gamma}(b) = C^1 \oplus_{b_1} C^2 \oplus_{b_2} ... \oplus_{b_{\gamma-1}} C^{\gamma}$ for all $\gamma = 1, ..., \Gamma$ and all $b \in \mathbb{R}_+^{\Gamma}$.

By definition,

$$e_i (C^*, x) = f_i (N_0, C^* \oplus_x (\{0\}, 0)) - f_i (N_0, C^* \oplus_0 (\{0\}, 0)).$$

Under SEP, we have that

$$f_i \left(\bigcup_{r=1}^{\gamma} N^r, C^{\gamma} (b) \oplus_0 (\{0\}, 0) \right) = f_i \left(\bigcup_{r=1}^{\gamma-1} N^r, C^{\gamma-1} (b) \oplus_{b_{\gamma-1}} (\{0\}, 0) \right)$$

for all $\gamma = \gamma_i + 1, ..., \Gamma$ and all $b \in \mathbb{R}_+^{\Gamma}$. Now,

$$\begin{aligned} & \sum_{\gamma=\gamma_i}^{\Gamma} e_i \left(C^{\gamma} (a'), a'_{\gamma}\right) \\ = \; & \sum_{\gamma=\gamma_i}^{\Gamma} \left[f_i \left(\bigcup_{r=1}^{\gamma} N^r, C^{\gamma} (a') \oplus_{a'_{\gamma}} (\{0\}, 0) \right) - f_i \left(\bigcup_{r=1}^{\gamma} N^r, C^{\gamma} (a') \oplus_0 (\{0\}, 0) \right) \right] \\ = \; & f_i \left(\bigcup_{r=1}^{\Gamma} N^r, C^{\Gamma} (a') \oplus_{a'_{\Gamma}} (\{0\}, 0) \right) - f_i \left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i} (a') \oplus_0 (\{0\}, 0) \right) \end{aligned}$$

and

$$\begin{aligned}
&\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a), a_{\gamma}\right)\\
=\ &\sum_{\gamma=\gamma_i}^{\Gamma}\left[f_i\left(\bigcup_{r=1}^{\gamma} N^r, C^{\gamma}(a)\oplus_{a_{\gamma}}(\{0\},0)\right)-f_i\left(\bigcup_{r=1}^{\gamma} N^r, C^{\gamma}(a)\oplus_{0}(\{0\},0)\right)\right]\\
=\ &f_i\left(\bigcup_{r=1}^{\Gamma} N^r, C^{\Gamma}(a)\oplus_{a_{\Gamma}}(\{0\},0)\right)-f_i\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a)\oplus_{0}(\{0\},0)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a'), a'_{\gamma}\right)-\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a), a_{\gamma}\right)\\
=\ &f_i\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a)\oplus_{0}(\{0\},0)\right)-f_i\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a')\oplus_{0}(\{0\},0)\right)\\
&+f_i\left(\bigcup_{r=1}^{\Gamma} N^r, C^{\Gamma}(a')\oplus_{a'_{\Gamma}}(\{0\},0)\right)-f_i\left(\bigcup_{r=1}^{\Gamma} N^r, C^{\Gamma}(a)\oplus_{a_{\Gamma}}(\{0\},0)\right).
\end{aligned}$$

Under CM,

$$f_i\left(\bigcup_{r=1}^{\Gamma} N^r, C^{\Gamma}(a')\oplus_{a'_{\Gamma}}(\{0\},0)\right)\geq f_i\left(\bigcup_{r=1}^{\Gamma} N^r, C^{\Gamma}(a')\oplus_{a'_{\Gamma}}(\{0\},0)\right).$$

We now prove that

$$f_i\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a)\oplus_{0}(\{0\},0)\right)=f_i\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a')\oplus_{0}(\{0\},0)\right).$$

For $\gamma_i = 1$, $C^1(a) = C^1(a') = C^1$ and the result holds trivially. Assume $\gamma_i > 2$. Then, $N^1 \cup ... \cup N^{\gamma_i - 1}$ and N^{γ_i} are two separable components in both $mcstp\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a)\oplus_{0}(\{0\},0)\right)$ and $\left(\bigcup_{r=1}^{\gamma_i} N^r, C^{\gamma_i}(a')\oplus_{0}(\{0\},0)\right)$. Besides, the restriction of C^* to N^{γ_i} coincides in both $mcstp$. Under SEP, we obtain the equality.

Hence,

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a'), a'_{\gamma}\right)-\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a), a_{\gamma}\right)\geq 0. \ \blacksquare$$

7.6 Proof of Proposition 5.1

(1) Using an obligation function o we can arrive at a cost allocation as follows. We compute a $mcst$ following Kruskal's algorithm (Kruskal, 1956), which consists in to construct a tree

by sequentially adding edges with the lowest cost and without introducing cycles. The cost of each edge selected by Kruskal's algorithm is divided among the agents who benefit from adding this edge. Each of these agents pays the difference between her obligation to two groups, one in which she belonged before the edge was added and the one after. We now define an obligation rule, f^o, formally.

Given a network g we define $P(g) = \{T_k(g)\}_{k=1}^{n(g)}$ as the partition of N_0 in *connected components* induced by g. Namely, $P(g)$ is the only partition of N_0 satisfying the following two properties: Firstly, if $i, j \in T_k(g)$, i and j are connected in g. Secondly, if $i \in T_k$, $j \in T_l$, and $k \neq l$, then i and j are not connected in g. Given a network g, let $S(P(g), i)$ denote the element of $P(g)$ to which i belongs to.

Given a *mcstp* (N_0, C), let $g^{|N|}$ be a tree obtained applying Kruskal's algorithm to (N_0, C), and for each $p = 1, ..., |N|$, (i^p, j^p) is the edge selected by Kruskal's algorithm at Stage p, and g^p the set of edges selected by Kruskal's algorithm at stages $1, ..., p$. For each $i \in N$, we define the obligation rule associated with the obligation function o as

$$f_i^o(N_0, C) = \sum_{p=1}^{|N|} c_{i^p j^p} \left(o_i \left(S \left(P \left(g^{p-1} \right), i \right) \right) - o_i \left(S \left(P \left(g^p \right), i \right) \right) \right)$$

where by convention, $o_i(T) = 0$ if $0 \in T$.

Tijs *et al* (2006) prove that f^o is well defined, namely, it is independent of the *mcst* obtained following Kruskal's algorithm.

We prove that if f^o is an obligation rule, then $f^o = f^e$ where $e(C^*, x) = x o_i(N)$ for each (N, C^*) and x.

We proceed by induction on the number of agents. If $|N| = 1$ the result holds trivially. Assume that $f^o = f^e$ when $|N| < q$ and we prove it when $|N| = q$.

Let (N_0, C) be a *mcstp*. Since f^o and f^e satisfy CM, it is enough to prove that $f^o(N_0, C^*) = f^e(N_0, C^*)$.

Proposition 3.1 in Bergantiños and Vidal-Puga (2007) says the following. (N_0, C^*) is an irreducible problem if and only if there exists a *mcst* t in (N_0, C^*) that satisfies the following two conditions:

$(A1)$ $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ where $\pi_0 = 0$ (the source).

$(A2)$ Given $\pi_p, \pi_q \in N_0$ with $p < q$, $c^*_{\pi_p \pi_q} = \max_{s | p < s \leq q} \left\{ c^*_{\pi_{s-1} \pi_s} \right\}$.

Let t be such *mcst*. Without loss of generality we assume that $\pi_s = s$ for each $s = 1, ..., |N|$. We consider two cases.

1. There exists $s > 1$ such that $c^*_{s-1,s} \geq c^*_{r-1,r}$ for all $r = 1, ..., |N|$. Let $S = \{1, ..., s-1\}$. By Proposition 3.1 in Bergantiños and Vidal-Puga (2007) we have that $\{(r-1, r)\}_{r=1}^{s-1}$ can be obtained applying Kruskal's algorithm to $\left(S_0, C^*_{S_0}\right)$ and $\{(0, s)\} \cup \{(r-1, r)\}_{r=s+1}^{|N|}$ can be obtained applying Kruskal's algorithm to $\left((N \backslash S)_0, C^*_{(N \backslash S)_0} \right)$. Thus, $m(N_0, C^*) = m\left(S_0, C^*_{S_0}\right) + m\left((N \backslash S)_0, C^*_{(N \backslash S)_0} \right)$.

Let $i \in S$. Since f^o and f^e satisfy SEP, we deduce that

$$f_i^o(N_0, C^*) = f_i^o\left(S_0, C_{S_0}^*\right) \text{ and } f_i^e(N_0, C^*) = f_i^e\left(S_0, C_{S_0}^*\right).$$

By induction hypothesis $f_i^o\left(S_0, C_{S_0}^*\right) = f_i^e\left(S_0, C_{S_0}^*\right)$. Hence, $f_i^o(N_0, C^*) = f_i^e(N_0, C^*)$. Similarly, we can prove that $f_i^o(N_0, C^*) = f_i^e(N_0, C^*)$ when $i \in N \backslash S$.

2. $c_{01}^* > c_{r-1,r}^*$ for all $r = 2, ..., |N|$. Let $\alpha = c_{01}^* - \max_{r=2,...,|N|}\left\{c_{r-1,r}^*\right\}$. Let (N_0, C') be the irreducible problem associated with t and C' where $c'_{01} = c_{01}^* - \alpha$ and $c'_{r-1,r} = c_{r-1,r}^*$ for all $r = 2, ..., |N|$.

   Since C' is under the conditions of the previous case, we have that $f^o(N_0, C') = f^e(N_0, C')$. Thus, it is enough to prove that for all $i \in N$,

$$f_i^o(N_0, C^*) - f_i^o(N_0, C') = f_i^e(N_0, C^*) - f_i^e(N_0, C').$$

   Fix $i \in N$. We first compute $f_i^o(N_0, C^*) - f_i^o(N_0, C')$. We can apply Kruskal's algorithm to both C^* and C' in such a way that:

   - The edge selected at each stage belongs to t. Namely, for each $p = 1, ..., |N|$, $(i^p(C^*), j^p(C^*)) \in t$ and $(i^p(C'), j^p(C')) \in t$.
   - The edge selected at each stage is the same in both problems. Namely, for each $p = 1, ..., |N|$, $(i^p(C^*), j^p(C^*)) = (i^p(C'), j^p(C'))$.
   - The last edge selected is $(0, 1)$. Namely, $\left(i^{|N|}(C^*), j^{|N|}(C^*)\right) = \left(i^{|N|}(C'), j^{|N|}(C')\right) = (0, 1)$.

   Thus, $f_i^o(N_0, C^*) - f_i^o(N_0, C') = c_{01}^* o_i(N) - c'_{01} o_i(N) = \alpha o_i(N)$.

   We now compute $f_i^e(N_0, C^*) - f_i^e(N_0, C')$. Notice that if S is a neighborhood of i in (N_0, C'), then S is also a neighborhood of i in (N_0, C^*). Besides, N is the unique neighborhood of i in (N_0, C^*) which is not a neighborhood of i in (N_0, C'). Thus,

$$f_i^e(N_0, C^*) - f_i^e(N_0, C') = c_{0i}^* - (\delta_N - e_i(C_N^*, \delta_N)) - c_{0i}'^*.$$

   Since $\delta_N = \alpha$,

$$f_i^e(N_0, C^*) - f_i^e(N_0, C') = e_i(C_N^*, \alpha) = \alpha o_i(N).$$

Using arguments similar to those used above we can prove that if f^e is associated with some e as in the statement, then $f^e = f^o$ where $o(N) = e(C^*, 1)$. Notice that, by hypothesis, $o(N)$ does not depend on C^*.

(2) I is a trivial consequence of part (1) and the definition of optimistic weighted Shapley rules.

(3) I is a trivial consequence of part (1) and the definition of the folk rule. ■

7.7 Proof of Proposition 5.2

We prove that the extra-costs function e satisfies the ANM property, which implies, under Theorem 3, that f^e satisfies CM and PM.

Consider a disjoint sequence $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset \mathcal{C}^*$, $i \in N^{\gamma_i} \subset N$ with $\gamma_i \neq 2$, $a \in \mathbb{R}_+^\Gamma$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^\gamma$ for all $\gamma = 1, ..., \Gamma - 1$, and $y \in [0, a_2]$ ($y \geq 0$ when $\Gamma = 1$). We will prove that

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^\gamma\left(a'\right), a'_\gamma\right) \geq \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^\gamma\left(a\right), a_\gamma\right).$$

If $\Gamma = 1$ the result is obvious. Assume now that $\Gamma > 1$. Since $a'_\gamma = a_\gamma$ when $\gamma \geq 3$,

$$\begin{aligned} e_i\left(C^\gamma\left(a'\right), a'_\gamma\right) &= \int_0^{a'_\gamma} o_i^x\left(N^1 \cup ... \cup N^\gamma\right) dx \\ &= \int_0^{a_\gamma} o_i^x\left(N^1 \cup ... \cup N^\gamma\right) dx = e_i\left(C^\gamma\left(a\right), a_\gamma\right) \end{aligned}$$

for all $\gamma \geq 3$.

In particular, if $\gamma_i \geq 3$ the inequality holds. Hence, we assume $i \in N^1$. We know that $e_i\left(C^\gamma\left(a'\right), a'_\gamma\right) = e_i\left(C^\gamma\left(a\right), a_\gamma\right)$ for all $\gamma \geq 3$. Thus, it is enough to prove that

$$\sum_{\gamma=1}^{2} e_i\left(C^\gamma\left(a'\right), a'_\gamma\right) \geq \sum_{\gamma=1}^{2} e_i\left(C^\gamma\left(a\right), a_\gamma\right).$$

We make some computations:

$$\begin{aligned} e_i\left(C^1\left(a'\right), a'_1\right) &= \int_0^{a'_1} o_i^x\left(N^1\right) dx = \int_0^{a_1+y} o_i^x\left(N^1\right) dx \\ e_i\left(C^2\left(a'\right), a'_2\right) &= \int_0^{a'_2} o_i^x\left(N^1 \cup N^2\right) dx = \int_0^{a_2-y} o_i^x\left(N^1 \cup N^2\right) dx \\ e_i\left(C^1\left(a\right), a_1\right) &= \int_0^{a_1} o_i^x\left(N^1\right) dx, \text{ and} \\ e_i\left(C^2\left(a\right), a_2\right) &= \int_0^{a_2} o_i^x\left(N^1 \cup N^2\right) dx. \end{aligned}$$

Thus, the inequality holds if and only if

$$\int_0^{a_1+y} o_i^x\left(N^1\right) dx + \int_0^{a_2-y} o_i^x\left(N^1 \cup N^2\right) dx \geq \int_0^{a_1} o_i^x\left(N^1\right) dx + \int_0^{a_2} o_i^x\left(N^1 \cup N^2\right) dx.$$

Equivalently,

$$\int_{a_1}^{a_1+y} o_i^x\left(N^1\right) dx \geq \int_{a_2-y}^{a_2} o_i^x\left(N^1 \cup N^2\right) dx$$

which is a particular case of the condition given in (3). ■

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