A solution for bargaining problems with coalition structure

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Published in July 2007 in *Mathematical Social Sciences*[¶]

Abstract

In this paper we study the restriction, to the class of bargaining problems with coalition structure, of several values which have been proposed on the class of non-transferable utility games with coalition structure. We prove that all of them coincide with the solution independently studied in Chae and Heidhues (2004) and Vidal-Puga (2005a). Several axiomatic characterizations and two noncooperative mechanisms are proposed.

Keywords: Coalition structure; Bargaining; Values

JEL classification: C71

1 Introduction

In many economic and political situations, agents do not act individually but are partitioned into unions, groups, or coalitions. Examples include political parties in a Parliament, wage bargaining between firms and labor unions, tariff bargaining between countries, bargaining between the member states of a federated country, etc.

Assuming that cooperation is carried out, one may wonder how the benefit is shared between the coalitions and between the members inside each coalition. Game Theory has

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[¶]DOI: 10.1016/j.mathsocsci.2007.03.001.

addressed this issue. Several solutions have been proposed for several kinds of games. Classically, there are two possible justifications for a solution: one comes from the axiomatic approach and the other one comes from the non-cooperative approach. In the axiomatic approach the objective is to characterize the solution using nice properties. In the non-cooperative approach the objective is to describe a natural non-cooperative game whose equilibria coincide with the solution.

In this paper we focus on bargaining problems (Nash, 1950). Chae and Heidhues (2004) and Vidal-Puga (2005a) describe two values in bargaining problems with coalition structure. Chae and Heidhues (2004) follow an axiomatic approach whereas Vidal-Puga (2005a) follows a non-cooperative approach. Both values generalize the Nash solution. We prove that both values coincide. We call this value δ.

We study δ and we find three kinds of results. It is well-known that bargaining problems can be expressed as games with non-transferable utility (NTU games). We prove that five values presented in the literature for NTU games with coalition structure coincide with δ when we restrict to bargaining problems. We also follow the axiomatic approach and we present three new characterizations of δ. Finally, we follow the non-cooperative approach and we present a natural non-cooperative mechanism. We prove that this mechanism has a unique subgame perfect equilibrium payoff that approaches δ. Let us clarify these three issues.

In games with transferable utility (TU games) and coalition structure, Owen (1977) proposes a value, which is an extension of the Shapley value (Shapley, 1953). Casas-Méndez, García-Jurado, van den Nouweland, and Vázquez-Brage (2003) extend the $\tau - value$ (Tijs, 1981) to TU games with coalition structure. It is well-known that TU games can be expressed as NTU games.

In NTU games with coalition structure there are several values. Winter (1991) introduces the game coalition structure value which coincides with the Owen value in TU games with coalition structure and with the Harsanyi value (Harsanyi, 1963) in NTU games. Bergantiños and Vidal-Puga (2005) introduce two values: the consistent coalitional value and the random order coalitional value. Both values coincide with the Owen value in TU games with coalition structure and with the consistent value (Maschler and Owen, 1989, 1992) in NTU games. Following the classical $\lambda - transfer$ procedure we can extend values from TU games to NTU games. In particular, in differential games Krasa, Tememi and Yannelis (2003) extend the Owen value. Let λTC and $\tau - \lambda TC$ be the NTU values obtained when we extend the Owen value and the coalitional $\tau - value$ (Casas-Méndez *et al.*, 2003), respectively. We prove that, in bargaining problems with coalition structure, the five NTU coalitional values mentioned above coincide with δ.

We present three new axiomatic characterizations for δ. The first one uses the properties of Independence of Affine Transformations (IAT), Independence of Irrelevant Alternatives (IIA), and Unanimity Coalitional Game. This result is inspired by the characterization of the game coalition structure value (Winter, 1991). The second one uses IAT, IIA, Pareto Efficiency, Symmetry inside Coalitions, and Coalitional Symmetry. This result is inspired by the characterization of the Owen value (Owen, 1977). The third one uses IAT, IIA, Pareto Efficiency, Symmetry inside Coalitions, and Symmetry between Exchangeable Coalitions. This result is also inspired by the characterization of the Owen value (Owen, 1977).

Hart and Mas-Colell (1996) propose a non-cooperative mechanism in NTU games. The set of limit stationary subgame perfect equilibrium payoffs is contained in the consistent value. This mechanism has several rounds and in each round a proposer is randomly chosen among the active players. Vidal-Puga (2005a) adapts this mechanism when players are divided in coalitions. Hart and Mas-Colell's mechanism is played in two levels, first between players inside each coalition and second between coalitions. Vidal-Puga (2005a) proves that in bargaining problems with coalition structure there exists a unique stationary subgame perfect equilibrium payoff. In this paper we prove that δ is the unique limit stationary subgame perfect equilibrium. We present another mechanism for bargaining problems with coalition structure. The new mechanism is also a modification of the mechanism of Hart and Mas-Colell (1996). We prove that, in the new mechanism, δ is also the unique limit stationary subgame perfect equilibrium.

The paper is organized as follows. In Section 2, we introduce the notation and some previous results. In Section 3, we present the axiomatic characterizations of δ. In Section 4, we prove that the five NTU coalitional values coincide with δ in bargaining problems. In Section 5, we study the non-cooperative approach. Finally, we present some concluding remarks.

2 Preliminaries

Let A be a finite set. We denote by $|A|$ the number of elements in A. Let $x, y \in \mathbb{R}^A$. We say $y \leq x$ when $y_i \leq x_i$ for each $i \in A$ and $y < x$ when $y_i < x_i$ for each $i \in A$. We denote by xy the vector $(x_i y_i)_{i \in A}$ and by $x + y$ the vector $(x_i + y_i)_{i \in A}$. Given $T \subsetneq A$, x_T is the restriction of x to $\mathbb{R}^T$. We denote by $\mathbb{R}^A_+$ the set $\{x \in \mathbb{R}^A : x_i \geq 0 \text{ for every } i \in A\}$ and by $\mathbb{R}^A_{++}$ the set $\{x \in \mathbb{R}^A : x_i > 0 \text{ for every } i \in A\}$. Given $\gamma \in \mathbb{R}^A_{++}$, $\frac{1}{\gamma}$ is the vector $\left(\frac{1}{\gamma_i}\right)_{i \in A}$. For every $S \subseteq \mathbb{R}^A$ and $\gamma, \beta \in \mathbb{R}^A$, we define $\gamma S + \beta = \{\gamma x + \beta : x \in S\}$. Given $\theta \in \mathbb{R}$ and $x \in \mathbb{R}^A$, we define θx as the vector $(\theta x_i)_{i \in A}$.

We consider $N = \{1, ..., n\}$ the set of *players*.

A *coalition structure* $\mathcal{C}$ over N is a partition of the player set, *i.e.*,

$$\mathcal{C} = \{C_1, ..., C_p\} \subsetneq 2^N \text{ where } \cup_{C_q \in \mathcal{C}} C_q = N \text{ and } C_q \cap C_r = \varnothing \text{ whenever } q \neq r.$$

Each $C_q \in \mathcal{C}$ is called a *coalition*. We denote by $c \in \mathbb{R}^N$ the vector whose ith coordinate is given by $c_i = |C_q|$ if $i \in C_q$.

A *transferable utility (TU) game* is a pair (N, v) where v is a *characteristic function* that assigns to each subset $T \subseteq N$ a number $v(T) \in \mathbb{R}$, with $v(\phi) = 0$, which represents the total utility players in T can get by themselves when cooperate. A *TU game with coalition structure* is a triple $(N, v, \mathcal{C})$ where (N, v) is a TU game and $\mathcal{C}$ is a coalition structure over N.

The *Owen value* (Owen, 1977) is a function Ow which assigns to each *TU* game with coalition structure $(N, v, \mathcal{C})$ a vector $Ow(N, v, \mathcal{C}) \in \mathbb{R}^N$. The Owen value generalizes the *Shapley value* (Sh) (Shapley, 1953), *i.e.* when $\mathcal{C} = \{N\}$ or $\mathcal{C} = \{\{1\}, \ldots, \{n\}\}$, $Ow(N, v, \mathcal{C}) = Sh(N, v)$.

Given (N, v) a TU game the Shapley value (Shapley,1953) is given by

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n-|S|-1)!}{n!}[v(S \cup i) - v(S)] \text{ for all } i \in N.$$

To state the definition of the Owen value we introduce additional notation. Let $\Pi(N)$ be the set of all orders on N. We say that $\pi \in \Pi(N)$ is *admissible* with respect to the coalition structure $\mathcal{C}$ if for any $i, j, k \in N$, $i, k \in C_q \in \mathcal{C}$, and $\pi(i) < \pi(j) < \pi(k)$ imply that $j \in C_q$, where $\pi(i)$, $\pi(j)$, $\pi(k)$ denote the position of i, j, and k in the order π, respectively. We denote by $\Pi^{\mathcal{C}}$ the set of all admissible orders on N with respect to $\mathcal{C}$.

Given $(N, v, \mathcal{C})$ a TU game with coalition structure the Owen value (Owen, 1977) is defined as:

$$Ow_i(N, v, \mathcal{C}) = \frac{1}{|\Pi^{\mathcal{C}}|} \sum_{\pi \in \Pi^{\mathcal{C}}} [v(P_i^\pi \cup i) - v(P_i^\pi)] \text{ for all } i \in N$$

where $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$ is the set of *predecessors* of player i in π.

A *bargaining problem* over N is a pair (S, d) where $d \in S \subsetneq \mathbb{R}^N$, there exists $x \in S$ such that $x > d$, and

A1. S is closed, convex, *comprehensive* (if $x \in S$ and $y \leq x$ then $y \in S$), and *bounded above* (*i.e.* for all $x \in S$ the set $\{y \in S : y \geq x\}$ is compact).

A2. The boundary of S, ∂S, is *smooth* (on each point of the boundary there exists a unique outward vector) and *nonlevel* (the outward vector on each point of the boundary has all its coordinates positive).

We denote by Λ the bargaining problem $(\Delta, 0)$ with

$$\Delta = \left\{ x \in \mathbb{R}^N : \sum_{i=1}^{n} x_i \leq 1 \right\}.$$

We call $\Lambda = (\Delta, 0)$ the unanimity bargaining problem.

The *Nash solution* of a bargaining problem (Nash, 1950) is the unique point $N(S,d) \in \partial S$ satisfying

$$\prod_{i \in N} (N_i(S,d) - d_i) = \max_{x \in S,\ x \geq d} \prod_{i \in N} (x_i - d_i). \tag{1}$$

A *bargaining problem with coalition structure* is a triple $(S, d, \mathcal{C})$ where (S,d) is a bargaining problem and $\mathcal{C}$ is a coalition structure. By $\mathcal{B}(N)$ we represent the class of all bargaining problems with coalition structure where N is the set of agents.

A *solution* of a bargaining problem with coalition structure is a map which assigns to every $(S, d, \mathcal{C}) \in \mathcal{B}(N)$ an element of S.

In this context, Chae and Heidhues (2004) characterize the solution defined by the unique point $\delta(S, d, \mathcal{C}) \in \partial S$ satisfying

$$\prod_{i \in N} (\delta_i(S, d, \mathcal{C}) - d_i)^{\frac{1}{c_i}} = \max_{x \in S,\ x \geq d} \prod_{i \in N} (x_i - d_i)^{\frac{1}{c_i}}. \tag{2}$$

This solution is the *weighted Nash solution* (Kalai, 1977), N^w, where for any $i \in N$, $w_i = \frac{1}{pc_i}$ with $p = |\mathcal{C}|$ and $c_i = |C_q|$ if $i \in C_q$.

A *non-transferable utility (NTU) game* is a pair (N, V) where V is a correspondence which assigns to each coalition $T \subseteq N$ a subset $V(T) \subsetneq \mathbb{R}^T$. This set represents all the possible payoffs that members of T can obtain for themselves when play cooperatively. For each $T \subsetneq N$, we assume that $V(T)$ satisfies $A1$ and that $V(N)$ satisfies A1 and A2. A payoff configuration $\{x^T\}_{T \subseteq N}$ is a family of vectors such that $x^T \in \mathbb{R}^T$ for every $T \subseteq N$.

NTU games generalize both TU games and bargaining problems. Any *TU* game (N, v) can be expressed as an NTU game (N, V) with

$$V(T) = \left\{ x \in \mathbb{R}^T : \sum_{i \in T} x_i \leq v(T) \right\} \text{ for all } T \subseteq N.$$

We say that (N, V) is a *hyperplane game* if for all $T \subseteq N$ there exists $\lambda^T \in \mathbb{R}^T_{++}$ satisfying

$$V(T) = \left\{ x \in \mathbb{R}^T : \sum_{i \in T} \lambda_i^T x_i \leq v(T) \right\} \tag{3}$$

for some $v : 2^N \to \mathbb{R}$. Notice that each TU game is a hyperplane game (just take $\lambda_i^T = 1$ for each $T \subseteq N$ and $i \in T$).

Any bargaining problem (S, d) can be expressed as an NTU game (N, V) with

$$V(T) = \left\{x \in \mathbb{R}^T : x \leq d_T\right\} \text{ for all } T \subsetneq N \tag{4}$$

and $V(N) = S$.

An *NTU game with coalition structure* is a triple $(N, V, \mathcal{C})$ where (N, V) is an NTU game and $\mathcal{C}$ is a coalition structure over N. By $\mathcal{NTU}(N)$ we denote the class of all NTU games with coalition structure where N is the set of agents.

A *value* Γ is a correspondence which assigns to each NTU game with coalition structure $(N, V, \mathcal{C})$ a subset $\Gamma(N, V, \mathcal{C}) \subseteq V(N)$.

Notice that a solution on $\mathcal{B}(N)$ can be considered as a value which assigns to each $(S, d, \mathcal{C})$ a singleton.

We say a value Γ *generalizes the Owen value* if $\Gamma(N, v, \mathcal{C}) = \{Ow(N, v, \mathcal{C})\}$ for each TU game with coalition structure $(N, v, \mathcal{C})$.

We say that a value Γ *generalizes the Nash solution* if $\Gamma(S, d, \mathcal{C}) = \{N(S, d)\}$ for every bargaining problem with coalition structure $(S, d, \mathcal{C})$ when $\mathcal{C} = \{N\}$ or $\mathcal{C} = \{\{1\}, \ldots, \{n\}\}$.

We say that a value Γ *generalizes the solution* δ if $\Gamma(S, d, \mathcal{C}) = \{\delta(S, d, \mathcal{C})\}$ for every bargaining problem with coalition structure $(S, d, \mathcal{C})$.

Next we recall the definitions of the values we consider in this paper: the Game with Coalition Structure (GCS) value (Winter, 1991), the Consistent Coalitional (CC) value (Bergantiños and Vidal-Puga, 2005), the Random-Order Coalitional (ROC) value (Bergantiños and Vidal-Puga, 2005), the λ-Transfer Coalitional (λTC) value, and the τ-λ Transfer Coalitional ($\tau - \lambda TC$) value. Even though these values are defined in the context of NTU games with coalitional structure, we present the formal definitions in the context of bargaining problems with coalition structure. Let $(S, d, \mathcal{C}) \in \mathcal{B}(N)$.

The *GCS value*, Φ^{GCS}, was introduced by Winter (1991) as a generalization of the Owen value for TU games with coalition structure and the Harsanyi value (Harsanyi, 1963) for NTU games. We say that $x \in \mathbb{R}^N$ is an element of the *GCS value* for $(S, d, \mathcal{C})$ if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that λ supports S at x and moreover $x_i = \sum_{T \subseteq N : i \in T} y_i^T$ where $(y^T)_{T \subseteq N}$ is defined inductively as follows:

$$y^{\varnothing} = 0,$$

and for every $\varnothing \neq T \subseteq N$, given $y^{T'}$ defined for all $T' \subsetneq T$, then

$$z_i^T = \sum_{T' \subsetneq T : i \in T'} y_i^{T'} \text{ for every } i \in N, \text{ and}$$

$$y^T = \begin{cases} \frac{1}{\lambda_T} \frac{1}{c_T} \max\left\{t \in \mathbb{R} : z^T + \frac{1}{\lambda_T} \frac{1}{c_T} t \leq d_T\right\} & \text{if} \quad T \subsetneq N \\ \\ \frac{1}{\lambda_T} \frac{1}{c_T} \max\left\{t \in \mathbb{R} : z^T + \frac{1}{\lambda_T} \frac{1}{c_T} t \in S\right\} & \text{if} \quad T = N. \end{cases}$$

Then, $y^{\{i\}} = d_i$ for each $i \in N$. For every $T \subsetneq N$ with $|T| \geq 2$, $z_i^T = d_i$ for every $i \in T$ and $y_i^T = 0$ for every $i \in T$. For $T = N$, we have

$$z^N = d, \quad \text{and} \quad y^N = \frac{1}{\lambda} \frac{1}{c} \max\left\{t \in \mathbb{R} : d + \frac{1}{\lambda} \frac{1}{c} t \in S\right\}.$$

Hence,

$$x = y^N = d + \frac{1}{\lambda} \frac{1}{c} \max\left\{t \in \mathbb{R} : d + \frac{1}{\lambda} \frac{1}{c} t \in S\right\}, \tag{5}$$

and we get that x belongs to $\Phi^{GCS}(S, d, \mathcal{C})$. We will denote the set of points which satisfies (5) as $\Phi^{GCS}(S, d, \mathcal{C})$.

In case that the bargaining problem with coalition structure is given by $(H_\lambda, d, \mathcal{C})$ where $\lambda \in \mathbb{R}^N_{++}$ and

$$H_\lambda = \left\{x \in \mathbb{R}^N : \sum_{i \in N} \lambda_i x_i \leq 1\right\}, \tag{6}$$

$\Phi^{GCS}(H_\lambda, d, \mathcal{C})$ is the unique vector which satisfies (5).

The *CC value*, Φ^{CC}, and the *ROC value*, Φ^{ROC}, were introduced by Bergantiños and Vidal-Puga (2005) as a generalization of the Owen value for TU games with coalition structure and the consistent value (Maschler and Owen, 1989, 1992) for NTU games. Following Vidal-Puga (2005a) we first present an expression for any element of the CC value corresponding to any $(S, d, \mathcal{C})$. Let $\{\lambda^T \in \mathbb{R}^T_{++} : T \subseteq N\}$ be a family of vectors and let $x \in \partial S$ be such that λ^N supports S at x. We recursively build a payoff configuration $\{x^T\}_{T \subseteq N}$ as

$$x_i^{\{i\}} = d_i, \quad \text{for every } i \in N,$$

given $x^{T'}$ for any $T' \subsetneq T \subsetneq N$, and $i \in T \cap C_q = C'_q$,

$$\begin{aligned} x_i^T = \; & \frac{1}{|\mathcal{C}_T| \left|C'_q\right| \lambda_i^T} \left(\sum_{C'_r \in \mathcal{C}_T \setminus C'_q} \left(\sum_{j \in C'_q} \lambda_j^T x_j^{T \setminus C'_r} - \sum_{j \in C'_r} \lambda_j^T x_j^{T \setminus C'_q} \right) \right) \\ & + \frac{1}{|C'_q| \lambda_i^T} \left(\sum_{j \in C'_q \setminus \{i\}} \lambda_i^T x_i^{T \setminus \{j\}} - \sum_{j \in C'_q \setminus \{i\}} \lambda_j^T x_j^{T \setminus \{i\}} \right) \\ & + \frac{1}{|\mathcal{C}_T| |C'_q| \lambda_i^T} \sum_{j \in T} \lambda_j^T d_j \end{aligned}$$

where $\mathcal{C}_T = \{C_r \cap T : C_r \in \mathcal{C}\}$, and for $T = N$ and $i \in N$,

$$\begin{aligned} x_i^N = \; & \frac{1}{pc_i\lambda_i^N}\left(\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N x_j^{N\setminus C_r} - \sum_{j\in C_r}\lambda_j^N x_j^{N\setminus C_q}\right)\right) \\ & + \frac{1}{c_i\lambda_i^N}\left(\sum_{j\in C_q\setminus\{i\}}\lambda_i^N x_i^{N\setminus\{j\}} - \sum_{j\in C_q\setminus\{i\}}\lambda_j^N x_j^{N\setminus\{i\}}\right) \\ & + \frac{1}{pc_i\lambda_i^N}\textstyle\sum_{j\in T}\lambda_j^N x_j. \end{aligned}$$

By doing some algebra, we obtain that $x^T = d^T$ for every $T \subsetneq N$. If $x^N = x$ we say that x is a CC value for $(S, d, \mathcal{C})$ and it holds that

$$x = d + \frac{1}{\lambda^N}\frac{1}{p}\frac{1}{c}\left(\sum_{j\in N}\lambda_j^N x_j - \sum_{j\in N}\lambda_j^N d_j\right). \tag{7}$$

We will denote the set of points which satisfies (7) as $\Phi^{CC}(S, d, \mathcal{C})$.
In case that the bargaining problem with coalition structure is given by $(H_\lambda, d, \mathcal{C})$ where $\lambda \in \mathbb{R}^N_{++}$ and H_λ is defined as in (6), $\Phi^{CC}(H_\lambda, d, \mathcal{C})$ is the unique vector which satisfies (7).

Next we present the definition of the ROC value (Bergantiños and Vidal-Puga, 2005). Let $\{\lambda^T \in \mathbb{R}^T_{++} : T \subseteq N\}$ be a family of vectors and let $x \in \partial S$ be such that λ^N supports S at x. Let us consider $\pi \in \Pi^{\mathcal{C}}$. For each $T \subseteq N$ and $i \in T$, the marginal contribution of player i in the order π is

$$e_i^T(\pi) = \max\left\{y_i \in \mathbb{R} : \sum_{j\in P_i^\pi\cap T}\lambda_j^T e_j^T(\pi) + \lambda_i^T y_i \le \sum_{j\in(P_i^\pi\cap T)\cup\{i\}}\lambda_j^T d_j\right\}$$

whenever $T \subsetneq N$ or $T = N$ and $\pi(i) < n$, and

$$e_i^T(\pi) = \max\left\{y_i \in \mathbb{R} : \sum_{j\in P_i^\pi\cap T}\lambda_j^T e_j^T(\pi) + \lambda_i^T y_i \le \sum_{j\in(P_i^\pi\cap T)\cup\{i\}}\lambda_j^T x_j\right\}$$

when $T = N$ and $\pi(i) = n$.

We obtain a payoff configuration $(x^T)_{T\subseteq N}$ as

$$x^T = \frac{1}{|\Pi^{\mathcal{C}_T}|}\sum_{\pi\in\Pi^{\mathcal{C}_T}} e^T(\pi), \quad \text{for every } T \subseteq N.$$

In case that $x^N = x$, we say that x is a ROC value for $(S, d, \mathcal{C})$. We denote by $\Phi^{ROC}(S, d, \mathcal{C})$ the ROC value of $(S, d, \mathcal{C})$.

Let us take $i \in N$. Notice that $e_i^N(\pi) = d_i$ for all $\pi \in \Pi^{\mathcal{C}}$ unless $\pi(i) = n$. Whenever $\pi(i) = n$,

$$e_i^N(\pi) = \frac{1}{\lambda_i^N}\left(\sum_{j\in N}\lambda_j^N x_j - \sum_{j\in N\setminus\{i\}}\lambda_j^N d_j\right).$$

Counting all possible orders and doing some algebra,

$$\begin{aligned} x_i &= \frac{(pc_i - 1)\, d_i}{pc_i} + \frac{1}{\lambda_i^N pc_i}\left(\sum_{j\in N}\lambda_j^N x_j - \sum_{j\in N\setminus\{i\}}\lambda_j^N d_j\right) \\ &= \frac{pc_i\lambda_i^N d_i + \sum_{j\in N}\lambda_j^N x_j - \sum_{j\in N}\lambda_j^N d_j}{\lambda_i^N pc_i} \\ &= d_i + \frac{\sum_{j\in N}\lambda_j^N x_j - \sum_{j\in N}\lambda_j^N d_j}{\lambda_i^N pc_i}. \end{aligned}$$

This expression coincides with (7). Then, we prove that $\Phi^{CC}(S,d,\mathcal{C}) = \Phi^{ROC}(S,d,\mathcal{C})$.

Given a value for TU games, Shapley (1969) proves, via a fixed-point argument, that one can always find a vector λ of weights, one for each player, such that when each player's utility is multiplied by his weight, the resulting game will have the property that the value for the associated TU game (as presented in (8) below) is feasible in the NTU game.

Since the Shapley reasoning may be applied to any value, we apply the λ-transfer procedure to the Owen value and the coalitional τ value (Casas-Méndez *et al*, 2003).

The λTC value generalizes the Owen value for TU games with coalition structure and the Shapley NTU value (Shapley, 1969) for NTU games.

Given a bargaining problem with coalition structure $(S,d,\mathcal{C})$, we say that $x \in \mathbb{R}^N$ is a *λ-Transfer Coalitional (λTC) value* if $x \in \partial S$, there exists $\lambda \in \mathbb{R}^N_{++}$ such that λ supports S at x, and

$$\lambda x = Ow\left(N, v^\lambda, \mathcal{C}\right)$$

where

$$v^\lambda(T) = \begin{cases} \sum_{i\in T}\lambda_i d_i & \text{if } T \subsetneq N \\ \max\left\{\sum_{j\in N}\lambda_j x_j : x \in S\right\} & \text{if } T = N \end{cases}. \tag{8}$$

We denote by $\Phi^{\lambda TC}(S,d,\mathcal{C})$ the set of λTC values for $(S,d,\mathcal{C})$.

The $\tau - \lambda TC$ value generalizes the coalitional τ value for TU games with coalition structure (Casas-Méndez *et al*, 2003) and the τ value for NTU games (Borm *et al*, 1992).

Given $(S,d,\mathcal{C}) \in \mathcal{B}(N)$, we say that $x \in \mathbb{R}^N$ is a τ-λTC value if $x \in \partial S$, there exists $\lambda \in \mathbb{R}^N_{++}$ such that λ supports S at x, and

$$\lambda x = \tau(N, v^\lambda, \mathcal{C})$$

where v^λ is the TU game defined in (8). If $(S,d,\mathcal{C})$ is a bargaining problem with coalition structure, we denote by $\Phi^{\tau\lambda TC}(S,d,\mathcal{C})$ the set of $\tau - \lambda TC$ values for $(S,d,\mathcal{C})$.

Next table makes a matching of solutions to problems we consider:

	Without Coalition Structure	With Coalition Structure
Bargaining problems	Nash solution Weighted Nash solution	δ
TU games	Shapley value τ value	Owen value Coalitional τ value
NTU games	Harsanyi value Consistent value λ-transfer value	GCS value CC value ROC value λTC value τ - λTC value

Table 1: Matching solutions to problems

3 Characterizations of the solution δ

In this section we present three axiomatic characterizations of the solution δ.

We first define the following concepts, which will be used later. Let $(S, d, \mathcal{C}) \in \mathcal{B}(N)$.

1. Given $i, j \in N$, we say that i and j are symmetric if and only if $d_i = d_j$ and for every $x \in S$, the element y with $y_i = x_j$, $y_j = x_i$, and $y_k = x_k$, for any $k \in N \backslash \{i, j\}$, belongs to S.

2. Given $C_r \in \mathcal{C}$, we say that C_r is a homogeneous coalition if any pair of agents $i, j \in C_r$ are symmetric agents.

3. We say that (S, d) is a symmetric bargaining problem if any pair of agents $i, j \in N$ are symmetric.

4. Two different homogeneous coalitions $C_r, C_s \in \mathcal{C}$ are exchangeable if and only if

   (a) $d_i = d_j$ for any $i \in C_r$ and $j \in C_s$, and

(b) for any $x \in S$ with $x_i = z_r$ for every $i \in C_r$ and $x_i = z_s$ for every $i \in C_s$, we have that y, where $y_i = z_s$ for every $i \in C_r$, $y_j = z_r$ for every $j \in C_s$, and $y_k = x_k$ for every $k \in N \backslash (C_r \cup C_s)$, belongs to S.

5. If a coalition C_j is homogeneous and $j^* \in C_j$, Chae and Heidhues (2004) define the reduced bargaining problem $(S^j, d^j, \mathcal{C}^j) \in \mathcal{B}(N^j)$ as follows.

$$N^j = (N \backslash C_j) \cup \{j^*\}$$

$$S^j = \{x \in \mathbb{R}^{N^j} : \exists y \in S \text{ such that } y_i = x_i \ \forall i \in N \backslash C_j \text{ and } y_i = x_{j^*} \ \forall i \in C_j\}$$

$$\mathcal{C}^j = \{C_1, \cdots, C_{j-1}, \{j^*\}, C_{j+1}, \cdots, C_p\}, \text{ and}$$

$$d_i^j = d_i \text{ for all } i \in N^j.$$

We formulate some reasonable properties of a solution defined on $\mathcal{B}(N)$. Let φ be an arbitrary solution defined on $\mathcal{B}(N)$ and let $(S, d, \mathcal{C}) \in \mathcal{B}(N)$.

- Independence of irrelevant alternatives (**IIA**). Let us take $(S', d, \mathcal{C}) \in \mathcal{B}(N)$ such that $S' \subseteq S$ and $\varphi(S, d, \mathcal{C}) \in S'$, then $\varphi(S', d, \mathcal{C}) = \varphi(S, d, \mathcal{C})$.

- Invariance with respect to affine transformations (**IAT**). Given $\gamma \in \mathbb{R}^N_{++}$, and $\beta \in \mathbb{R}^N$, it holds that $\varphi(\bar{S}, \bar{d}, \mathcal{C}) = \gamma\varphi(S, d, \mathcal{C}) + \beta$, where $\bar{S} = \gamma S + \beta$ and $\bar{d} = \gamma d + \beta$.

- Pareto efficiency (**PE**). There is no $x \in S \setminus \{\varphi(S, d, \mathcal{C})\}$ such that $x_i \geq \varphi_i(S, d, \mathcal{C})$ for every $i \in N$.

- Unanimity coalitional game (**UCG**). Given the unanimity bargaining problem $(\Delta, 0)$, for each coalition structure $\mathcal{C}$, we have

$$\varphi_i(\Delta, 0, \mathcal{C}) = \frac{1}{pc_i}$$

  for every $i \in N$ where $c_i = |C_q|$ if $i \in C_q$ and $\mathcal{C} = \{C_1, \cdots, C_p\}$.

- Symmetry inside coalitions (**SIC**). Given $C_q \in \mathcal{C}$, let $i, j \in C_q$ be two symmetric agents, then $\varphi_i(S, d, \mathcal{C}) = \varphi_j(S, d, \mathcal{C})$.

- Symmetry between exchangeable coalitions (**SEC**). Given any pair of exchangeable coalitions C_r, C_s, then $\varphi_i(S, d, \mathcal{C}) = \varphi_j(S, d, \mathcal{C})$ for any $i \in C_r$ and $j \in C_s$.

- Coalitional symmetry (**CS**). Given the unanimity bargaining problem $(\Delta, 0)$, for each coalition $\mathcal{C}$, we have

$$\sum_{i \in C_r} \varphi_i(\Delta, 0, \mathcal{C}) = \sum_{i \in C_s} \varphi_i(\Delta, 0, \mathcal{C})$$

for every $C_r, C_s \in \mathcal{C}$.

- Symmetry (**SYM**) (Chae and Heidhues, 2004). If $\mathcal{C} = \{\{1\}, \cdots, \{n\}\}$ and (S, d) is symmetric, then for any two players i, j, one has $\varphi_i(S, d, \mathcal{C}) = \varphi_j(S, d, \mathcal{C})$.

- Representation of a homogeneous group (**RHG**) (Chae and Heidhues, 2004). If a coalition C_j is homogeneous and $j^* \in C_j$, then $\varphi_{j^*}(S, d, \mathcal{C}) = \varphi_{j^*}(S^j, d^j, \mathcal{C}^j)$.

Independence of irrelevant alternatives, invariance with respect to affine transformations, and Pareto efficiency are well-known properties.

Aumann (1985) defined the property of unanimity to characterize the Shapley-NTU value. This property says that the unanimity game[1] of a coalition has a unique value given by the equal split of the available amount. Hart (1985) also used this property to characterize the Harsanyi value in the context of NTU games. De Clippel, Peters, and Zank (2004) also use this property in the characterization of the egalitarian Kalai-Samet solution (Kalai and Samet, 1985). Winter (1991) used the property of unanimity games in his characterization of the GCS value. The unanimity coalitional game property has the same flavour in the context of bargaining problems with coalition structure.

The property of symmetry inside coalitions establishes that two symmetric agents of the same coalition obtain the same value. According to the property of symmetry between exchangeable coalitions, all members of two exchangeable coalitions receive the same amount. The property of coalitional symmetry has the same flavour that symmetry inside coalitions but applied to coalitions.

The properties of symmetry and representation of homogeneous group are not used in our characterizations of δ but they appear in the characterization of δ given by Chae and Heidhues (2004). Our symmetry properties (**SIC**, **SEC**, and **CS**) differ from the property of symmetry proposed by Chae and Heidhues (2004). The property of representation of a homogeneous group says that a member of a homogeneous coalition receives what he would receive if he became a representative member bargaining on behalf of the coalition, that is, a homogeneous coalition can be replaced by a member to whom bargaining is delegated. **RHG** relates the payoff of a player in two different problems whereas the

[1]Given $T \subseteq N$, the unanimity game of the coalition T is the TU game defined as $u_T(R) = 1$ if $T \subseteq R \subseteq N$ and $u_T(R) = 0$, otherwise.

property of symmetry between exchangeable coalitions (**SEC**) compares the payoff of several players in the same problem.

Remark 3.1 *We want to point out that there is a conceptual difference between the properties defined above. **IIA**, **IAT**, **PE**, **SIC**, **SEC**, **SYM**, and **RHG** hold for the set of all bargaining problems. **UCG** and **CS** only hold for unanimity bargaining problems.*

Next we provide our characterizations of the solution δ using these properties.

Theorem 3.1 1.- The solution δ is the unique solution defined on $\mathcal{B}(N)$ which satisfies **IIA**, **IAT**, and **UCG**.

2.- The solution δ is the unique solution defined on $\mathcal{B}(N)$ which satisfies **PE**, **IIA**, **IAT**, **SIC**, and **CS**.

3.- The solution δ is the unique solution defined on $\mathcal{B}(N)$ which satisfies **PE**, **IIA**, **IAT**, **SIC**, and **SEC**.

Before the formal proof of this result it is worthily to describe the uniqueness part. Since we use **IAT** and **IIA**, by the same reasoning as in Nash (1950) it is enough to prove the uniqueness in unanimity bargaining problems. In characterization 1, **UCG** proposes a sharing in unanimity bargaining problems. For proving the uniqueness part of characterizations 2 and 3 we use two lemmas. The first lemma says that **PE**, **SIC,** and **CS** imply **UCG**. The second lemma says that **PE**, **IAT**, **SIC,** and **SEC** imply **UCG**. Thus, the uniqueness part of characterizations 2 and 3 follows from characterization 1.

We next present two lemmas and later the proof of the theorem.

Lemma 3.1 Any solution φ defined on $\mathcal{B}(N)$ which satisfies **PE**, **SIC**, and **CS** also satisfies **UCG**.

Proof. Let φ be a solution defined on $\mathcal{B}(N)$ which satisfies **PE**, **SIC**, and **CS**. Let us consider $(\Lambda, \mathcal{C}) \in \mathcal{B}(N)$. For every $C_r \in \mathcal{C}$, we have that any two agents $i, j \in C_r$ are symmetric. By **SIC**, $\varphi_i(\Lambda, \mathcal{C}) = \varphi_j(\Lambda, \mathcal{C})$ for every $i, j \in C_r$ and $C_r \in \mathcal{C}$. Moreover, since the solution φ satisfies **CS**, for every $C_r, C_s \in \mathcal{C}$, it holds

$$c_i \varphi_i(\Lambda, \mathcal{C}) = \sum_{k \in C_r} \varphi_k(\Lambda, \mathcal{C}) = \sum_{k \in C_s} \varphi_k(\Lambda, \mathcal{C}) = c_j \varphi_j(\Lambda, \mathcal{C})$$

with $i \in C_r$ and $j \in C_s$.

Finally, taking into account that the solution φ satisfies **PE**, we get, for any $i \in N$,

$$1 = \sum_{j \in N} \varphi_j(\Lambda, \mathcal{C}) = p c_i \varphi_i(\Lambda, \mathcal{C}).$$

Then, for every $i \in N$,

$$\varphi_i(\Lambda, \mathcal{C}) = \frac{1}{pc_i}.$$

■

Lemma 3.2 Any solution φ defined on $\mathcal{B}(N)$ which satisfies **PE**, **IAT**, **SIC**, and **SEC** also satisfies **UCG**.

Proof. Let us consider the bargaining problem with coalition structure $(H_\lambda, 0, \mathcal{C})$ where $\lambda = \frac{1}{p}\frac{1}{c}$ and H_λ is defined by (6). If $|\mathcal{C}| = 1$, the bargaining problem $(H_\lambda, 0)$ is symmetric. Otherwise, any pair of coalitions $C_r, C_s \in \mathcal{C}$ are exchangeable. Since the solution φ satisfies **PE**, **SIC**, and **SEC**, it holds

$$\varphi_i(H_\lambda, 0, \mathcal{C}) = \varphi_j(H_\lambda, 0, \mathcal{C}) = 1 \text{ for every } i \in C_r, j \in C_s \text{ and } C_r, C_s \in \mathcal{C}.$$

Moreover, applying the affine transformation defined by $\lambda \in \mathbb{R}^N_{++}$ and $\beta = 0$ to $(H_\lambda, 0, \mathcal{C})$, we obtain the bargaining problem with coalition structure $(\Lambda, \mathcal{C})$. Since the solution φ satisfies **IAT**, we have

$$\varphi_i(\Lambda, \mathcal{C}) = \frac{1}{pc_i} \text{ for every } i \in N.$$

■

Proof. of Theorem 3.1 First we will see that the solution δ satisfies these properties. The solution δ satisfies **IIA**, **IAT**, and **PE** (Chae and Heidhues, 2004). Since δ is a weighted Nash solution, it assigns the vector of weights to the unanimity bargaining problem (Kalai, 1977). Thus, given the structure of the weights, δ satisfies **UCG**. Furthermore, the total amount that a coalition receives in $(\Delta, 0, \mathcal{C})$ is the same and we prove that δ also satisfies **CS**.

Next, we see that it also satisfies **SIC**. Let us assume that this does not happen. Since δ satisfies **IAT**, we take a bargaining problem with a coalition structure $(S, 0, \mathcal{C}) \in \mathcal{B}(N)$. Let $C_q \in \mathcal{C}$ and $i, j \in C_q$ such that i and j are symmetric. Let us assume that $\delta_i(S, 0, \mathcal{C}) \neq \delta_j(S, 0, \mathcal{C})$. We define the point $\bar{x} \in \mathbb{R}^N$ as

$$\begin{aligned} \bar{x}_i &= \tfrac{1}{2}\left(\delta_i(S, 0, \mathcal{C}) + \delta_j(S, 0, \mathcal{C})\right) = \bar{x}_j \text{ and} \\ \bar{x}_k &= \delta_k(S, 0, \mathcal{C}) \text{ for every } k \in N \setminus \{i, j\}. \end{aligned} \tag{9}$$

This point $\bar{x}$ belongs to S because i and j are symmetric and S is a convex set. Furthermore,

$$\bar{x}_i \bar{x}_j - \delta_i(S, 0, \mathcal{C})\,\delta_j(S, 0, \mathcal{C}) = \frac{1}{4}\left(\delta_i(S, 0, \mathcal{C}) - \delta_j(S, 0, \mathcal{C})\right)^2 > 0. \tag{10}$$

Moreover, since $i, j \in C_q$, (9), and (10), it holds

$$\prod_{k \in N} \bar{x}_k^{\frac{1}{c_k}} > \prod_{k \in N} \delta_k(S, 0, \mathcal{C})^{\frac{1}{c_k}}.$$

This is a contradiction with respect to the definition of δ. Then, the solution δ satisfies **SIC**.

Let us check that it also satisfies **SEC**. Let $(S, 0, \mathcal{C}) \in \mathcal{B}(N)$. If $|\mathcal{C}| > 1$, let us take C_r, C_s two exchangeable coalitions. Since δ satisfies **SIC** we have

$$\delta_i(S,0,\mathcal{C}) = \delta_j(S,0,\mathcal{C}) \text{ for every } i,j \in C_r \text{ and}$$
$$\delta_i(S,0,\mathcal{C}) = \delta_j(S,0,\mathcal{C}) \text{ for every } i,j \in C_s.$$

Let us define the vector $z \in \mathbb{R}^N$ as

$$\begin{array}{lll} z_i = \delta_i(S,0,\mathcal{C}) & \text{if} & i \notin C_r \cup C_s \\ z_i = \delta_j(S,0,\mathcal{C}) & \text{if} & i \in C_r \text{ with } j \in C_s \\ z_i = \delta_j(S,0,\mathcal{C}) & \text{if} & i \in C_s \text{ with } j \in C_r. \end{array}$$

Since C_r and C_s are exchangeable, $z \in S$. Then, given $i \in C_r$ and $j \in C_s$,

$$\prod_{k \in C_r} z_k^{\frac{1}{c_k}} \prod_{k \in C_s} z_k^{\frac{1}{c_k}} = \delta_j(S,0,\mathcal{C})\,\delta_i(S,0,\mathcal{C}) = \prod_{k \in C_r} \delta_i(S,0,\mathcal{C})^{\frac{1}{c_k}} \prod_{k \in C_s} \delta_j(S,0,\mathcal{C})^{\frac{1}{c_k}}$$

and

$$\prod_{k \in N} z_k^{\frac{1}{c_k}} = \prod_{k \in N} \delta_k(S,0,\mathcal{C})^{\frac{1}{c_k}} = \max_{x \in S, x \geq 0} \prod_{k \in N} x_k^{\frac{1}{c_k}}.$$

Thus, z and $\delta(S,0,\mathcal{C})$ are solutions of the maximization problem (2). Since this solution is unique, we have $z = \delta(S,0,\mathcal{C})$. In particular, $\delta_i(S,0,\mathcal{C}) = \delta_j(S,0,\mathcal{C})$ for every $i \in C_r$ and $j \in C_s$.

Next we prove the uniqueness of the solution in each case.

1.- Let us consider a solution φ defined on the class $\mathcal{B}(N)$ which satisfies **IIA**, **IAT**, and **UCG**. Let $(S, d, \mathcal{C}) \in \mathcal{B}(N)$. Because δ satisfies **IAT**, we assume $d = 0 \in \mathbb{R}^N$ and $\delta(S,d,\mathcal{C}) = (1,\ldots,1) = e$.

There exists a hyperplane which separates S and the set

$$\left\{ x \in \mathbb{R}^N : \prod_{i \in N} x_i^{\frac{1}{c_i}} > 1 \right\}.$$

Let us assume that $\lambda \in \mathbb{R}^N_{++}$ defines such hyperplane. Since S is a convex set and e is the solution of the maximization problem (2), $\sum_{i \in N} \lambda_i x_i \leq 1$ for every $x \in S$. Thus, we consider the bargaining problem with coalition structure given by $(H_\lambda, 0, \mathcal{C})$ where H_λ is defined as in (6). The set H_λ is obtained from Δ by the affine transformation defined as $\gamma = \frac{1}{\lambda}$ and $\beta = 0$. Since δ and φ satisfy **IAT** and **UCG**, it holds

$$\varphi(H_\lambda, 0, \mathcal{C}) = \delta(H_\lambda, 0, \mathcal{C}) = \frac{1}{p}\frac{1}{\lambda}\frac{1}{c}. \tag{11}$$

By the definition of the solution δ and because $S \subseteq H_\lambda$,

$$1 = \max_{x \in S, x \geq 0} \prod_{i \in N} x_i^{\frac{1}{c_i}} \leq \max_{x \in H_\lambda, x \geq 0} \prod_{i \in N} x_i^{\frac{1}{c_i}} \leq 1.$$

Then,

$$\delta(H_\lambda, 0, \mathcal{C}) = \delta(S, 0, \mathcal{C}) = e \in S. \tag{12}$$

From (11) and (12),

$$\varphi(H_\lambda, 0, \mathcal{C}) = e \in S.$$

Since $S \subseteq H_\lambda$, $\varphi(H_\lambda, 0, \mathcal{C}) \in S$, and φ satisfies **IIA**, we have $\varphi(S, 0, \mathcal{C}) = \varphi(H_\lambda, 0, \mathcal{C})$. Then, $\varphi(S, 0, \mathcal{C}) = e = \delta(S, 0, \mathcal{C})$.

2.- By Lemma 3.1, any solution φ which satisfies **PE**, **IIA**, **IAT**, **SIC**, and **CS** also satisfies **IIA**, **IAT**, and **UCG**. In these conditions, as we have previously proved, the solution φ coincides with δ.

3.- Let us take any solution φ which satisfies all these properties. By Lemma 3.2, any solution φ which satisfies **PE**, **IIA**, **IAT**, **SIC**, and **SEC** also satisfies **IIA**, **IAT**, and **UCG**. Using Item 1 of this Theorem, we get that φ coincides with δ. ■

We analyze the independence of the properties in Theorem 3.1.

1. The properties **IIA**, **IAT**, and **UCG** are independent.
   (a) The Nash solution satisfies **IIA** and **IAT**, but not **UCG**.
   (b) The weighted Kalai-Smorodinsky solution (Gutiérrez-López, 1993) with weights given by $w_i = \frac{1}{pc_i}$ for each $i \in N$, is defined as

   $$\eta_i(S, d, \mathcal{C}) = d_i + \hat{t}\frac{u_i}{pc_i} \tag{13}$$

   where for each $i \in N$,

   $$u_i = \max\{t \in \mathbb{R} : (d_1, \ldots, d_{i-1}, t, d_{i+1}, \ldots, d_n) \in S\}, \text{ and}$$
   $$\hat{t} = \max\left\{t \in \mathbb{R}_{++} : \left(d_1 + t\frac{u_1}{pc_1}, \ldots, d_n + t\frac{u_n}{pc_n}\right) \in S\right\}$$

   satisfies **IAT** and **UCG**, but not **IIA**.
   (c) The solution ν^0 which assigns to any $i \in N$ the number

   $$\nu_i^0(S, d, \mathcal{C}) = d_i + \frac{\hat{t}}{pc_i} \tag{14}$$

   where

   $$\hat{t} = \max\left\{t \in \mathbb{R}_{++} : \left(d_1 + \frac{t}{pc_1}, \ldots, d_n + \frac{t}{pc_n}\right) \in S\right\}$$

   satisfies **IIA** and **UCG**, but not **IAT**.

2. The properties **PE**, **IIA**, **IAT**, **SIC**, and **CS** are independent.
   (a) The solution ν^1 which assigns to each bargaining problem with coalition structure $(S, d, \mathcal{C})$ the vector d satisfies **IIA**, **IAT**, **SIC**, and **CS**, but not **PE**.
   (b) The weighted Kalai-Smorodinsky solution defined in (13) satisfies **PE**, **IAT**, **SIC**, and **CS**, but not **IIA**.
   (c) The solution defined in (14) satisfies **PE**, **IIA**, **SIC**, and **CS**, but not **IAT**.
   (d) Let N^w be the weighted Nash solution where w is a vector of weights such that $w_i \neq w_j$ for any $i, j \in C_q$ and $\sum_{i \in C_q} w_i = \frac{1}{p}$, for each coalition $C_q \in \mathcal{C}$. This solution satisfies **PE**, **IIA**, **IAT**, and **CS**, but not **SIC**.
   (e) The Nash solution satisfies **PE**, **IIA**, **IAT**, and **SIC**, but not **CS**.

3. The properties **PE**, **IIA**, **IAT**, **SIC**, and **SEC** are independent.
   (a) The solution ν^1 defined above satisfies **IIA**, **IAT**, **SIC**, and **SEC**, but not **PE**.
   (b) The solution ν^2 defined as

$$\nu^2 (S, d, \mathcal{C}) = \begin{cases} \delta (S, d, \mathcal{C}) & \text{if} \quad |\mathcal{C}| > 1 \\ \eta (S, d, \mathcal{C}) & \text{if} \quad |\mathcal{C}| = 1 \end{cases}$$

   satisfies **PE**, **IAT**, **SIC**, and **SEC**, but not **IIA**.
   (c) The solution ν^3 defined as

$$\nu_i^3 (S, d, \mathcal{C}) = d_i + \hat{t}, \text{ for every } i \in N$$

   where $\hat{t}$ is given by

$$\hat{t} = \max \{ t \in \mathbb{R}_{++} : (d_1 + t, \ldots, d_n + t) \in S \}$$

   satisfies **PE**, **IIA**, **SIC**, and **SEC**, but not **IAT**.
   (d) Let w be a vector of weights such that there exist $i, j \in N$ with $w_i \neq w_j$. The solution ν^4 defined as

$$\nu^4 (S, d, \mathcal{C}) = \begin{cases} \delta (S, d, \mathcal{C}) & \text{if} \quad |\mathcal{C}| > 1 \\ N^w (S, d, \mathcal{C}) & \text{if} \quad |\mathcal{C}| = 1 \end{cases}$$

   satisfies **PE**, **IAT**, **IIA**, and **SEC**, but not **SIC**.
   (e) The Nash solution satisfies **PE**, **IIA**, **IAT**, and **SIC**, but not **SEC**.

Chae and Heidhues (2004) characterize δ as the unique solution satisfying **PE**, **IIA**, **IAT**, **SYM**, and **RHG**. We would like to mention that there is no relationship between our characterizations of δ and the characterization of δ given in Chae and Heidhues (2004).

It is clear that there is no relationship between our characterizations 1 and 2 and the one of Chae and Heidhues (2004).

There is no relationship between **SEC** and **RHG** as we illustrate next. For instance, the following solution ν^5 defined as

$$\nu^5(S,d,\mathcal{C}) = \begin{cases} \delta(S,d,\mathcal{C}) & \text{if} \quad |N| > 2 \\ \nu^3(S,d,\mathcal{C}) & \text{if} \quad |N| \leq 2 \end{cases}$$

satisfies **SEC** but not **RHG**.

We consider the solution ν^6 defined as

$$\nu_i^6(S,d,\mathcal{C}) = \begin{cases} d_i + t^* & \text{if} \quad i \in C_1 \\ d_i & \text{if} \quad i \notin C_1 \end{cases}$$

where $\mathcal{C} = \{C_1,...,C_p\}$ and $t^* = \max\left\{t \in \mathbb{R}_{++} : \left((d_i+t)_{i\in C_1}, (d_i)_{i\in N\setminus C_1}\right) \in S\right\}$. ν^6 satisfies **RHG** but not **SEC**.

Thus, there is no relationship between our characterization 3 and the one of Chae and Heidhues (2004).

4 About δ and some NTU-values

In this section we show that the following values, the Game with Coalition Structure (GCS) value, the Consistent Coalitional (CC) value, the Random-Order Coalitional (ROC) value, the λ-Transfer Coalitional (λTC) value, and the τ-λ Transfer Coalitional ($\tau-\lambda TC$) value, generalize the solution δ.

Theorem 4.1 The values Φ^{GCS}, Φ^{CC}, Φ^{ROC}, $\Phi^{\lambda TC}$, and $\Phi^{\tau\lambda TC}$ generalize the solution δ.

Proof. Let $(S,d,\mathcal{C}) \in \mathcal{B}(N)$.

Claim 1. $\{\delta(S,d,\mathcal{C})\} = \Phi^{GCS}(S,d,\mathcal{C})$.

From the characterization of each point belonging to $\Phi^{GCS}(S,d,\mathcal{C})$ proposed in (5), it holds that Φ^{GCS} satisfies **IAT**. Since δ also satisfies **IAT**, we assume $d = 0$ and $\delta(S,0,\mathcal{C}) = (1,\ldots,1) = e$.

Let us assume that the supporting hyperplane of S at e is defined by $\lambda \in \mathbb{R}^N_{++}$. As a consequence of (11) and (12), and doing some algebra,

$$e = \frac{1}{\lambda}\frac{1}{c}\max\left\{t \in \mathbb{R} : \frac{1}{\lambda}\frac{1}{c}t \in S\right\}.$$

By (5), $\delta(S,0,\mathcal{C}) = e \in \Phi^{GCS}(S,0,\mathcal{C})$.

Let us take $x \in \Phi^{GCS}(S,0,\mathcal{C})$. Let $\lambda \in \mathbb{R}^N_{++}$ be the vector which defines the supporting hyperplane of S at x. Let us consider $(H_\lambda, 0, \mathcal{C}) \in \mathcal{B}(N)$ with H_λ defined as in (6). Then, $\delta(H_\lambda,0,\mathcal{C}) \in \Phi^{GCS}(H_\lambda,0,\mathcal{C})$. Moreover, $x \in \Phi^{GCS}(H_\lambda,0,\mathcal{C})$ because $x \in \Phi^{GCS}(S,0,\mathcal{C}) \cap H_\lambda$. Since $\Phi^{GCS}(H_\lambda,0,\mathcal{C})$ is a singleton, $\delta(H_\lambda,0,\mathcal{C}) = x$. Since $S \subseteq H_\lambda$, $\delta(H_\lambda,0,\mathcal{C}) \in S$, and δ satisfies **IIA**, we have

$$x = \delta(H_\lambda,0,\mathcal{C}) = \delta(S,0,\mathcal{C}) = e,$$

and the claim is proved.

Claim 2. $\{\delta(S,d,\mathcal{C})\} = \Phi^{CC}(S,d,\mathcal{C})$.

It follows from similar reasoning as we did in *Claim 1.* Notice that Φ^{CC} satisfies **IAT**, and assuming that $d = 0$ and $\delta(S,0,\mathcal{C}) = e$, we obtain that $\delta(S,0,\mathcal{C})$ satisfies (7).

Claim 3. $\{\delta(S,d,\mathcal{C})\} = \Phi^{ROC}(S,d,\mathcal{C})$.

We have previously seen that $\Phi^{ROC}(S,d,\mathcal{C}) = \Phi^{CC}(S,d,\mathcal{C})$.

Claim 4. $\{\delta(S,d,\mathcal{C})\} = \Phi^{\lambda TC}(S,d,\mathcal{C})$.

For every $\lambda \in \mathbb{R}^N_{++}$ such that the game v^λ defined as in (8) is well-defined, the Owen value for v^λ is given by

$$Ow_i\left(N, v^\lambda, \mathcal{C}\right) = \lambda_i d_i + \frac{v^\lambda(N) - \sum_{j\in N} \lambda_j d_j}{pc_i} \text{ for every } i \in N.$$

By *Claim 2* and (7)

$$\delta_i(S,d,\mathcal{C}) = d_i + \frac{1}{\lambda_i}\frac{\sum_{j\in N} \lambda_j (x_j - d_j)}{pc_i} \text{ for every } i \in N,$$

and thus $\Phi^{\lambda TC}(S,d,\mathcal{C}) = \Phi^{CC}(S,d,\mathcal{C}) = \{\delta(S,d,\mathcal{C})\}$.

Claim 5. $\{\delta(S,d,\mathcal{C})\} = \Phi^{\tau\lambda TC}(S,d,\mathcal{C})$.

It follows from a similar reasoning that *Claim 4*, because, for every $\lambda \in \mathbb{R}^N_{++}$ such that the game v^λ is well-defined,

$$\tau_i\left(N, v^\lambda, \mathcal{C}\right) = \lambda_i d_i + \frac{v^\lambda(N) - \sum_{j\in N} \lambda_j d_j}{pc_i} \text{ for every } i \in N.$$

■

5 A non-cooperative perspective

In the context of *NTU* games, Hart and Mas-Colell (1996) design a simple non-cooperative mechanism of negotiation between n players. Applied to bargaining problems, this mechanism is as follows: In each round, a player is randomly chosen to propose a payoff. If all the other players agree, the mechanism finishes with this payoff. If at least a player disagrees, the mechanism is repeated with probability $\rho \in [0,1)$. With probability $1-\rho$, the proposer leaves the mechanism and thus each player gets his disagreement payoff.

In Theorem 3 in Hart and Mas-Colell (1996), it is shown that the above mechanism (when applied to bargaining problems) yields the Nash bargaining solution as ρ approaches 1.

Vidal-Puga (2005a) adapts this mechanism when players are divided in coalitions. Hart and Mas-Colell's mechanism is played in two levels, first between players inside each coalition and second between coalitions. In the first level, players inside the same coalition decide (following Hart and Mas-Colell's mechanism) which proposal to use in the second level.

Formally:

Mechanism I First, a *proposer* $i \in C_1$ is randomly chosen out of coalition $C_1 \in \mathcal{C}$, being each player equally likely to be chosen. Player i proposes a feasible payoff, *i.e.* a point in S. The members of $C_1 \backslash \{i\}$ are then asked in some prespecified order. If one of the members of $C_1 \backslash \{i\}$ rejects the proposal, then with probability ρ the mechanism is repeated under the same conditions, and with probability $1-\rho$ the mechanism finishes in disagreement. If all the members of $C_1 \backslash \{i\}$ accept the proposal, then the same procedure is repeated with coalition C_2, and so on. If there is no rejection, one of the proposals is chosen at random, being each proposal equally likely to be chosen. Say the proposal of coalition C_q is chosen. Then, the members of $N \backslash C_q$ are asked in some prespecified order. If one of the members of $N \backslash C_q$ rejects the proposal, then with probability ρ the mechanism is repeated under the same conditions, and with probability $1-\rho$ the mechanism finishes in disagreement. If the mechanism finishes in disagreement, the final payoff is d.

This structure in two levels appears in many situations where negotiations are carried out by agents who are the delegates of larger coalitions. Delegates begin to negotiate among them not before agreeing their proposals with their respective coalitions.

However, it may be possible an inverse structure: a coalition is first chosen to make a proposal, and only then they choose a proposer to make the offer.

Formally:

Mechanism II First, a coalition C_q out of $\mathcal{C}$ is randomly chosen, being each coalition equally likely to be chosen. Then, a *proposer* i is randomly chosen out of C_q, being each player equally likely to be chosen. Player i proposes a feasible payoff, *i.e.*

a point in S. The members of $N\backslash\{i\}$ are then asked in some prespecified order. If one of the members of $N\backslash\{i\}$ rejects the proposal, then with probability ρ the mechanism is repeated under the same conditions, and with probability $1-\rho$ the mechanism finishes in disagreement. In the latter case, the final payoff is d.

This procedure is the adaptation to bargaining problems of the mechanism that appears in Section 4.4. in Vidal-Puga (2002).

Clearly, each player $i\in N$ is chosen as proposer with probability $\mu^i=\frac{1}{pc_i}$.

This mechanism also generalizes Hart and Mas-Colell's bargaining mechanism (applied to bargaining problems) when the coalition structure is trivial. However, it is not equivalent to the mechanism in Vidal-Puga (2005a). In particular, it does not implement the Owen value when applied to a TU game with coalition structure, as Example 43 in Vidal-Puga (2002) shows.

As in Hart and Mas-Colell (1996) and Vidal-Puga (2005a), we work with stationary strategies. This means that the proposal of an agent is independent of the previous history. Notice that a stationary subgame perfect equilibrium is also optimal against non-stationary strategies.

Theorem 5.1 If $(S,d,C)\in\mathcal{B}(N)$, in the two above mechanisms there exists a stationary subgame perfect equilibrium for each $\rho\in[0,1)$. Moreover, as ρ approaches 1, any stationary subgame perfect equilibrium payoff converges to $\delta(S,d,\mathcal{C})$.

From now on, when we say equilibrium, we mean stationary subgame perfect equilibrium.

As it can be readily checked from the proof of Theorem 5.1 below, both mechanisms yield the same equilibrium payoff *for any* ρ, and not only in the limit. However, this is not generally true for the class of TU games. See Example 43 in Vidal-Puga (2002). The reason is that, for Mechanism I, the players do not know which proposal will be selected in the game between coalitions. When a player is substitutable for someone else outside the coalition, it may happen that one of his coalition partners makes a proposal that leaves him a negative payoff. The substitutable player nevertheless agrees because his final payoff is nonnegative in expected terms (he hopes another proposal be chosen in the play between coalitions). See Example 14 in Vidal-Puga (2005a). For Mechanism II, however, no negative payoff is bound to be accepted in the game inside a coalition. This cancels out some disadvantage of a substitutable player. For the class of bargaining problems, however, no player is substitutable and hence the result.

The proof for Mechanism I comes from Theorem 12 in Vidal-Puga (2005a), *Claim 2* in the proof of Theorem 4.1, and an analogous reasoning as in the proof of Proposition 5.5 below. Hence, we concentrate on Mechanism II.

In order to prove Theorem 5.1 for Mechanism II, we need further notation.

Given $\rho \in [0,1)$, let $a^i(\rho)$ be the proposal of player i when he is the proposer. Let

$$a(\rho) := \sum_{i \in N} \mu^i a^i(\rho) \in \mathbb{R}^N$$

be the final payoff when all the proposals are due to be accepted. When there is no ambiguity, we write a and a^i instead of $a(\rho)$ and $a^i(\rho)$, respectively.

Proposition 5.1 Given $\rho \in [0,1)$, the proposals in any equilibrium of a bargaining problem with coalition structure $(S, d, \mathcal{C})$ are characterized by

P1 $a^i(\rho) \in \partial S$ for each $i \in N$ and

P2 $a^i_j(\rho) = \rho a_j(\rho) + (1-\rho) d_j$ for each $j \neq i$.

Moreover, the proposals are always accepted and $a^i(\rho) \geq d$ for each $i \in N$.

This Proposition is similar to Proposition 1 in Hart and Mas-Colell (1996). However, in Hart and Mas-Colell the vector a is the average of the a^i's. In this case, a is a weighted average with weights given by the μ^i's.

Proof. Assume we are in equilibrium. Let $b \in \mathbb{R}^N$ be the expected final payoff. Each player $i \in N$ can guarantee himself a payoff of at least d_i by proposing always d and accepting only proposals which give him no less than d_i. Thus, $b \geq d$.

We must prove that conditions P1 and P2 hold. We proceed by two Claims:

Claim (A): Assume the proposer is $i \in C_q$. Then, all players in $N \setminus \{i\}$ accept a^i if $a^i_j > \rho b_j + (1-\rho) d_j$ for each $j \neq i$. If $a^i_j < \rho b_j + (1-\rho) d_j$ for some $j \neq i$, then the proposal is rejected.

Notice that, in the case of rejection, the expected payoff of a player $j \neq i$ is $\rho b_j + (1-\rho) d_j$.

We assume without loss of generality that $i = 1$ and $(2, ..., n)$ is the order in which the players in $N \setminus \{i\}$ are asked.

If the game reaches player n, *i.e.* there has been no previous rejection, his optimal strategy involves accepting the proposal if a^i_n is higher than $\rho b_n + (1-\rho) d_n$ and rejecting it if it is lower than $\rho b_n + (1-\rho) d_n$. Player $n-1$ anticipates reaction of player n. Hence, if $a_n > \rho b_n + (1-\rho) d_n$, $a_{n-1} > \rho b_{n-1} + (1-\rho) d_{n-1}$, and the game reaches player $n-1$, he accepts the proposal. If $a_n < \rho b_n + (1-\rho) d_n$, then player $n-1$ is indifferent between accepting or rejecting the proposal, since he knows player n is bound to reject the proposal should the game reach him. In any case, the proposal is rejected. By going backwards, we prove the result for all players in $N\backslash\{i\}$.

Claim (B): Assume the proposer is player i. Then, his proposal is accepted.

Assume the proposal of player i is rejected. This means the final payoff for player i is $\rho b_i + (1-\rho)\, d_i$.

We define a new proposal a^i for player i as follows. Since $b \in S$ and d belongs to the interior of S, by convexity $\rho b + (1-\rho)\, d$ belongs to the interior of S. Thus, it is possible to find $\varepsilon > 0$ such that $\rho b + (1-\rho)\, d + (\varepsilon, ..., \varepsilon)$ belongs to S. Let $a^i = \rho b + (1-\rho)\, d + (\varepsilon, ..., \varepsilon)$. By *Claim (A)*, this offer is accepted and the final payoff for player i is $\rho b_i + (1-\rho)\, d_i + \varepsilon$. This contradiction proves *Claim (B)*.

Since all the proposals are accepted, and each player i has probability μ^i to be chosen as proposer, we can assure that $b = a$.

We show now that P1 and P2 hold.

Suppose P1 does not hold, *i.e.* there exists a player i such that a^i is not Pareto optimal. Thus, a^i belongs to the interior of S; so, there exists $\varepsilon > 0$ such that $a^i + (\varepsilon, ..., \varepsilon) \in S$.

Notice that, since the proposal a^i of player i is accepted (*Claim (B)*), by *Claim (A)* we know that $a^i_j \geq \rho a_j + (1-\rho) d_j$ for each $j \neq i$. So, if player i changes his proposal to $a^i + (\varepsilon, ..., \varepsilon)$, it is bound to be accepted and his expected final payoff improves by $\mu^i \varepsilon > 0$. This contradiction proves P1.

Suppose P2 does not hold. Let $j_0 \neq i$ be a player such that $a^i_{j_0} = \rho a_{j_0} + (1-\rho) d_{j_0} + \alpha$ with $\alpha \neq 0$. By *Claim* (*A*) and *Claim (B)*, $\alpha > 0$.

Let $x \in \mathbb{R}^N$ be defined by $x_{j_0} = \alpha$ and $x_j = 0$ for all $j \neq j_0$. By comprehensiveness and nonlevelness, we have $a^i - x$ belongs to the interior of S. Thus, there exists $\varepsilon > 0$ such that

$$\widehat{a}^i := a^i - x + (\varepsilon, ..., \varepsilon)$$

belongs to S. Suppose player i changes his proposal to $\widehat{a}^i$. Let $\widehat{a}^j = a^j$ for all $j \neq i$. The new average $\widehat{a} = \sum_{i \in N} \mu^i \widehat{a}^i$ satisfies

$\widehat{a}^i_i = a^i_i - x_i + \varepsilon = a^i_i + \varepsilon > a^i_i$,

$\widehat{a}^i_{j_0} = a^i_{j_0} - x_{j_0} + \varepsilon = \rho a_{j_0} + (1-\rho) d_{j_0} + \alpha - \alpha + \varepsilon > \rho a_{j_0} + (1-\rho) d_{j_0}$, and

$\widehat{a}^i_j = a^i_j - x_i + \varepsilon = a^i_j + \varepsilon > a^i_j \geq \rho a_j + (1-\rho)\, d_j$ for all $j \neq i, j_0$.

Thus, by *Claim (A)*, the new proposal of player i is due to be accepted. Also, player i improves his expected payoff. This contradiction proves P2.

Conversely, we show that proposals $(a^i)_{i \in N}$ satisfying P1 and P2 can be supported as an equilibrium.

First, we prove that $a^i \geq d$ for all $i \in N$. By convexity, $x = \rho a + (1-\rho) d$ belongs to S. Fix $i \in N$, by P2, we have $a^i_j = x_j$ for all $j \neq i$. We conclude that $a^i \geq x$ because $a^i \in \partial S$ and $x \in S$. Hence:

$$a_j = \sum_{i\in N} \mu^i a^i_j \geq \sum_{i\in N} \mu^i x_j = \sum_{i\in N} \mu^i (\rho a_j + (1-\rho) d_j) = \rho a_j + (1-\rho) d_j$$

and thus $(1-\rho) a_j \geq (1-\rho) d_j$, *i.e.* $a_j \geq d_j$.

Fix a player $i \in N$. If he rejects the proposal from a proposer $j \neq i$, his expected final payoff is $\rho a_i + (1-\rho) d_i$. Thus, his expected final payoff is the same as that the other player is offering. Since the rest of the players accept the proposal, he does not improve his expected final payoff by rejecting it. If the proposer is player i himself, the strategies of the other players do not allow him to decrease his proposal to any of them (since it would be rejected by *Claim (A)*). Moreover, increasing one or more of his offers to the other players keeping the rest unaltered implies his own payment decreases (by P1 and nonlevelness). Finally, by offering an unacceptable proposal, he may be dropped out and his expected final payment becomes d_i, which does not improve his final payoff because $a^i_i \geq d_i$. Thus, the proposals do form an equilibrium. ■

Proposition 5.2 Let $S = \left\{ x \in \mathbb{R}^N : \sum_{i\in N} \lambda_i x_i \leq \xi \right\}$ for some $\lambda \in \mathbb{R}^N_{++}$ and $\xi \in \mathbb{R}$. Assume a set of proposals $(a^i)_{i\in N}$ *satisfies P1 and P2. Then* $a = \delta(S, d, \mathcal{C})$, *i.e.*

$$\lambda_i a_i = \lambda_i d_i + \mu^i \left(\sum_{j\in N} \lambda_j a_j - \sum_{j\in N} \lambda_j d_j \right)$$

for each $i \in N$.

Proof. Fix $i \in C_q$. Then,

$$\lambda_i a_i = \lambda_i \sum_{j\in N} \mu^j a^j_i = \lambda_i \sum_{j\neq i} \mu^j a^j_i + \mu^i \lambda_i a^i_i.$$

By P1,

$$\begin{aligned} \lambda_i a_i &= \lambda_i \sum_{j\neq i} \mu^j a^j_i + \mu^i \left(\xi - \sum_{j\neq i} \lambda_j a^i_j \right) \\ &= \lambda_i \sum_{j\in N} \mu^j a^j_i + \mu^i \left(\xi - \sum_{j\in N} \lambda_j a^i_j \right). \end{aligned}$$

By P2,

$$\begin{aligned} \lambda_i a_i &= \lambda_i \sum_{j\in N} \mu^j (\rho a_i + (1-\rho) d_i) + \mu^i \left(\xi - \sum_{j\in N} \lambda_j (\rho a_j + (1-\rho) d_j) \right) \\ &= \rho \lambda_i a_i + (1-\rho) \lambda_i d_i + \mu^i \left(\xi - \rho \sum_{j\in N} \lambda_j a_j - (1-\rho) \sum_{j\in N} \lambda_j d_j \right). \end{aligned}$$

Since $a^i \in \partial S$ and $\sum_{j\in N} \mu^j = 1$, we have $\sum_{j\in N} \lambda_j a_j = \xi$. Hence,

$$\lambda_i a_i = \rho\lambda_i a_i + (1-\rho)\lambda_i d_i + \mu^i \left((1-\rho)\xi - (1-\rho)\sum_{j\in N} \lambda_j d_j\right).$$

Hence,

$$(1-\rho)\lambda_i a_i = (1-\rho)\lambda_i d_i + (1-\rho)\mu^i \left(\xi - \sum_{j\in N} \lambda_j d_j\right)$$

and dividing by $(1-\rho)$,

$$\lambda_i a_i = \lambda_i d_i + \mu^i \left(\xi - \sum_{j\in N} \lambda_j d_j\right)$$

which completes the proof because $\xi = \sum_{j\in N} \lambda_j a_j$. ■

Corollary 5.1 Assume $S = \left\{x \in \mathbb{R}^N : \sum_{i\in N} \lambda_i x_i \le \xi\right\}$ for some $\lambda \in \mathbb{R}^N_{++}$, $\xi \in \mathbb{R}$. *Then, for each* $\rho \in [0,1)$, *there exists a unique equilibrium payoff, which equals* $\delta(S, d, \mathcal{C})$.

Proof. Immediate from Proposition 5.1 and Proposition 5.2. ■

Proposition 5.3 Let $(S, d, \mathcal{C}) \in \mathcal{B}(N)$. Then, for *each* $\rho \in [0,1)$, there exists an equilibrium.

Proof. By Proposition 5.1, we only need to prove that there exist proposals satisfying P1 and P2.

Let $K = \{x \in S : x \ge d\}$. This set is nonempty ($d \in K$), closed (because S is closed), and bounded. Thus, K is a compact set. Furthermore, K is convex (because S is convex).

We define n functions $\alpha^i : K \to K$ as follows. Given $i \in N$, $\alpha^i_j(x) := \rho x_j + (1-\rho)d_j$ for each $j \ne i$ and $\alpha^i_i(x)$ is defined in such a way that $\alpha^i(x) \in \partial S$.

These functions are well-defined because $y := \rho x + (1-\rho)d$ belongs to K (by convexity) and $\alpha^i(x)$ equals y in all coordinates but i's, which we increase until reaching the boundary of S.

Also, because of the smoothness of S the functions α^i are continuous. By the convexity of the domain, $\sum_{i\in N} \mu^i \alpha^i(x) \in K$ for each $x \in K$. Hence, the function $\alpha : K \to K$ defined as $\alpha(x) = \sum_{i\in N} \mu^i \alpha^i(x)$ for all $x \in K$ is well-defined and continuous. Since K is a non-empty, compact, and convex subset of the Euclidean space, under Kakutani fixed point theorem there exists a vector $a \in K$ satisfying $a = \sum_{i\in N} \mu^i \alpha^i(a)$.

We define $a^i = \alpha^i(a)$ for each $i \in N$. It is trivial to see that $(a^i)_{i\in N}$ satisfies P1 and P2. ■

Proposition 5.4 Let $(S, d, \mathcal{C}) \in \mathcal{B}(N)$ and let $(a^i)_{i\in N}$ be the proposals in equilibrium. Then, there exists $M \in \mathbb{R}$ such that $\left|a_j^i - a_j\right| \leq M(1-\rho)$ for all $i, j \in N$.

Proof. Fix $i \in N$. Given $j \in N\backslash\{i\}$, by P2:

$$\left|a_j^i - a_j\right| = |\rho a_j + (1-\rho)\, d_j - a_j| = (1-\rho)\,|a_j - d_j|\,.$$

We define

$$M_1^i = \max\{|a_j - d_j| : j \in N\backslash\{i\}\,, \rho \in [0,1)\}\,.$$

Notice that a_j depends on ρ. This maximum is well-defined because $a_j \geq d_j$ for all $j \in N\backslash\{i\}$, $a \in K = \{x \in S : x \geq d\}$, and K is compact.

We have then $\left|a_j^i - a_j\right| \leq M_1^i(1-\rho)$ for all $j \in N\backslash\{i\}$.

We now study $|a_i^i - a_i|$. We know that $a_i = \sum_{j\in N} \mu^j a_i^j$. Then,

$$a_i^i = \frac{1}{\mu^i}\left(a_i - \sum_{j\neq i}\mu^j a_i^j\right).$$

So,

$$\begin{aligned}
\left|a_i^i - a_i\right| &= \frac{1}{\mu^i}\left|a_i - \sum_{j\neq i}\mu^j a_i^j - \mu^i a_i\right| \\
&= \frac{1}{\mu^i}\left|a_i - \sum_{j\neq i}\mu^j\left(\rho a_i + (1-\rho)\, d_i\right) - \mu^i a_i\right| \\
&= \frac{1}{\mu^i}\left|a_i - \rho\sum_{j\in N}\mu^j a_i - (1-\rho)\sum_{j\neq i}\mu^j d_i - (1-\rho)\,\mu^i a_i\right|.
\end{aligned}$$

Since $\sum_{j\in N}\mu^j = 1$,

$$\begin{aligned}
\left|a_i^i - a_i\right| &= \frac{1}{\mu^i}\left|(1-\rho)\sum_{j\in N}\mu^j a_i - (1-\rho)\sum_{j\neq i}\mu^j d_i - (1-\rho)\,\mu^i a_i\right| \\
&= \frac{1-\rho}{\mu^i}\left|\sum_{j\neq i}\mu^j a_i - \sum_{j\neq i}\mu^j d_i\right| \\
&\leq \frac{1-\rho}{\mu^i}\sum_{j\neq i}\mu^j\,|a_i - d_i| \\
&= \frac{1-\rho}{\mu^i}\left(1-\mu^i\right)|a_i - d_i|\,.
\end{aligned}$$

Let

$$M_2^i = \frac{1-\mu^i}{\mu^i}\max\{|a_i - d_i| : \rho \in [0,1)\}\,.$$

Using similar arguments to those used with M_1^i we can argue that M_2^i is well-defined, for each $i \in N$.

So, we take $M^i = \max\{M_1^i, M_2^i\}$ and $M = \max\{M^i\}_{i\in N}$. ■

Proposition 5.5 Let $(S,d,\mathcal{C}) \in \mathcal{B}(N)$, and let $a(\rho)$ be an equilibrium payoff for each $\rho \in [0,1)$. Then, $a(\rho) \to \delta(S,d,\mathcal{C})$ when $\rho \to 1$.

Proof. Note that $a(\rho) \to \delta(S,d,\mathcal{C})$ means that for all $\varepsilon > 0$ there exists $\rho_0 \in [0,1)$ such that if $\rho > \rho_0$ then, $|a(\rho) - \delta(S,d,\mathcal{C})| < \varepsilon$.

Assume the result is not true. This means that there exists $\hat{\varepsilon} > 0$ such that for each $\rho_0 \in [0,1)$ it is possible to find $\rho > \rho_0$ satisfying

$$|a(\rho) - \delta(S,d,\mathcal{C})| \geq \hat{\varepsilon}.$$

Let $\{\rho_0^k\}_{k=0}^{\infty} \subsetneq [0,1)$ be a sequence with $\rho_0^k \to 1$. For each k, it is possible to find $\rho^k > \rho_0^k$ satisfying $|a(\rho^k) - \delta(S,d,\mathcal{C})| \geq \hat{\varepsilon}$. Since $\rho_0^k \to 1$ and $\rho^k > \rho_0^k$ for all k, we have $\rho^k \to 1$. Moreover, $|a(\rho^k) - \delta(S,d,\mathcal{C})| \geq \hat{\varepsilon}$ for all k.

Since $a(\rho^k) \geq d$ for each k and S is closed, there exists $a^* \geq d$ such that a^* is a limit point of $\{a(\rho^k)\}_{k=0}^{\infty}$, *i.e.* there exists a subsequence of $\{a(\rho^k)\}_{k=0}^{\infty}$ which converges to a^*. We can assure without loss of generality that $a(\rho^k) \to a^*$.

Since $\rho^k \to 1$, by Proposition 5.4, $a^i(\rho^k) \to a^*$ for each $i \in N$. Since $a^i(\rho) \in \partial S$ for each $\rho \in [0,1)$, $i \in N$ and ∂S is closed, we conclude that $a^* \in \partial S$.

Let λ be the unit length vector normal to ∂S at a^*. We associate to each ρ^k a bargaining problem with coalitional structure $(S_k, d, \mathcal{C})$ as follows:

Given k, there exists at least one hyperplane on $\mathbb{R}^N$ containing the n points $\{a^i(\rho^k) : i \in N\}$. If there are more than one hyperplane, we take the one whose unit length outward orthogonal vector λ^k is the closest to λ.

We define:

$$S_k = \left\{ x \in \mathbb{R}^N : \sum_{j\in N} \lambda_j^k x_j \leq \sum_{j\in N} \lambda_j^k a_j^i(\rho), i \in N \right\}.$$

The half-space S_k is well-defined because $\sum_{j\in N} \lambda_j^k a_j^i(\rho) = \sum_{j\in N} \lambda_j^k a_j^{i'}(\rho)$ for all $i, i' \in N$.

Since $a^i(\rho^k) \to a^*$ for all $i \in N$, by the smoothness of ∂S, $\lambda^k \to \lambda$. Therefore,

$$S_k \to S' = \left\{ x \in \mathbb{R}^N : \sum_{j\in N} \lambda_j x_j \leq \sum_{j\in N} \lambda_j a_j^* \right\}.$$

By Proposition 5.1, the proposals $\{a^i(\rho^k) : i \in N\}$ satisfy P1 and P2 for $(S,d,\mathcal{C})$. But these properties are the same for $(S_k,d,\mathcal{C})$. Thus, by Proposition 5.1, $a(\rho^k)$ is an equilibrium payoff for $(S_k,d,\mathcal{C})$. By Proposition 5.2, this implies that $a(\rho^k) = \delta(S_k,d,\mathcal{C})$. Hence, given $i \in N$,

$$a_i(\rho^k) = d_i + \frac{\mu^i}{\lambda_i^k}\left(\sum_{j\in N}\lambda_j^k a_j(\rho^k) - \sum_{j\in N}\lambda_j^k d_j\right)$$

and thus

$$a_i^* = d_i + \frac{\mu^i}{\lambda_i}\left(\sum_{j\in N}\lambda_j a_j - \sum_{j\in N}\lambda_j d_j\right).$$

Hence $a^* = \delta(S, d, \mathcal{C})$. But this contradicts that $|a(\rho^k) - \delta(S, d, \mathcal{C})| \geq \hat{\varepsilon}$ for each $k = 0, 1, \ldots$. This proves the result. ■

6 Concluding Remarks

Chae and Heidhues (2004) and Vidal-Puga (2005a) describe two values in bargaining problems with coalition structure. We prove that both values coincide.

We study this value (called δ) and we find three kinds of results. Firstly, we present three new characterizations of δ. Secondly, we prove that five values presented in the literature for *NTU* games with coalition structure coincide with δ when we restrict to bargaining problems. Thirdly, we present a new non-cooperative mechanisms with a unique stationary subgame perfect equilibrium payoff that approaches δ.

The Harsanyi paradox (Harsanyi, 1977) says that an individual can be worse off bargaining as a member of a coalition than bargaining alone. This paradox makes some solutions inadequate for some situations. Nevertheless, in other situations this is not so relevant. For instance, when coalitions are fixed and agents can not leave them. A good example could be a group of countries (considered as coalitions of local governments) bargaining about the reduction of greenhouse gas emissions. Most of the solutions of the literature have this paradox. Chae and Heidhues (2004) prove that δ has this paradox. Recently, Chae and Moulin (2004) and Vidal-Puga (2005b) found solutions without the Harsanyi paradox.

Acknowledgements. The authors wish to thank interesting suggestions and comments made by an Editor in charge and an anonymous referee. Financial support from the Ministerio de Educación y Ciencia, and FEDER through projects SEJ2005-07637-C02-01 and SEJ2005-07637-C02-02, and from Xunta de Galicia through Projects PGIDT03PXI20701PN and PGIDIT06PXIC300184PN is gratefully acknowledged.

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