The optimistic TU game in minimum cost spanning tree problems*

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Abstract

We associate an optimistic TU game with each minimum cost spanning tree problem. We define the worth of a coalition S as the cost of connecting agents in S to the source assuming that agents in $N \backslash S$ are already connected to the source, and agents in S can connect through agents in $N \backslash S$. We study the Shapley value of this new game.

Keywords: minimum cost spanning tree problems, optimistic TU game, Shapley value.

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). Imagine that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly.

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There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004) study a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Other examples include communication networks, such as telephone, Internet, or cable television.

The literature on *mcstp* starts by defining algorithms for constructing minimal cost spanning trees (*mt*). We can mention, for instance, the papers of Kruskal (1956) and Prim (1957).

Other important issue is how to allocate the cost associated with an *mt* among agents. Bird (1976) and Dutta and Kar (2004) introduce two rules based on Prim's algorithm. Feltkamp, Tijs, and Muto (1994) introduce the Equal Remaining Obligation rule (ERO) based on Kruskal's algorithm. ERO is called the $P - value$ in Branzei *et al* (2004).

Bird (1976) associates with each *mcstp* a cooperative game with transferable utility (TU game). According to Bird, the worth of a coalition is the cost of connection, assuming that the rest of the agents are not present. Hence, this worth takes the classical "stand alone" interpretation. The worth of a coalition is simply the best they can do without other players' contribution.

In this paper, we associate with each *mcstp* a different TU game. We define the worth of a coalition as the cost of connection, assuming that the rest of the agents are already connected, and that connection is possible through them at no charge.

Both TU games compute the cost of connecting agents to the source. The former takes a pessimistic point of view because it assumes, given a coalition, that the rest of the agents are not connected. The latter takes an optimistic point of view because it assumes that the rest of the agents are already connected.

In general there is no relationship between the optimistic and the pessimistic TU game. However, it is possible to find a relationship in an interesting class of problems. An *mcstp* is *irreducible* if reducing the cost of any arc, the total cost of connection is also reduced. Given an *mcstp*, Bird (1976) defined the irreducible problem associated with it. We prove that, in irreducible problems, both TU games are dual (two TU games v, w are *dual* if $v(S) + w(N \setminus S)$ is constant for all S).

We apply this result to study the important issue of cost sharing. A cost sharing rule allocates the cost associated with an *mt* between the agents.

An idea is to use a solution concept in the field of *TU* games and applying it in the *mcstp*. The core and the nucleolus of the pessimistic *TU* game are studied in Granot and Huberman (1981, 1984). The Shapley value of the pessimistic *TU* game is studied in Kar (2002). Bergantiños and Vidal-Puga (2005a) define the rule φ, of the *mcstp* C, as the Shapley value of the pessimistic *TU* game of the irreducible form associated with C. Bergantiños and Vidal-Puga (2005b) prove that φ coincides with *ERO*. Moreover, in irreducible problems, the rule presented by Bird (1976) coincides with φ.

We define two rules in *mcstp* using the optimistic *TU* game. The first rule is the Shapley value of the optimistic *TU* game. The second one is the Shapley value of the optimistic *TU* game associated with the irreducible problem.

We thus have four rules in *mcstp* based on the Shapley value. We prove that, in fact, the Shapley value of the optimistic *TU* game coincides with the Shapley value of the optimistic *TU* game associated with the irreducible form, and with the Shapley value of the pessimistic *TU* game associated with the irreducible form. The classical Shapley value (as defined by Kar (2002)) differs from these three.

Finally, we present a new characterization of this rule using a property of equal contributions.

The paper is organized as follows. In Section 2 we introduce *mcstp*. In Section 3 we introduce the optimistic *TU* game and present the main result. In Section 4 we study the four Shapley values. In the Appendix we give the proof of some of the results.

2 The minimum cost spanning tree problem

In this section we introduce minimum cost spanning tree problems.

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let Π_N be the set of all permutations over N. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.* $Pre(i, \pi) = \{j \in N : \pi(j) < \pi(i)\}$. Given $S \subset N$, let π_S denote the order induced by π among agents in S.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* (N_0, C), we define the *mcstp* induced by C for $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of g are called *arcs*.

Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network such that for all $i \in N$ there is a unique path from i to the source. If t is a tree, we usually write $t = \{(i^0, i)\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0.

Let $\mathcal{G}^N$ denote the set of all networks over N_0. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from i to 0 in the network.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree t over N_0 such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it is not necessarily unique. Given an *mcstp* (N_0, C), we denote the cost associated with any *mt* t in (N_0, C) as $m(N_0, C)$.

Given an *mcstp*, Prim (1957) introduced an algorithm for solving the problem of connecting all agents to the source, such that the total cost of creating the network is minimal. The idea of this algorithm is quite simple: starting from the source, we construct a network by consecutively connecting agents with the lowest cost to agents already connected

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take an arc $(0,i)$ such that $c_{0i} = \min_{i \in N} \{c_{0i}\}$. If there are several arcs $(0,i)$ satisfying this condition, select any of them. Now, $S^1 = \{0,i\}$ and $g^1 = \{(0,i)\}$.

Stage $p+1$: Assume we have defined $S^p \subset N_0$ and $g^p \in \mathcal{G}^N$. We now define S^{p+1} and g^{p+1}. Take an arc (j,i) with $j \in S^p$ and $i \in N_0 \setminus S^p$ such that $c_{ji} = \min_{k \in S^p, l \in N_0 \setminus S^p} \{c_{kl}\}$. If there are several arcs (j,i) satisfying this condition, select any of them. Now, $S^{p+1} = S^p \cup \{i\}$ and $g^{p+1} = g^p \cup \{(j,i)\}$.

This process is completed in n stages. We say that g^n is a tree obtained via Prim's algorithm. Notice that this algorithm leads to a tree, but that this is not always unique.

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source between them. We now briefly introduce some of the rules studied in the literature.

A (*cost allocation*) *rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$ for each *mcstp* (N_0, C). As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

Notice that we implicitly assume that the agents build an *mt*. As far as we know, all the rules proposed in the literature make this assumption.

A *game with transferable utility*, briefly a *TU game*, is a pair (N, v) where $v : 2^N \to \mathbb{R}$ satisfies that $v(\varnothing) = 0$. $Sh(N, v)$ denotes the Shapley value (Shapley (1953)) of (N, v).

A quite standard approach for defining rules in some problems is using *TU* games. We first associate with each problem a *TU* game. We then compute a solution for *TU* games (Shapley value, core, ...) in the associated *TU* game. Thus, the rule in the original problem is defined as the solution applied to the *TU* game associated with the original problem. This approach was already applied in *mcstp*.

Bird (1976) associated a *TU* game (N, v_C) with each *mcstp* (N_0, C). For each coalition $S \subseteq N$,

$$v_C(S) = m(S_0, C).$$

We define, in *mcstp*, the rule Sh^1 as the Shapley value of the associated *TU* game, *i.e.*

$$Sh^1(N_0, C) = Sh(N, v_C).$$

This rule was studied in Kar (2002).

An *mcstp* (N_0, C) is *irreducible* if reducing the cost of any arc, the cost of connecting all agents to the source $(m(N_0, C))$ is also reduced. In Bergantiños and Vidal-Puga (2005a) we proved that (N_0, C) is irreducible if and only if there exists an *mt* t in (N_0, C) satisfying the following two conditions:

$(A1)$ $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ where $i_0 = 0$ (the source).

$(A2)$ Given $i_p, i_q \in N_0$, $p < q$, then $c_{i_p i_q} = \max_{p<r\leq q} \left\{c_{i_{r-1} i_r}\right\}$.

Given an *mcstp* (N_0, C), Bird (1976) defined the irreducible form (N_0, C^*) associated with (N_0, C). We define the rule Sh^2 as the Shapley value of the *TU* game associated with the irreducible form, *i.e.*

$$Sh^2(N_0, C) = Sh(N, v_{C^*}).$$

In Bergantiños and Vidal-Puga (2005b) we proved that Sh^2 coincides with the *ERO* rule (Feltkamp *et al.* (1994)).

3 The optimistic game

In many class of problems it is possible to associate two *TU* games to each problem in the class: a pessimistic game and an optimistic game. An example could be queuing problems, where a set of agents stands to receive a service. No two agents can be served simultaneously. Each agent has a constant per unit of time waiting cost. A queue has to be organized, but monetary compensations may be set up for those who have to wait. Maniquet (2003) defined the worth of a coalition S as the sum of its waiting cost in an efficient queue if they had the power to be served before agents in $N\backslash S$. Maniquet is taking an optimistic approach. Chun (2006) defined the worth of a coalition S as the sum of its waiting cost in an efficient queue, assuming that members of S are served after the members of $N\backslash S$. Chun is taking a pessimistic approach.

In this section, we associate an "optimistic" *TU* game $\left(N, v_C^+\right)$ with each *mcstp* (N_0, C).

Given $S \subseteq N$, Bird (1976) defined the worth of coalition S, $v_C(S)$, as the minimal cost of connecting all agents of S to the source, assuming that agents in $N\backslash S$ are out. This is a pessimistic approach because agents in $N\backslash S$ also want to be connected to the source.

Alternatively, we can take an optimistic approach. We can define the worth of coalition S, $v_C^+(S)$, as the minimal cost of connecting all agents of

S to the source, assuming that the agents in $N \setminus S$ are already connected to the source, and that the agents in S can connect to the source through them.

Given an *mcstp* (N_0, C), $S, T \subseteq N$, $S \cap T = \varnothing$, $\left(S_0, C^{+T}\right)$ is the *mcstp* obtained from (N_0, C) assuming that the agents in S have to be connected to the source, the agents in T are already connected to the source, and the agents in S can connect to the source through agents in T. Formally, $c_{ij}^{+T} = c_{ij}$ for all $i, j \in S$ and $c_{0i}^{+T} = \min_{j \in T_0} c_{ji}$ for all $i \in S$.

We now associate a *TU* game $\left(N, v_C^+\right)$ with each *mcstp* (N_0, C). For each $S \subseteq N$,

$$v_C^+ (S) = m \left(S_0, C^{+(N \setminus S)}\right).$$

Notice that given $S \subseteq N$, $v_C^+ (S)$ is the minimal cost of connecting all the agents of S to the source, assuming that agents of $N \setminus S$ are already connected.

Example 1. Let (N_0, C) be such that $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 10 & 100 \\ 10 & 0 & 2 \\ 100 & 2 & 0 \end{pmatrix}.$$

We now compute v_C and v_C^+.

S	$v_C(S)$	$v_C^+(S)$
{1}	10	2
{2}	100	2
{1,2}	12	12

This example shows that v_C and v_C^+ could be different.

We say that two *mcstp* (N_0, C) and (N_0, C') are *tree-equivalent* if there exists a tree t such that, firstly, t is an *mt* for both (N_0, C) and (N_0, C') and secondly, $c_{ij} = c'_{ij}$ for all $(i, j) \in t$.

In Bergantiños and Vidal-Puga (2005a) we proved that (N_0, C) and (N_0, C^*) (its irreducible form) are tree-equivalent.

In the next theorem we give some results about v_C^+.

Theorem 1. (a) If (N_0, C) is irreducible, for all $S \subset N$

$$v_C (S) + v_C^+ (N \setminus S) = m (N_0, C).$$

(b) If (N_0, C) and (N_0, C') are tree-equivalent, then $v_C^+ = v_{C'}^+$.
(c) $C^* = C'^*$ if and only if $v_C^+ = v_{C'}^+$.
Proof. See the Appendix.

Theorem 1(a) says that (N, v) and (N, v^+) are dual games in irreducible problems. This result it is not true when (N_0, C) is not an irreducible problem. In Example 1, $v_C(\{2\}) = 100$, $v_C^+(\{1\}) = 2$, and $m(N_0, C) = 12$.

It is clear from Theorem 1(b) that we can compute the optimistic game v_C^+ from an *mt* t of C. We now present an algorithm to do this:

Let (N_0, C) be an *mcstp* and let $t = \{(i^0, i)\}_{i \in N}$ be an *mt* such that i^0 represents the first player in the (unique) path in t from node i to the source. We start with $i_0 = 0$, $S^0 = \{i_0\}$ and $v^a(N) = m(N_0, C)$.

Stage 1: Take $i_1 \in N \backslash S^0$ such that $i_1^0 = 0$ and $c_{0i_1} = \min\{c_{0i} : (0, i) \in t\}$. If there are several arcs $(0, i)$ satisfying this condition, select any of them. Define $S^1 = \{i_0, i_1\}$, $c_{i_0 i_1}^a = c_{0i_1}$, and $v^a(N \backslash \{i_1\}) = m(N_0, C) - c_{i_0 i_1}^a$.

Stage $p+1$: Assume we have defined $S^p = \{i_0, i_1, ..., i_p\} \subset N_0$, $c_{i_q i_r}^a \in \mathbb{R}_+$ when $q, r \in \{0, 1, ..., p\}$, and $v^a(S)$ when $N \backslash S \subset S^p$. Take $i_{p+1} \in N \backslash S^p$ such that $i_{p+1}^0 \in S^p$ and

$$c_{i_{p+1}^0 i_{p+1}} = \min\left\{c_{i^0 i} : i^0 \in S^p \text{ and } i \in N \backslash S^p\right\}.$$

If there are several arcs (i^0, i) satisfying this condition, select any one of them.

Define $S^{p+1} = S^p \cup \{i_{p+1}\}$, and $c_{i_q i_{p+1}}^a = \max\left\{c_{i_q i_p}^a, c_{i_{p+1}^0 i_{p+1}}\right\}$ for each $q = 0, ..., p$.

Let $S \subset N$ be such that $N \backslash S \subset S^{p+1}$ and $i_{p+1} \in N \backslash S$. Assume that $N \backslash S = \left\{i_{q_1}, ..., i_{q_{n-s}}\right\}$ where $q_{r-1} \leq q_r$ for all $r = 2, ..., n-s$. Define $v^a(S) = m(N_0, C) - \sum_{r=1}^{n-s} c_{i_{q_{r-1}} i_{q_r}}^a$ where $q_0 = 0$.

The next Proposition says that with this algorithm we compute the irreducible form and the optimistic game of an *mcstp*.

Proposition 1. For each *mcstp* (N_0, C) and each *mt* $t = \{(i^0, i)\}_{i \in N}$, $C^a = C^*$ and $v^a = v_C^+$.

Proof. Following Bergantiños and Vidal-Puga (2005a), we say that "the agents in C connect to the source via t' in the order $\pi = (\pi_1, ..., \pi_n)$ following

Prim's algorithm" if t' is obtained through Prim's algorithm and the selected arc in stage p is $\left(\pi_p^0, \pi_p\right)$, for each p. In Bergantiños and Vidal-Puga (2005a) we proved that C^* can be computed as $c^*_{\pi_q \pi_p} = \max\limits_{s:q<s\leq p} \left\{c_{\pi_s^0 \pi_s}\right\}$.

Because of the definition of the algorithm, it is trivial to see that the agents in C connect to the source via t in the order $(i_1, ..., i_n)$ following Prim's algorithm.

We now prove that given $p, q \in \{1, ..., n\}$ such that $q < p$, we have that $c^a_{i_q i_p} = c^*_{i_q i_p}$. We use an induction argument. It is trivial to see that $c^a_{i_0 i_1} = c^*_{i_0 i_1} = c_{i_1^0 i_1}$. Assume that $c^a_{i_q i_p} = c^*_{i_q i_p}$ when $p \leq \alpha$. We prove this when $p = \alpha + 1$. Take $q < \alpha + 1$. By the induction hypothesis $c^a_{i_q i_\alpha} = c^*_{i_q i_\alpha}$. Thus,

$$\begin{aligned} c^a_{i_q i_p} &= \max\left\{c^a_{i_q i_\alpha}, c_{i_p^0 i_p}\right\} = \max\left\{c^*_{i_q i_\alpha}, c_{i_p^0 i_p}\right\} \\ &= \max\left\{\max_{s:q<s\leq\alpha}\left\{c_{i_s^0 i_s}\right\}, c_{i_p^0 i_p}\right\} \\ &= \max_{s:q<s\leq\alpha+1}\left\{c_{i_s^0 i_s}\right\} = c^*_{i_q i_p}. \end{aligned}$$

We now prove that $v^a = v^+$. Recall that (N_0, C) and (N_0, C^*) are tree-equivalent. Under Theorem $1(b)$, $v^+_C = v^+_{C^*}$. Under Theorem $1(a)$, for each $S \subset N$, $v^+_C(S) = m(N_0, C) - v_{C^*}(N \backslash S)$.

Assume that $N \backslash S = \left\{i_{q_1}, ..., i_{q_{n-s}}\right\}$ where $q_{r-1} \leq q_r$ for all $r = 2, ..., n-s$. In Bergantiños and Vidal-Puga (2005a) we proved that $v_{C^*}(N \backslash S) = \sum\limits_{r=1}^{n-s} c^*_{i_{q_{r-1}} i_{q_r}}$ where $q_0 = 0$. Since $C^* = C^a$ we conclude that $v^a = v^+_C$. ■

4 The Shapley value

In Section 2, we defined two rules for *mcstp* based on the Shapley value of the pessimistic game: $Sh^1(N_0, C) = Sh(N, v_C)$ and $Sh^2(N_0, C) = Sh(N, v_{C^*})$.

We now introduce two rules based on the Shapley value of the optimistic game. For all *mcstp* (N_0, C), we define

$$\begin{aligned} Sh^3(N_0, C) &= Sh\left(N, v^+_C\right) \text{ and} \\ Sh^4(N_0, C) &= Sh\left(N, v^+_{C^*}\right). \end{aligned}$$

For Example 1, the four rules are

Rule	Agent 1	Agent 2
$Sh^1(N_0, C)$	-39	51
$Sh^2(N_0, C)$	6	6
$Sh^3(N_0, C)$	6	6
$Sh^4(N_0, C)$	6	6

In this example $Sh^2(N_0, C) = Sh^3(N_0, C) = Sh^4(N_0, C)$. We now prove that this is true in general.

Theorem 2. For all *mcstp* (N_0, C),

$$Sh^2(N_0, C) = Sh^3(N_0, C) = Sh^4(N_0, C).$$

Proof. Let (N_0, C) be an *mcstp*. Recall that (N_0, C) and (N_0, C^*) are tree-equivalent. According to Theorem 1(b), $v_C^+ = v_{C^*}^+$. Thus, $Sh^3(N_0, C) = Sh^4(N_0, C)$.

According to Theorem 1(a), $v_{C^*}(S) + v_{C^*}^+(N \setminus S) = m(N_0, C)$ for all $S \subseteq N$. Hence, $Sh^2(N_0, C) = Sh^4(N_0, C)$ follows directly from self-duality of the Shapley value (see, e.g. Kalai and Samet (1987)). ■

Because of Theorem 2 we can define the rule φ as

$$\varphi(N_0, C) = Sh(N, v_{C^*}) = Sh\left(N, v_C^+\right) = Sh\left(N, v_{C^*}^+\right).$$

We now present an axiomatic characterization of this rule. Myerson (1980) introduced the property of balanced contributions in TU games. The next property is inspired by Myerson's property.

We say that a rule ψ satisfies *Equal Contributions* (EC) if for all $i, j \in N$, $i \neq j$,

$$\psi_i(N_0, C) - \psi_i\left((N \setminus \{j\})_0, C^{+j}\right) = \psi_j(N_0, C) - \psi_j\left((N \setminus \{i\})_0, C^{+i}\right).$$

EC says that the impact of the connection of agent j on agent's i cost is equal to the impact of the connection of agent i on agent's j cost.

The next theorem characterizes φ as the only rule satisfying EC.

Theorem 3. The rule φ is the only rule satisfying EC.

Proof. We first prove that φ satisfies EC.

For all $i \in N$, we denote $N^{-i} = N \setminus \{i\}$ and $N_0^{-i} = N_0 \setminus \{i\}$.

Given a TU game (N, v), Myerson (1980) proved that the Shapley value satisfies

$$Sh_i(N, v) - Sh_i\left(N^{-j}, v\right) = Sh_j(N, v) - Sh_j\left(N^{-i}, v\right)$$

for all $i, j \in N$, $i \neq j$.

Take $i, j \in N$, $i \neq j$. By Claim 1 of the proof of Theorem 1, $v_C^+(S) = v_{C^{+j}}^+(S)$ for all $S \subseteq N^{-j}$. Since $\varphi_i\left(N^{-j}, C^{+j}\right) = Sh_i\left(N^{-j}, v_{C^{+j}}^+\right)$, we have $\varphi_i\left(N^{-j}, C^{+j}\right) = Sh_i\left(N^{-j}, v_C^+\right)$.

Applying Myerson's result to the TU game $\left(N, v_C^+\right)$, we obtain that φ satisfies EC.

We now prove uniqueness. Let ψ be a rule satisfying EC. We prove that $\psi = \varphi$ by induction on $|N|$. If $|N| = 1$ the equality is trivial. Assume that $\psi = \varphi$ when $|N| \leq \alpha - 1$. We prove that $\psi = \varphi$ when $|N| = \alpha$.

Given $i, j \in N$, for simplicity, we write $\varphi_i = \varphi_i(N_0, C)$, $\psi_i = \psi_i(N_0, C)$, $\varphi_i^{+j} = \varphi_i\left(N_0^{-j}, C^{+j}\right)$, and $\psi_i^{+j} = \psi_i\left(N_0^{-j}, C^{+j}\right)$.

Since ψ satisfies EC,

$$\sum_{j \in N \setminus \{i\}} \psi_i - \sum_{j \in N \setminus \{i\}} \psi_i^{+j} = \sum_{j \in N \setminus \{i\}} \psi_j - \sum_{j \in N \setminus \{i\}} \psi_j^{+i}.$$

Thus,

$$n\psi_i = m(N_0, C) + \sum_{j \in N \setminus \{i\}} \psi_i^{+j} - \sum_{j \in N \setminus \{i\}} \psi_j^{+i}.$$

Since φ also satisfies EC,

$$n\varphi_i = m(N_0, C) + \sum_{j \in N \setminus \{i\}} \varphi_i^{+j} - \sum_{j \in N \setminus \{i\}} \varphi_j^{+i}.$$

Under the induction hypothesis, for all $i, j \in N$, $\psi_i^{+j} = \varphi_i^{+j}$ and $\psi_j^{+i} = \varphi_j^{+i}$. Thus, $\varphi_i = \psi_i$ for all $i \in N$. ■

A rule ψ satisfies *Equal Treatment* (ET) if given (N_0, C) and (N_0, C') such that $c_{lk} = c'_{lk}$ for all $(l, k) \neq (i, j)$,

$$\psi_i(N_0, C) - \psi_i(N_0, C') = \psi_j(N_0, C) - \psi_j(N_0, C').$$

ET says that if only the cost between agents i and j changes, both agents must win (or lose) the same.

Kar (2002) characterized Sh^1 as the only rule satisfying *Efficiency*, *Absence of Cross Subsidization*, *Group independence*, and *Equal Treatment*.

It follows from Theorem 3 that Sh^1 does not satisfy EC. The next example shows that φ does not satisfy ET.

Example 2. Let (N_0, C) be such that $N = \{1, 2\}$,

$$C = \begin{pmatrix} 0 & 5 & 14 \\ 5 & 0 & 10 \\ 14 & 10 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 5 & 14 \\ 5 & 0 & 12 \\ 14 & 12 & 0 \end{pmatrix}.$$

Computations reveal that $\varphi(N_0, C) = (5, 10)$ and $\varphi(N_0, C') = (5, 12)$. Nevertheless, $Sh^1(N_0, C) = (3, 12)$ and $Sh^1(N_0, C') = (4, 13)$.

We have two rules for *mcstp* based on the Shapley value of an associated TU game: Sh^1 and φ. Both rules are very different, as we can see in the examples. The rule Sh^1 is defined through the pessimistic TU game. The rule φ can be defined through the pessimistic TU game and both optimistic TU games.

One may wonder which is the fairest rule (Sh^1 or φ)? We strongly believe that φ is a more suitable rule in *mcstp*. See Bergantiños and Vidal-Puga (2005a) for a detailed discussion of this issue.

There exist many problems for which authors have proposed rules through both: optimistic TU games and pessimistic TU games. We conclude the section comparing *mcstp* with bankruptcy problems and queuing problems.

For bankruptcy problems the Shapley value of the pessimistic TU game and the Shapley value of the optimistic TU game coincide. See, for instance, Thomson (2003). The reason is that both games are dual, like in irreducible *mcstp*.

For queuing problems the Shapley value of both games differ, like in *mcstp*. Maniquet (2003) studied the Shapley value of the optimistic game. He provided several axiomatic characterizations. Chun (2006) studied the Shapley value of the pessimistic game, which he called the *reverse rule*. He provided axiomatic characterizations of the reverse rule. These characterizations are obtained by replacing some properties in Maniquet's characterization by their "reverse".

5 Appendix

We prove Theorem 1.

(a) Assume, without loss of generality, that $t = \{(i-1,i)\}_{i=1}^{n}$ is the tree associated with C satisfying $(A1)$ and $(A2)$. Let $S = \left\{i_1, ..., i_{|S|}\right\}$, where $i_{p-1} \leq i_p$ for all $p = 2, ..., |S|$.

For each $p = 1, ..., |S|$, we define:

$$S^p = \{(i-1,i) \in t : i_{p-1} < i \leq i_p\}.$$

If $p = 1$, let $i_0 = 0$. Define, for each $p = 1, ..., |S|$, $c_{k^p l^p} = \max_{(k,l) \in S^p} c_{kl}$. If there are several arcs satisfying this condition, select any of them.

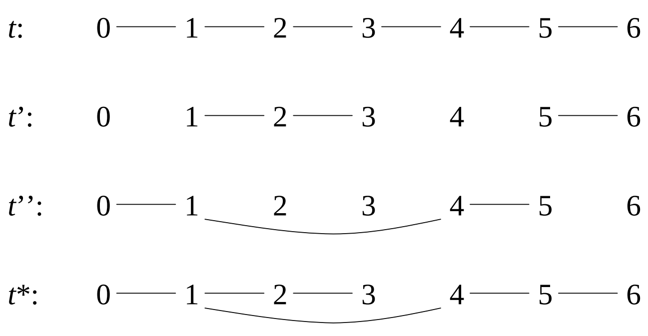
Construct

$$\begin{aligned} t' &= [t \backslash \{(k^p, l^p) : 1 \leq p \leq |S|\}], \\ t'' &= \{(i_{p-1}, i_p) : 1 \leq p \leq |S|\}, \text{ and} \\ t^* &= t' \cup t''. \end{aligned}$$

We clarify these definitions with an example.

Example 3. Let $N = \{1,2,3,4,5,6\}$ and $S = \{1,4,5\}$. Thus, $S^1 = \{(0,1)\}$, $S^2 = \{(1,2),(2,3),(3,4)\}$, and $S^4 = \{(4,5)\}$. Hence, $(k^1, l^1) = (0,1)$ and $(k^3, l^3) = (4,5)$. Assume that $(k^2, l^2) = (3,4)$.Thus, t, t', t'', and t^* are given by

Figure 1.



It is immediate that t'' is a tree in (S_0, C) and t^* a tree in (N_0, C). Since C is an irreducible matrix, if we take agents of S as the source, t' can be considered as a tree in $\left((N\backslash S)_0, C^{+S}\right)$.

Therefore,

$$\begin{aligned} v_C^+(N\backslash S) + v_C(S) &= m\left((N\backslash S)_0, C^{+S}\right) + m(S_0, C) \\ &\leq c\left((N\backslash S)_0, C^{+S}, t'\right) + c(S_0, C, t'') \\ &= c(N_0, C, t^*). \end{aligned}$$

As C is an irreducible matrix, for each $p = 1, ..., |S|$, $c_{k^p l^p} = c_{i_{p-1} i_p}$. Thus,

$$c(N_0, C, t^*) = c(N_0, C, t).$$

Hence,

$$v_C^+(N\backslash S) + v_C(S) \leq m(N_0, C).$$

We now prove that $m(N_0, C) \leq v_C^+(N\backslash S) + v_C(S)$. Let t' be an mt in $\left((N\backslash S)_0, C^{+S}\right)$ and let t'' be an mt in (S_0, C). It is possible to find a tree t''' in (N_0, C) such that

$$c\left((N\backslash S)_0, C^{+S}, t'\right) + c(S_0, C, t'') = c(N_0, C, t''').$$

Thus,

$$m(N_0, C) \leq c(N_0, C, t''') = v_C^+(N\backslash S) + v_C(S).$$

(b) Assume that $t = \{(i^0, i)\}_{i=1}^n$ is an mt in (N_0, C) and (N_0, C') such that $c_{i^0 i} = c'_{i^0 i}$ for all $i = 1, ..., n$. For all $i \in N$, $i^0 \in N_0$ is the first node in the unique path from i to the source.

We proceed by induction on $|N|$. If $|N| = 1$ the result is trivial. Assume that the result holds when $|N| \leq \alpha - 1$. We now prove it when $|N| = \alpha$.

In order to simplify the notation, for all $i \in N$ we denote $N^{-i} = N \setminus \{i\}$.

We prove several claims.

Claim 1. For all *mcstp* (N_0, C), $S \subset N$, and all $j \in N \setminus S$,

$$\left(S_0, C^{+(N\backslash S)}\right) = \left(S_0, \left(C^{+j}\right)^{+\left(N^{-j}\backslash S\right)}\right).$$

Proof. Let $i, k \in S$ be such that $i \neq 0$ and $k \neq 0$. Thus,

$$c_{ik}^{+(N\backslash S)} = c_{ik} = \left(c_{ik}^{+j}\right)^{+\left(N^{-j}\backslash S\right)}.$$

Given $i \in S$,

$$\begin{aligned} c_{0i}^{+(N\setminus S)} &= \min_{k\in(N\setminus S)_0}\{c_{ki}\} \\ &= \min\left\{\min_{k\in(N^{-j}\setminus S)}\{c_{ki}\}, \min\{c_{0i}, c_{ji}\}\right\} \\ &= \min\left\{\min_{k\in(N^{-j}\setminus S)}\left\{c_{ki}^{+j}\right\}, c_{0i}^{+j}\right\} \\ &= \min_{k\in(N^{-j}\setminus S)_0}\left\{c_{ki}^{+j}\right\} \\ &= \left(c_{0i}^{+j}\right)^{+\left(N^{-j}\setminus S\right)}. \blacksquare \end{aligned}$$

Claim 2. Let t^* be an mt in (N_0, C) and $j \in N$. Let $g = \{(i_{p-1}, i_p)\}_{p=1}^{r}$ be the unique path in t^* from 0 $(= i_0)$ to j $(= i_r)$. Let q be such that $c_{i_{q-1}i_q} = \max_{p=1,\dots,r}\left\{c_{i_{p-1}i_p}\right\}$. Given $A_j^* = \{(j,k) : (j,k) \in t^* \setminus \{(i_{q-1}, i_q)\}\}$,

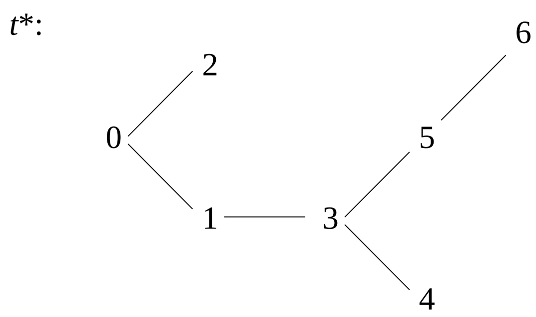
$$t_j^* = \left(t^* \setminus A_j^*\right) \cup \left\{(0,k) : (j,k) \in A_j^*\right\} \setminus \{i_{q-1}, i_q\} \quad (1)$$

is an mt in $\left(N_0^{-j}, C^{+j}\right)$.

Proof. First, we clarify the definitions given above with an example.

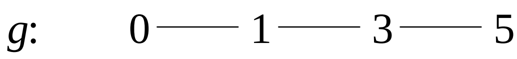
Example 4. Let $N = \{1,2,3,4,5,6\}$, $j = 5$, and t^* the tree given by Figure 2 below.

Figure 2.



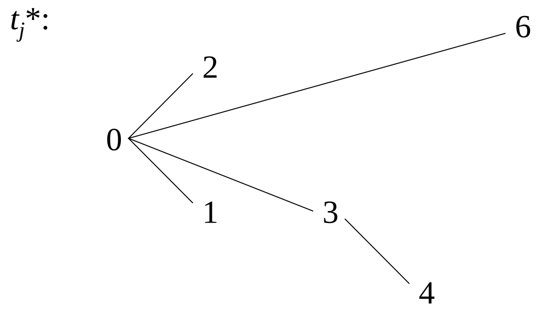
Thus, g is given by

Figure 3.



Assume that $(i_{q-1}, i_q) = (1,3)$. Then, $A_j^* = \{(3,5), (5,6)\}$ and t_j^* is given by

Figure 4.



First, we note that each arc $(0,k)$ in (1) for $\left(N_0^{-j}, C^{+j}\right)$ corresponds to the arc (j,k) for (N_0, C) (notice that j becomes a source itself when connected). Hence,

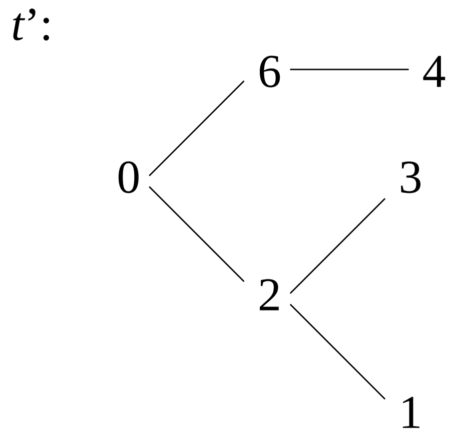
$$c\left(N_0, C, t^*\right) = c\left(N_0^{-j}, C^{+j}, t_j^*\right) + c_{i_{q-1}i_q}.$$

Suppose that t_j^* is not an mt in $\left(N_0^{-j}, C^{+j}\right)$. There exists a tree t' in $\left(N_0^{-j}, C^{+j}\right)$ such that

$$c\left(N_0^{-j}, C^{+j}, t'\right) < c\left(N_0^{-j}, C^{+j}, t_j^*\right).$$

Assume that, in Example 4, t' is given by

Figure 5.



Let S_j denote the set of agents in N^{-j} who are connected to the source in t' through agent j. We now define S_j formally. For each $i \in N^{-j}$, let $\left\{ \left(0, l_1^i\right), \left(l_1^i, l_2^i\right), ..., \left(l_{s-1}^i, i\right) \right\}$ be the unique path in t' from the source to i. We define

$$S_j = \left\{ i \in N^{-j} : c_{0l_1^i}^{+j} = c_{jl_1^i} \right\}.$$

Let $(i_{t-1}, i_t) \in g \subseteq t^*$ be such that $i_{t-1} \in N_0^{-j} \setminus S_j$ and $i_t \in S_j \cup \{j\}$.

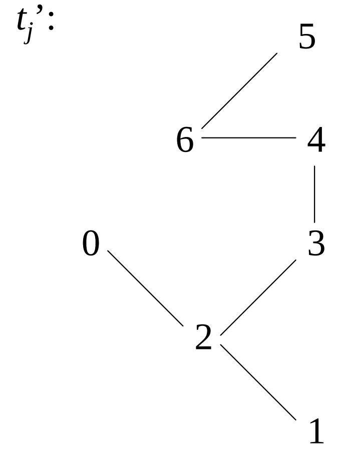
Let $A'_j = \left\{ (0, l) \in t' : c_{0l}^{+j} = c_{jl} \right\}$. Thus,

$$t'_j = \left(t' \setminus A'_j\right) \cup \left\{ (j, l) : (0, l) \in A'_j \right\} \cup \{(i_{t-1}, i_t)\}$$

is a tree in (N_0, C).

We clarify these definitions in Example 4. Assume that $S_5 = \{4, 6\}$. Let $(i_{t-1}, i_t) = (3, 4)$. Thus, $A'_5 = \{(0, 6)\}$ and t'_j is given by

Figure 6.



Since

$$\begin{aligned} c\left(N_0, C, t'_j\right) &= c\left(N_0^{-j}, C^{+j}, t'\right) + c_{i_{t-1}i_t}, \\ c\left(N_0, C, t^*\right) &= c\left(N_0^{-j}, C^{+j}, t^*_j\right) + c_{i_{q-1}i_q}, \text{ and} \\ c_{i_{t-1}i_t} &\le c_{i_{q-1}i_q} \end{aligned}$$

we deduce that

$$c\left(N_0, C, t'_j\right) < c\left(N_0, C, t^*\right)$$

which is a contradiction because t^* is an mt of (N_0, C). ■

Claim 3. For all $j \in N$, $\left(N_0^{-j}, C^{+j}\right)$ and $\left(N_0^{-j}, C'^{+j}\right)$ are tree-equivalent.

Proof. Given $j \in N$, let t_j be the mt in $\left(N_0^{-j}, C^{+j}\right)$ obtained from the mt t in (N_0, C) as in the statement of Claim 2. Similarly, let t'_j be the mt in $\left(N_0^{-j}, C'^{+j}\right)$ obtained from the mt t in (N_0, C') as in the statement of Claim 2.

It is not difficult to see that $t_j = t'_j$. Moreover, for all $(i,k) \in t_j$, $c_{ik}^{+j} = c_{ik}'^{+j}$. Thus, $\left(N_0^{-j}, C^{+j}\right)$ and $\left(N_0^{-j}, C'^{+j}\right)$ are tree-equivalent. ■

Claim 4. v_C^+ coincides with $v_{C'}^+$.

Proof. We prove that $v_C^+(S) = v_{C'}^+(S)$ for all $S \subseteq N$. If $S = N$,

$$v_C^+(N) = m(N_0, C) = m(N_0, C') = v_{C'}^+(N).$$

Assume that $S \neq N$. Take $j \in N \setminus S$. Under Claim 1, $v_C^+(S) = v_{C^{+j}}^+(S)$ and $v_{C'}^+(S) = v_{C'^{+j}}^+(S)$.

Under Claim 3, $\left(N_0^{-j}, C^{+j}\right)$ and $\left(N_0^{-j}, C'^{+j}\right)$ are tree-equivalent. Under the induction hypothesis, $v_{C^{+j}}^+(S) = v_{C'^{+j}}^+(S)$. Thus, $v_C^+(S) = v_{C'}^+(S)$. ■

(c) $(\Longrightarrow)$ Assume that $C^* = C'^*$. By Theorem 1(b), $v_C^+ = v_{C^*}^+$ and $v_{C'}^+ = v_{C'^*}^+$. Thus, $v_C^+ = v_{C'}^+$.

$(\Longleftarrow)$ We first prove the following claim.

Claim 5. If C is an irreducible matrix, for all $i, j \in N$,

$$v_C(\{i,j\}) = \min\{c_{0i}, c_{0j}\} + c_{ij}.$$

Proof. Let $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ be the mt given by conditions $(A1)$ and $(A2)$. Assume, *wlog*, that $i = i_p$, $j = i_q$ and $p < q$. Bergantiños and Vidal-Puga (2005a) proved that $v_C(\{i,j\}) = c_{0i} + c_{ij}$. By $(A2)$, $c_{0i} \leq c_{0j}$. Thus, $v_C(\{i,j\}) = \min\{c_{0i}, c_{0j}\} + c_{ij}$. ■

Assume that $v_C^+ = v_{C'}^+$. Then,

$$m(N_0, C) = v_C^+(N) = v_{C'}^+(N) = m(N_0, C').$$

By Theorem 1(b), $v_{C^*}^+ = v_C^+ = v_{C'}^+ = v_{C'^*}^+$. By Theorem 1$(a)$, $v_{C^*} = v_{C'^*}$. In particular, for any $i \in N$,

$$c_{0i}^* = v_{C^*}(\{i\}) = v_{C'^*}(\{i\}) = c_{0i}'^*.$$

Given $i, j \in N$, by Claim 5,

$$\begin{aligned} c_{ij}^* &= v_{C^*}(\{i,j\}) - \min\left\{c_{0i}^*, c_{0j}^*\right\} \\ &= v_{C'^*}(\{i,j\}) - \min\left\{c_{0i}'^*, c_{0j}'^*\right\} = c_{ij}'^*. \end{aligned}$$

This finishes the proof of Theorem 1. ■

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