Additivity in minimum cost spanning tree problems*

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Abstract

We characterize a rule in minimum cost spanning tree problems using an additivity property and some basic properties. If the set of possible agents has at least three agents, these basic properties are symmetry and separability. If the set of possible agents has two agents, we must add positivity.

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1 Introduction

Imagine that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the *source*. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix C, which specifies the connection costs between each pair agent-agent and agent-source. There are many situations that can be modeled in this way. For instance, several towns may draw power from a common power plant,

and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004, 2005) study a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Other examples include communication networks, such as telephone, Internet, or cable television.

We assume that the agents construct a minimum cost spanning tree (mt). The question is how to divide the cost associated with the mt between the agents. A *rule* determines an allocation.

In this paper, we characterize a rule using a property of additivity. In its most natural formulation in $mcstp$, additivity is very demanding and no rule satisfies it. Hence, we introduce a restricted additivity. Other properties used in the characterization are symmetry, positivity, and separability. Symmetry says that symmetric agents must pay the same. Positivity says that every agent must pay at least zero.

Separability appears in Megiddo (1978), Granot and Huberman (1981), and Granot and Maschler (1998) with the name of decomposition, and in Bergantiños and Vidal-Puga (2007) with the name of separability. Two subsets of agents can connect to the source separately or jointly. If there are no savings when they connect jointly, separability says that the agents must pay the same in both circumstances.

Our results are the following. If the set of possible agents has at least three members, then there is a unique rule satisfying restricted additivity, symmetry, and separability. If the set of possible agents has exactly two members, then there is a unique rule satisfying positivity, restricted additivity, symmetry, and separability.

The rule we obtain in these characterizations is well known in the literature of $mcstp$. This rule was first introduced by Feltkamp, Tijs, and Muto (1994) and studied later by Brânzei, Moretti, Norde, and Tijs (2004) and Bergantiños and Vidal-Puga (2004, 2006, 2007).

Our paper is very related to the paper of Brânzei *et al* (2004) because they also characterize this rule using an additivity property and other two properties: *equal treatment* and *upper bound contributions*. These properties are very related to the concept of C-component. A C-component is a maximal coalition of agents that can be connected among themselves at zero cost. A rule satisfies *equal treatment* if all the members of a C-component receive the same. A rule satisfies *upper bound contributions* if the aggregate allocation assigned to the members of a C-component is not more than their connection cost to the source.

In Section 2 we introduce the model. In Section 3 we present our results.

2 Minimum cost spanning tree problems

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let Π_N be the set of all permutations over N. Given $\pi \in \Pi_N$, let π_p denote the agent at position $p \in \{1, ..., |N|\}$ in the order π.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$.

A *minimum cost spanning tree problem*, briefly a *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix. Given a *mcstp* (N_0, C), we denote the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of g are called *arcs*. Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^l$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$. A *tree* is a network such that for all $i \in N$ there is a unique path from i to the source. If t is a tree, we usually write $t = \{(i^0, i)\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0.

Let $\mathcal{G}^N$ denote the set of all networks over N_0. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from i to 0 in the network.

Given a *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as $c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}$. When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree t over N_0 such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it is not necessarily unique. Given a *mcstp* (N_0, C), we denote the cost associated with any *mt* as $m(N_0, C)$.

A (*cost allocation*) *rule* is a function ψ that assigns to each *mcstp* (N_0, C) a vector $\psi(N_0, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$.

A *coalitional game with transferable utility*, briefly a *TU game*, is a pair (N, v) where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. $Sh(N, v)$ denotes the Shapley value (Shapley, 1953) of (N, v). It is possible to associate a *TU* game (N, v_C) with each *mcstp* (N_0, C)

as follows. For each $S \subset N$, $v_C(S) = m(S_0, C)$.

In this paper we focus on a rule first introduced by Feltkamp *et al* (1994), which we denote as φ. This rule can be defined as

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

where C^* is the irreducible matrix. For all $i, j \in N_0$, $c_{ij}^* = \max_{(k,j) \in g_{ij}} \{c_{kl}\}$ where g_{ij} is the unique path connecting i and j in an *mt* t. See Bergantiños and Vidal-Puga (2007) for more details.

Finally, we define *C-components* following Norde, Moretti, and Tijs (2004). This concept will be used in some of the proofs. Given a *mcstp* (N_0, C) and $S \subset N_0$, we say that $i, j \in N_0$, $i \neq j$ are *(C, S)-connected* if there exists a path g from i to j satisfying that $g \in \mathcal{G}^S$ and $c_{kl} = 0$ for all $(k, l) \in g$. We say that $S \subset N_0$ is a *C-component* if two conditions hold. First, for all $i, j \in S$, i and j are *(C, S)-connected*. Second, S is maximal, *i.e.* if $S \subsetneq T$ there exist $i, j \in T$, $i \neq j$ such that i and j are not *(C, T)-connected*.

Clearly, the set of *C-components* is a partition of N_0.

3   The axiomatic characterization

In its most natural definition, we say that a rule ψ satisfies *additivity* if for all *mcstp* (N_0, C) and (N_0, C'), $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$. However, no rule satisfies this property. Take

$$C = \begin{pmatrix} 0 & 10 & 50 \\ 10 & 0 & 10 \\ 50 & 10 & 0 \end{pmatrix}, C' = \begin{pmatrix} 0 & 50 & 10 \\ 50 & 0 & 10 \\ 10 & 10 & 0 \end{pmatrix}, \text{ and } C + C' = \begin{pmatrix} 0 & 60 & 60 \\ 60 & 0 & 20 \\ 60 & 20 & 0 \end{pmatrix}.$$

Notice that $m(N_0, C) = m(N_0, C') = 20$ and $m(N_0, C + C') = 80$. By definition, $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$ for all *mcstp* (N_0, C). Then, $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$ is not possible.

Hence, we should only claim additivity when the pair of problems are "similar", and not in any case. Moreover, we also want to claim additivity in a large subclass of problems.

In *mcstp* there exists an additivity property called *cone-wise positive linearity* (CPL), which has been introduced by Brânzei *et al* (2004). We say that ψ satisfies CPL if $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$ for all *mcstp* (N_0, C) and (N_0, C')

satisfying that there exists an order for the arcs $\sigma : \{(i,j)\}_{i,j \in N_0, i<j} \rightarrow \left\{1, 2, ..., \frac{n(n+1)}{2}\right\}$ such that if $i, j, k, l \in N_0$ with $i < j$, $k < l$, and $\sigma(i,j) \leq \sigma(k,l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$.

Notice that according to this definition[1] two problems are "similar" when, ordering the arcs by their cost, we can obtain the same order in C and C'.

We now introduce our additivity property. If we want to claim $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$ we need, at least, that $m(N_0, C + C') = m(N_0, C) + m(N_0, C')$.

Let t be an mt in $(N_0, C + C')$. It is easy to see that t is an mt in both (N_0, C) and (N_0, C'). Assume that we order the arcs in t in non-decreasing cost. If we obtain the same order in (N_0, C) and (N_0, C'), then we claim additivity on these problems. This is our idea of "similar" problems.

Formally, we say that ψ satisfies *restricted additivity* (RA) if

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$$

for all $mcstp$ (N_0, C) and (N_0, C') satisfying that there exists an mt $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C), (N_0, C'), and $(N_0, C + C')$ and an order $\pi \in \Pi_N$ such that $c_{\pi_1^0 \pi_1} \leq c_{\pi_2^0 \pi_2} \leq ... \leq c_{\pi_n^0 \pi_n}$ and $c'_{\pi_1^0 \pi_1} \leq c'_{\pi_2^0 \pi_2} \leq ... \leq c'_{\pi_n^0 \pi_n}$.

It is straightforward to check that RA implies CPL.

We now introduce the "basic" properties we use in our characterization results.

Given an $mcstp$ (N_0, C), we say that $i, j \in N$, $i \neq j$ are *symmetric* if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

We say that ψ satisfies *symmetry* (SYM) if for all $mcstp$ (N_0, C) and all pair of symmetric agents $i, j \in N$, $\psi_i(N_0, C) = \psi_j(N_0, C)$.

We say that ψ satisfies *positivity* (POS) if for all $mcstp$ (N_0, C) and all $i \in N$, $\psi_i(N_0, C) \geq 0$.

We say that ψ satisfies *separability* (SEP) if for all $mcstp$ (N_0, C) and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$,

$$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S \end{cases}$$

for all $i \in N$.

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[1]Brânzei *et al* (2004) define CPL in a little bit different way. Take $x, x' \geq 0$. They say that a rule ψ satisfies CPL if $\psi(N_0, xC + x'C') = x\psi(N_0, C) + x'\psi(N_0, C')$ when (N_0, C) and (N_0, C') are "similar". It is not difficult to check that the characterization of Brânzei *et al* (2004) also holds with our definition. We present it in a different way only because it is simpler.

Two subsets of the agents, S and $N \setminus S$, can connect to the source separately or can connect jointly. If there are no savings when they connect jointly, SEP says that agents must pay the same in both circumstances.

Proposition 1. φ satisfies SEP, SYM, and RA.

Proof. In Bergantiños and Vidal-Puga (2007), we proved that φ satisfies SEP. On the other hand, φ is the Shapley value of an associated game. From this, it is straightforward to prove that φ satisfies SYM.

We now prove that φ satisfies RA. Let (N_0, C) and (N_0, C') be two $mcstp$ and let $t = \{(i^0, i)\}_{i \in N}$ be an mt in (N_0, C), (N_0, C'), and $(N_0, C + C')$ satisfying that there exists an order $\pi \in \Pi_N$ such that $c_{\pi_1^0 \pi_1} \leq c_{\pi_2^0 \pi_2} \leq ... \leq c_{\pi_n^0 \pi_n}$ and $c'_{\pi_1^0 \pi_1} \leq c'_{\pi_2^0 \pi_2} \leq ... \leq c'_{\pi_n^0 \pi_n}$.

For all $i, j \in N_0$, $c_{ij}^* = \max\limits_{(k,l) \in g_{ij}} \{c_{kl}\}$, $c_{ij}'^* = \max\limits_{(k,l) \in g_{ij}} \{c'_{kl}\}$, and $(c + c')_{ij}^* = \max\limits_{(k,l) \in g_{ij}} \{c_{kl} + c'_{kl}\}$ where g_{ij} is the (unique) path in t connecting i and j. Hence, $(C + C')^* = C^* + C'^*$.

Applying Proposition 2.3 in Bergantiños and Vidal-Puga (2007), it is not difficult to deduce that for all $S \subset N$, $v_{(C+C')^*}(S) = v_{C^*}(S) + v_{C'^*}(S)$. Thus, the result follows from the additivity of the Shapley value. ∎

Assume that $\mathcal{N}$ has at least three agents. Then we have the following result:

Proposition 2. There is a unique rule satisfying SEP, SYM, and RA.

Proof. From Theorem 1 in Norde *et al* (2004), if (N_0, C) is a $mcstp$, then there exists a family $\{C^p\}_{p=1}^a$ of cost matrices satisfying three conditions:

1. $C = \sum\limits_{p=1}^a C^p$.

2. For each $p \in \{1, ..., a\}$ there exist $x^p \in \mathbb{R}$ and a network g^p such that $c_{ij}^p = x_p$ if $(i, j) \in g^p$ and $c_{ij}^p = 0$ otherwise.

3. There exists $\sigma : \{(i,j)\}_{i,j \in N_0, i<j} \to \left\{1, 2, ..., \frac{n(n+1)}{2}\right\}$ such that if $i, j, k, l \in N$ with $i < j$, $k < l$, and $\sigma(i,j) \leq \sigma(k,l)$, then $c_{ij} \leq c_{kl}$ and $c_{ij}^p \leq c_{kl}^p$ for all $p \in \{1, ..., a\}$.

Let ψ be a rule satisfying SYM, SEP, and RA. Since RA implies CPL, from conditions 1 and 3, $\psi(N_0, C) = \sum\limits_{p=1}^a \psi(N_0, C^p)$.

Hence, it is enough to prove that ψ is unique on the subclass of $mcstp$ (N_0, C) satisfying that there exist $x \in \mathbb{R}$ and a network g such that $c_{ij} = x$ if $(i, j) \in g$ and $c_{ij} = 0$ otherwise. Assume that (N_0, C) satisfies these conditions.

Let $\{(N_1)_0, N_2, ..., N_q\}$ be the partition of N_0 in C-components. It is straight-forward to check that $m(N_0, C) = \sum_{r=1}^{q} m((N_r)_0, C)$. Under SEP, $\psi_i(N_0, C) = \psi_i((N_r)_0, C)$ for all $r \in \{1, ..., q\}$ and $i \in N_r$.

Hence, we can assume that all the agents can be connected among themselves at zero cost. This implies that, given $t = \{(i^0, i)\}$ is an mt in (N_0, C), there exists $\alpha \in N$ such that $\alpha^0 = 0$ and $c_{i^0 i} = 0$ for all $i \in N \setminus \{\alpha\}$. We define C^1 and C^2 as follows:

$$c_{ij}^1 = \begin{cases} c_{ij} & \text{if } 0 \in \{i, j\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } c_{ij}^2 = \begin{cases} 0 & \text{if } 0 \in \{i, j\} \\ c_{ij} & \text{otherwise.} \end{cases}$$

It is not difficult to see that t is an mt in both (N_0, C^1) and (N_0, C^2). Moreover, $c_{i^0 i}^1 = c_{i^0 i}^2 = 0$ for all $i \in N \setminus \{\alpha\}$, $c_{0\alpha}^1 \geq 0 = c_{0\alpha}^2$, and $C = C^1 + C^2$. Under RA, $\psi(N_0, C) = \psi(N_0, C^1) + \psi(N_0, C^2)$.

Since $m(\{i\}_0, C^2) = 0$ for all $i \in N$ and $m(N_0, C) = 0$, under SEP, $\psi_i(N_0, C^2) = \psi_i(\{i\}_0, C^2) = 0$ for all $i \in N$. Hence, $\psi(N_0, C) = \psi(N_0, C^1)$.

Thus, it is enough to prove that ψ is unique on the subclass of $mcstp$ (N_0, C) where $c_{ij} = 0$ if $0 \notin \{i, j\}$ and $c_{0i} \in \{0, x\}$ for all $i \in N$. If $c_{0i} = 0$ for all $i \in N$, or $c_{0i} = x$ for all $i \in N$, all the agents are symmetric. Under SYM, $\psi_i(N_0, C)$ is 0 or $\frac{x}{|N|}$ for all $i \in N$. If there exist $j, k \in N$ such that $c_{0j} = x$ and $c_{0k} = 0$, we define $N^1 = \{i \in N \mid c_{0i} = x\} \cup \{k\}$ and $N^2 = \{i \in N \mid c_{0i} = 0\} \setminus \{k\}$. These sets are under the conditions of SEP and hence $\psi_i(N_0, C) = \psi_i(N_0^1, C)$ for all $i \in N^1$ and $\psi_i(N_0, C) = \psi_i(N_0^2, C)$ for all $i \in N^2$. Notice that (N_0^2, C) is in the same case as before.

Thus, it is enough to prove that ψ is unique on the subclass of $mcstp$ (N_0, C) where there exists $k \in N$ such that $c_{0i} = x \in \mathbb{R}$ if $i \neq k$ and $c_{ij} = 0$ otherwise.

For each $i \in N \setminus \{k\}$, let (N, C^i) such that $c_{0i}^i = x$ and $c_{jl}^i = 0$ otherwise. Notice that $C = \sum_{i \in N \setminus \{k\}} C^i$. Moreover, $t = \{(j-1, j)\}_{j=2}^n \cup \{(0, k)\}$ is an mt in (N_0, C^i) for all $i \in N \setminus \{k\}$. Under RA, $\psi(N_0, C) = \sum_{i \in N \setminus \{k\}} \psi(N_0, C^i)$.

Thus, it is enough to prove that for all $i \in N \setminus \{k\}$, ψ is unique in each problem (N_0, C^i). Take $i \in N \setminus \{k\}$. It is trivial to see that

$$m(N_0, C^i) = m(\{k, i\}_0, C^i) + \sum_{j \in N \setminus \{k, i\}} m(\{j\}_0, C^i).$$

Under SEP, $\psi_k(N_0, C^i) = \psi_k(\{k, i\}_0, C^i)$, $\psi_i(N_0, C^i) = \psi_i(\{k, i\}_0, C^i)$, and for all $j \in N \setminus \{k, i\}$, $\psi_j(N_0, C^i) = \psi_j(\{j\}_0, C^i) = 0$.

It only remains to prove that ψ is unique in the $mcstp$ $(\{i, k\}, C)$ where $c_{0k} = c_{ik} = 0$ and $c_{0i} = x$.

Since $m(\{i, k\}_0, C) = 0$, $\psi(\{i, k\}_0, C) = (y, -y)$. We prove that $y = 0$.

Let (N_0, C') be such that $N = \{i, j, k\}$, $c'_{0i} = x$, and $c'_{hl} = 0$ otherwise. We can find such a (N_0, C') because $\mathcal{N}$ has at least three members.

Hence, $m(N_0, C') = m(\{j\}_0, C') + m(\{i, k\}_0, C')$. Under SEP,

$$\psi_k(N_0, C') = \psi_k(\{i, k\}_0, C') = \psi_k(\{i, k\}_0, C) = -y.$$

Moreover, $m(N_0, C') = m(\{i, j\}_0, C') + m(\{k\}_0, C')$. Under SEP, $\psi_k(N_0, C') = \psi_k(\{k\}_0, C') = 0$. Hence, $y = 0$. $\blacksquare$

The next theorem is a trivial consequence of Propositions 1 and 2.

Theorem 1. If $\mathcal{N}$ has at least three agents, then φ is the unique rule satisfying SEP, SYM, and RA.

The properties used in Theorem 1 are independent.

Assume $N = \{1, ..., n\}$. For each $mcstp$ (N_0, C) and $i \in N$ we define $\psi_i^1(N_0, C) = v_{C^*}(\{1, ..., i\}) - v_{C^*}(\{1, ..., i - 1\})$. ψ^1 satisfies SEP and RA, but fails SYM.

Consider the egalitarian rule defined as $\psi_i^2(N_0, C) = m(N_0, C) / |N|$ for all $i \in N$. ψ^2 satisfies RA and SYM, but fails SEP.

Let ψ^3 be defined as

$$\psi_i^3(N_0, C) = \frac{1}{|\Pi'_N|} \sum_{\pi \in \Pi'_N} [v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))]$$

for all $i \in N$, where Π'_N is the subset of orders in which the agents with the cheapest cost to the source connect first, $i.e.$

$$\Pi'_N = \left\{\pi \in \Pi_N \mid c_{0\pi_q} \leq c_{0\pi_p} \text{ when } q < p\right\}.$$

ψ^3 satisfies SEP (Bergantiños and Vidal-Puga, 2007) and SYM, but fails RA.

The next theorem is the analogous to Theorem 1 when $\mathcal{N}$ has two agents.

Theorem 2. If $\mathcal{N}$ has two agents, then φ is the unique rule satisfying POS, SEP, SYM, and RA.

Proof. In Bergantiños and Vidal-Puga (2007), we proved that φ satisfies POS. By Proposition 1 we know that φ satisfies SEP, SYM, and RA.

We now prove the uniqueness. Let ψ be a rule satisfying POS, SYM, SEP, and RA. If $N = \{i\}$, by definition $\psi_i(N_0, C) = m(N_0, C) = c_{0i}$.

Assume that $N = \{i, j\}$. Using arguments similar to those used in the proof of Proposition 2, we can deduce that it is enough to prove that ψ is unique in the $mcstp$ $(\{i, j\}_0, C)$ where $c_{0i} = c_{ij} = 0$ and $c_{0j} = x$.

Since $m(N_0, C) = 0$, $\psi_i(N_0, C) = -\psi_j(N_0, C)$. Under POS, $\psi_i(N_0, C) = \psi_j(N_0, C) = 0$. $\blacksquare$

The properties used in Theorem 2 are also independent. When $\mathcal{N}$ has two agents, ψ^1 satisfies SEP, RA and POS but fails SYM; ψ^2 satisfies SYM, RA and POS but fails SEP; and ψ^3 satisfies SEP, SYM and POS but fails RA.

Assume, without loss of generality, that $N = \{i, j\}$ and $c_{0i} \leq c_{0j}$. We consider the rule ψ^4 defined as

$$
\begin{aligned}
\psi_i^4(N_0, C) &= \varphi_i(N_0, C) - \max\{0, c_{0j} - \max\{c_{0i}, c_{ij}\}\} \text{ and} \\
\psi_j^4(N_0, C) &= \varphi_j(N_0, C) + \max\{0, c_{0j} - \max\{c_{0i}, c_{ij}\}\}.
\end{aligned}
$$

ψ^4 satisfies SYM, SEP and RA, but fails POS.

In Bergantiños and Vidal-Puga (2007) we introduce, in $mcstp$, the property of Population Monotonicity (PM). We say that ψ satisfies PM if for all $mcstp$ (N_0, C), all $S \subset N$, and all $i \in S$, $\psi_i(N_0, C) \leq \psi_i(S_0, C)$. PM says that, if new agents join a society, no agent of the initial society can be worse off. Since PM implies SEP and φ satisfies PM, all the results of this paper hold with PM instead of SEP.

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