Values and coalition configurations

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Abstract. In this paper we consider coalition configurations (Albizuri et al., 2006), that is, families of coalitions not necessarily disjoint whose union is the grand coalition, and give a generalization of the Shapley value (1953) and the Owen value (1977) when coalition configurations form. This will be an alternative definition to the one given by Albizuri et al. (2006).

Key words: Shapley value, Owen value, coalition configurations

1. Introduction

There are negotiation situations in which some agents prefer to cooperate together than with others. There is a tool which has been employed to study these kind of negotiations: that of a coalition structure, that is, a partition of the set of agents into disjoint coalitions. Aumann and Dreze (1974) propose and study a value when agents form a coalition structure, and later on, Owen (1977) proposes and characterizes another modification of the Shapley (1953) value also when coalition structures are formed (see also Hart and Kurz, 1983). Even though other coalitional values have been studied (see Albizuri and Aurrekoetxea, 2006, Casajus, 2009, Gómez-Rúa and Vidal-Puga 2010, and references herein), the most widely used (see Gómez-Rúa and Vidal-Puga, 2013, and references herein) is the Owen (1977) value.

Aumann and Dreze (1974) considered a fixed coalition structure. Subsequent literature has considered instead rules that depend on the coalition structure. This is the case, for example, of Owen (1977) and Hart and Kurz (1983), and it has become the most usual way to define coalitional rules. A relevant exception, however, are the coalitional values defined by Levy and McLean (1989).

In Owen's (1977) approach, each member of the coalition structure bargains against the others to allocate the worth available to the grand coalition. Albizuri

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et al. (2006) consider the more general concept of coalition configuration to model negotiations in which players form coalitions not necessarily disjoint. A coalition configuration is defined as a family of coalitions not necessarily disjoint, whose union is the grand coalition. They generalize the Owen value (1977) (and therefore the Shapley value (1953)) with reference to coalition configurations. In fact they obtain two generalizations of the Owen value. The configuration value and the dual configuration value. Both values are dual each of the other. Let us present the second one by means of an example. The first one could be presented similarly.

Let $N = \{1, 2, 3\}$ be the set of players and consider the transferable utility game v on N which satisfies $v(1) = 0 = v(2) = v(3)$, $v(12) = 3$, $v(13) = 0$, $v(23) = 1$ and $v(123) = 5$. The dual configuration value ϕ^* associates a vector of outcomes with each coalition configuration. These outcomes can be calculated by means of orders and marginal contributions.

Suppose for example that players form coalition configuration $\mathcal{B}_1 = \{\{1, 2\}, \{2, 3\}\}$. Then we have to consider all the orders of the elements which form the two coalitions of $\mathcal{B}_1$, in such a way that the elements of the same coalition keep together. So we consider 1223, 1232, 2123, 2132, $2312, 2321, 3212$ and 3221. We interpret these orders as follows. Suppose that each player in each coalition of $\mathcal{B}_1$ has a representative associated with that coalition, and that these four representatives form a queue outside a room in such a way that all the representatives associated with a coalition are together. Then, these representatives enter in the room and form an order. We have in this way the orders above. When a representative of a player enters in the room a coalition forms if all the representatives of that player are in the room, being that coalition the one formed by the players whose representatives are all in the room. For example, given 1232, player 1 is given $v(1) = 0$ for when 1 enters coalition $\{1\}$ forms and 1 is given her contribution to the singleton coalition. When the first representative of player 2 enters in the room neither coalition forms for all the representatives of 2 are not yet in the room. Therefore, 2 is not given anything. Then 3 comes and coalition $\{1, 3\}$ forms, and 3 is given her marginal contribution to this coalition: $v(13) - v(1) = 0$. When the second representative of player 2 enters $\{1, 2, 3\}$ forms and 2 is given $v(123) - v(13) = 5$. If we suppose that all the orders are equally likely, the expected marginal contribution of a player is her dual configuration value associated with $\mathcal{B}_1$. The value is $\phi^*(\mathcal{B}_1, v) = \left(1, 3\frac{1}{2}, \frac{1}{2}\right)$.

Consider now the unanimity game u^N on N given by $u^N(N) = 1$ and $u^N(S) = 0$ otherwise. Then, an analogous reasoning as before leads to $\phi^*\left(\mathcal{B}_1, u^N\right) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

If $\mathcal{B}_2 = \{\{1, 2\}, \{2, 3\}, \{2\}\}$ is formed we can calculate the dual configuration value as before. Now we have to consider the orders 12232, 12322, 23122, ... In this case $\phi^*(\mathcal{B}_2, v) = \left(\frac{2}{3}, 4, \frac{1}{3}\right)$ and $\phi^*\left(\mathcal{B}_2, u^N\right) = \left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$.

Notice that in both cases player 2 has three representatives in $\mathcal{B}_2$ since she belongs to three coalitions. If player 2 were in four coalitions she would have four representatives and so on. So the more coalitions a player belongs to the more effort or weight she has.

In this work we do not allow any player (as player 2 here) to increase her weight by belonging to more and more coalitions. Each player will have a fix weight and this weight will be spread among the coalitions she belongs to. Think for example of agents that have a certain amount of time they share among the coalitions they belong to, the worth of these coalitions depending on the time spent by the agents inside the coalitions.

The weight of a player will be represented by a positive real number and we will consider all the possible weights for a player. So we have a family of alternative values. They will be called weighted bounded configuration values. The definition will be made by means of compositions of mappings, even though a definition by means on orders is also possible, as in Albizuri et al. (2006).

The paper is structured as follows. Section 2 is a preliminary one. In Section 3 we define the weighted bounded configuration values. In Section 4 we provide two axiomatic characterization which give all the weighted bounded configuration values. In Section 5 we focus on a specific weighted bounded configuration value. Finally, in Section 6 we summarize the results and compare them with those in Albizuri et al. (2006).

2. PRELIMINARIES

Given a finite set of players N, we denote by $\mathbf{G}^N$ the set formed by the cooperative transferable utility games with player set N, that is by the mappings $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. A *coalition* is a subset $S \subseteq N$, $S \neq \emptyset$. Given $i \in S \subseteq N$ and $T \subseteq N \backslash S$, let $S^{-i,T}$ denote the set $(S \backslash \{i\}) \cup T$. Given $v \in \mathbf{G}^N$ and $T \subseteq N$, $T \neq \emptyset$, we denote by $v_T \in \mathbf{G}^T$ the game on T such that $v_T(S) = v(S)$ for all $S \subseteq T$. A game $v \in \mathbf{G}^N$ is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T$. A coalition T is a *partnership* in $v \in \mathbf{G}^N$ if $v(C \cup S) = v(S)$ whenever $C \subset T$ and $S \subseteq N \backslash T$. For every coalition T we denote by u^T the unanimity game defined by

$$u^T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

A *coalition structure* of N is a family $\mathcal{B} = \{B_1, ..., B_m\}$ of coalitions of N such that $\bigcup_{q=1}^m B_q = N$, and $B_p \cap B_q = \emptyset$ if $B_p, B_q \in \mathcal{B}$ with $p \neq q$. We denote by $\mathbf{B}_0^N$ the set of coalition structures of N and $\mathbf{BG}_0^N = \mathbf{B}_0^N \times \mathbf{G}^N$.

If we fix $\mathcal{B} \in \mathbf{B}_0^N$, a *solution*[1] ψ on $\mathbf{G}^N$ is a function from $\mathbf{G}^N$ into $\mathbb{R}^N$. Vector

[1]This solution concept is defined, following Levy and McLean (1989), for a fixed coalition structure. Our solution, presented in the next section, is more general and it does not depend on any particular coalition configuration.

$\psi(v)$ represents the expectations of players in v when they form coalition structure $\mathcal{B}$.

Let us present the weighted coalition structure values with intercoalitional symmetry defined by Levy and McLean (1989).

First we need some notation. Let $\mathcal{B} = \{B_1, ..., B_m\}$ be a coalition structure of N. For $B_q \in \mathcal{B}$, denote by $\Sigma(B_q)$ the set of orders of B_q and let $\mathcal{A}_{\mathcal{B}}(N)$ be the set of tuples $(\sigma_1, ..., \sigma_m)$ such that for every $l = 1, ..., m$ it holds that

(1) $\sigma_l \in \Sigma(B_q)$ for some $q = 1, \ldots, m$, and

(2) $l' \neq l$ and $\sigma_{l'} \in \Sigma(B_q)$ implies $\sigma_l \notin \Sigma(B_q)$.

Each element $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ naturally induces an order R_α of N in which players in every $B_q \in \mathcal{B}$ appear successively. We denote by $R_\alpha^{B_q}$ the order $\sigma_l \in \Sigma(B_q)$.

Let $(\mathcal{B}, v) \in \mathbf{BG}_0^N$ and $\omega \in \mathbb{R}_+^N$.[2] Assume that players in N form a queue outside a room according to the following procedure. Players in every B_q are together, all the orders of coalitions $B_q \in \mathcal{B}$ are equally likely and players in every B_q are ordered as follows. One player in B_q is picked up at a time and she is placed in the front of the queue which is partially formed. Once a player is picked up she is not picked up any more and the probability of picking up player i is given by her weight ω_i divided by the weights of the players in B_q who are not yet in the queue. Then players proceed to enter in the room. When player i enters in the room a coalition S forms and player i is given her marginal contribution to this coalition, that is, $v(S) - v(S \backslash \{i\})$. The expected marginal contribution of player i is by definition the weighted coalition structure value with intercoalitional symmetry of player i in v associated with ω and $\mathcal{B}$.

This definition is due to Levy and McLean (1989). Let us formalize it. For every $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ and $i \in N$, let $P_\alpha[i]$ denote the set of players which are before player i in order R_α, including player i.

Let $(\mathcal{B}, v) \in \mathbf{BG}_0^N$ and $\omega \in \mathbb{R}_+^N$. For every $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ and $i \in N$ the *marginal contribution* of player i in order R_α is

$$C_i(v, R_\alpha) = v(P_\alpha[i]) - v(P_\alpha[i] \setminus \{i\}).$$

Consider the probability distribution $Q^{\omega,\mathcal{B}}$ on $\mathcal{A}_{\mathcal{B}}(N)$ such that

$$Q^{\omega,\mathcal{B}}(\alpha) = \frac{1}{|\mathcal{B}|!} \cdot \prod_{B_q \in \mathcal{B}} Q_q^\omega\left(R_\alpha^{B_q}\right),$$

where Q_q^ω is the probability distribution on $\Sigma(B_q)$ such that $Q_q^\omega(\sigma_l) = \prod_{t=1}^{|B_q|} \frac{\omega_t}{\sum_{s=1}^t \omega_s}$ for $\sigma_l = (1 \ldots |B_q|) \in \Sigma(B_q)$, where we simplify the writing of σ_l.

[2] We denote by $\mathbb{R}_+^N$ the set of $|N|$ -tuples with strictly positive components indexed by the elements in N.

The *weighted coalition structure value with intercoalitional symmetry* of player i in v associated with ω and $\mathcal{B}$ is defined by

$$\eta_i^{\omega,\mathcal{B}}(v) = \sum_{\alpha \in \mathcal{A}_{\mathcal{B}}(N)} Q^{\omega,\mathcal{B}}(\alpha) \cdot C_i(v, R_\alpha).$$

These solutions are characterized by Levy and McLean (1989) by means of the following axioms. Let $\mathcal{B} \in \mathbf{B}_0^N$ be fixed and let ψ denote a solution on $\mathbf{G}^N$.

Linearity. For every $v_1, v_2 \in \mathbf{G}^N$ and $\lambda, \mu \in \mathbb{R}$ it holds that

$$\psi(\lambda v_1 + \mu v_2) = \lambda \psi(v_1) + \mu \psi(v_2).$$

Efficiency. For every $v \in \mathbf{G}^N$ it holds that

$$\sum_{i \in N} \psi_i(v) = v(N).$$

$\mathcal{B}$-Positivity. If $v \in \mathbf{G}^N$ is monotonic and $C_i(v, R_\alpha) > 0$ for some $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$, then $\psi_i(v) > 0$.

Intercoalitional Symmetry. Let $v \in \mathbf{G}^N$. If $B_p, B_q \in \mathcal{B}$ are such that for every $\mathcal{C} \subseteq \mathcal{B} \setminus \{B_p, B_q\}$ it holds $v\left(B_p \cup \bigcup_{B_r \in \mathcal{C}} B_r\right) = v\left(B_q \cup \bigcup_{B_r \in \mathcal{C}} B_r\right)$, then

$$\sum_{i \in B_p} \psi_i(v) = \sum_{i \in B_q} \psi_i(v).$$

Intracoalitional Partnership. If $T \subseteq N$ is a partnership in v, then for every B_q such that $B_q \cap T \neq \emptyset$ and every $i \in B_q \cap T$ it holds that

$$\psi_i\left(u^T\right) \sum_{j \in B_q \cap T} \psi_j(v) = \psi_i(v) \sum_{j \in B_q \cap T} \psi_j\left(u^T\right).$$

Null Player Axiom. Let $v \in \mathbf{G}^N$. If $i \in N$ is a null player in v (i.e., if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$), then

$$\psi_i(v) = 0.$$

Theorem 1 [Levy and McLean, 1989]**.** *Let $\mathcal{B} \in \mathbf{B}_0^N$. A function $\psi : \mathbf{G}^N \to \mathbb{R}^N$ satisfies Efficiency, Linearity, the Null Player Axiom, $\mathcal{B}$-Positivity, Intercoalitional Symmetry and Intracoalitional Partnership if and only if there exists a vector $\omega \in \mathbb{R}_+^N$ such that $\psi = \eta^{\omega,\mathcal{B}}$.*

And finally, Levy and McLean (1989) proved that

$$\eta_i^{\omega,\mathcal{B}}\left(u^T\right) = \frac{1}{|\{B_p \in \mathcal{B} : B_p \cap T \neq \emptyset\}|} \cdot \frac{\omega_i}{\sum_{j \in B_q \cap T} \omega_j} \quad \text{if} \quad i \in B_q \cap T. \tag{1}$$

3. The weighted bounded configuration values

In this section we propose an alternative solution to the ones given in Albizuri et al. (2006). Again, players form coalitions and can be in more than one. In addition, we suppose that when a player is in more than one coalition, she can split into several players that represent her in the coalitions she belongs to.

Let N be a finite set of players. A *coalition configuration* of N is a family $\mathcal{B} = \{B_1, ..., B_m\}$ of different coalitions of N such that $\bigcup_{q=1}^{m} B_q = N$. The set of coalition configurations of N is denoted by $\mathbf{B}_1^N$. Given $\mathcal{B} \in \mathbf{B}_1^N$ and $i \in N$, let

$$\mathcal{B}^i = \{B_q \in \mathcal{B} : i \in B_q\}$$

be the set of coalitions in $\mathcal{B}$ that contain player i.

In order to consider the representative players, we define

$$[N, \mathcal{B}] = \{(i, B_q) : i \in B_q \in \mathcal{B}\}$$

for each $\mathcal{B} \in \mathbf{B}_1^N$. For simplicity, we write (i, q) instead of (i, B_q). Hence, each pair $(i, q) \in [N, \mathcal{B}]$ is a representative of player $i \in N$ when the coalition configuration is $\mathcal{B}$.

As opposed to the previous section, where the set of players is fixed, we define a finite set U of (non-split) potential players. We then denote as A the set of potential players, including the representatives. Namely, $A = U \cup U^*$ where

$$U^* = \left\{(i, q) : (i, q) \in [N, \mathcal{B}] \text{ for some } N \subseteq U, \text{ and } \mathcal{B} \in \mathbf{B}_1^N\right\}.$$

We restrict the coalition configurations to those that satisfy the two following conditions:

$$\left|\mathcal{B}^{(i,q)}\right| = 1 \text{ for all } (i, q) \in N \cap U^* \tag{2}$$

and

$$\left|\mathcal{B}^i\right| > 1 \text{ implies } (i, q) \notin N \text{ for all } (i, q) \in U^*. \tag{3}$$

The interpretation is as follows. There are two types of players, players in U and players in U^*, where players in U^* can represent players in U. The admissible set of players are the nonempty subsets of A. Each player $i \in U$ can be represented by some player $(i, q) \in U^*$ when player i joins coalition B_q in some coalition configuration. Each player (i, q) can be seen as a split player of player i, and she cannot split (this is the meaning of (2)). So, players in U^* cannot be in more than one coalition. Furthermore, player $i \in U$ cannot be with players that represent her when she is in more than one coalition (this is the meaning of (3)).

Hence, given $N \subseteq A$, we define

$$\mathbf{B}^N = \left\{\mathcal{B} \in \mathbf{B}_1^N : \mathcal{B} \text{ satisfies (2) and (3)}\right\}.$$

Moreover, we define $\mathbf{BG}^N = \mathbf{B}^N \times \mathbf{G}^N$ and $\mathbf{BG} = \bigcup_{N\subseteq A} \mathbf{BG}^N$.

With some abuse of notation, we define a *solution* on $\mathbf{BG}$ as a function ψ from $\mathbf{BG}$ into $\bigcup_{N\subseteq A} \mathbb{R}^N$ such that $\psi(\mathcal{B}, v) \in \mathbb{R}^N$ whenever $(\mathcal{B}, v) \in \mathbf{BG}^N$. Vector $\psi(\mathcal{B}, v)$ can be interpreted as the expected value of players in v when they form coalition configuration $\mathcal{B}$.

In this section, as in the previous one, we do not consider anonymity in the sense that symmetric players, even in the same coalition B_q, should receive the same. This lack of symmetry in the intracoalitional level can be justified in many situations. It may be the case that a greater effort is needed on the part of some players than on the part of some others in order for a common project to succeed. This example appears in Kalai and Samet (1987). It may also be the case that players are towns or cities with different number of inhabitants, or political parties with different number of seats in a parliament or members in a commission or management board. Having a larger number of seats or members may allow parties to increase their chances to make proposals and hence increase their relative power. For example, Hart and Mas-Colell (1996, item (e) on page 374) point out that the weighted Shapley value arises in equilibrium[3] in an offer-counteroffer model of bargaining when the weights determine the probability of making offers. Identification of probability of making offers and size has been suggested in Chae and Heidhues (2004) and further explored in Vidal-Puga (2012). Yet Kalai and Samet (1987, Corollary 2) provide a reasonable justification for identifying weight and size of constituencies players represent.

Identifying weight with either effort or size, it seems natural that it should decrease when a player participate in more than one coalition. We model this situation by considering a weight system such that the weight of a non-split player is divided among her representatives.

Formally, suppose that there exists a vector $\omega \in \mathbb{R}^A_+$ such that for each $N \subseteq A$, $i \in U \cap N$ and $\mathcal{B} \in \mathbf{B}^N$,

$$\sum_{B_q \in \mathcal{B}^i} \omega_{(i,q)} = \omega_i. \tag{4}$$

That is, every potential player has a weight and the weight of every non-split player is the sum of the weights of the split players associated with her.

Example 1. *Take $\omega_i = 1$ for all $i \in U$ and $\omega_{(i,q)} = \frac{1}{|\mathcal{B}^i|}$ for all $(i,q) \in U^*$. It is straightforward to check that such ω is well-defined and it satisfies (4).*

Example 2. *Let $\varpi \in \mathbb{R}^U_+$. Take $\omega_i = \varpi_i$ for all $i \in U$ and $\omega_{(i,q)} = \frac{1}{|\mathcal{B}^i|}\varpi_i$ for all $(i,q) \in U^*$. Again, it is straightforward to check that such ω is well-defined and it satisfies (4).*

[3]Stationary subgame perfect equilibrium.

Example 3. *Let* $\mu \in \mathbb{R}^A_+$*. Take* $\omega_i = \mu_i$ *for all* $i \in U$ *and* $\omega_{(i,q)} = \frac{\mu_{(i,q)}}{\sum_{B_r \in \mathcal{B}^i} \mu_{(i,r)}} \mu_i$ *for all* $(i,q) \in U^*$*. It is straightforward to check that such* ω *is well-defined and it satisfies (4).*

Given $\mathcal{B} \in \mathbf{B}^N$, the *weighted bounded configuration value* in $\mathbf{G}^N$ associated with ω and $\mathcal{B}$ is by definition the composition of mappings

$$\phi^{\omega,\mathcal{B}} : \mathbf{G}^N \longrightarrow \mathbf{G}^{[N,\mathcal{B}]} \longrightarrow \mathbb{R}^{[N,\mathcal{B}]} \longrightarrow \mathbb{R}^N$$

where the first mapping splits the players into their respective representatives, the second mapping is the weighted coalitional structure value of Levy and McLean (1989), and the third mapping assigns to each player the sum of her representatives' payoffs.

The formal definition of each mapping is as follows:

- Mapping $\mathbf{G}^N \longrightarrow \mathbf{G}^{[N,\mathcal{B}]}$ assigns to each $v \in \mathbf{G}^N$ a game $\widehat{v} \in \mathbf{G}^{[N,\mathcal{B}]}$ defined as

  $$\widehat{v}\left(\widehat{S}\right) = v\left(\left\{ i \in N : \bigcup_{B_q \in \mathcal{B}^i} \{(i,q)\} \subseteq \widehat{S} \right\}\right)$$

  for all $\widehat{S} \subseteq [N,\mathcal{B}]$.

- Mapping $\mathbf{G}^{[N,\mathcal{B}]} \longrightarrow \mathbb{R}^{[N,\mathcal{B}]}$ is the weighted coalition structure value $\eta^{\widehat{\omega},\widehat{\mathcal{B}}}$ where $\widehat{\mathcal{B}} = \left\{\widehat{B_q}\right\}_{B_q \in \mathcal{B}}$ with $\widehat{B_q} = \bigcup_{i \in B_q} \{(i,q)\}$ for all $B_q \in \mathcal{B}$, and $\widehat{\omega} \in \mathbb{R}^{[N,\mathcal{B}]}$ is defined as

  $$\widehat{\omega}_{(i,q)} = \begin{cases} \omega_{(i,q)} & \text{if } i \in U \\ \omega_{(j,p)} & \text{if } i = (j,p) \in U^* \end{cases}$$

  for all $(i,q) \in [N,\mathcal{B}]$.

- Mapping $\mathbb{R}^{[N,\mathcal{B}]} \longrightarrow \mathbb{R}^N$ assigns to each $x \in \mathbb{R}^{[N,\mathcal{B}]}$ the vector $y \in \mathbb{R}^N$ defined by $y_i = \sum_{B_q \in \mathcal{B}^i} x_{(i,q)}$ for all $i \in N$.

Definition 1. *The* weighted bounded configuration value ϕ^ω *in* $\mathbf{BG}$ *associated with* ω *is the corresponding solution defined by* $\phi^\omega(\mathcal{B}, v) = \phi^{\omega,\mathcal{B}}(v)$ *for all* $(B, v) \in \mathbf{BG}$.

Next Proposition states that the weighted bounded configuration value of a player is the sum of the Levy and McLean values of the split players associated with that player in an associated game with the natural associated coalition structure.

Proposition 2. *Let ϕ^{ω} be a weighted bounded configuration value. Then for every $v \in \mathbf{G}^N$ and $i \in N$ it holds that*

$$\phi_i^{\omega}(\mathcal{B}, v) = \sum_{B_q \in \mathcal{B}^i} \eta_{(i,q)}^{\widehat{\omega},\widehat{\mathcal{B}}}(\widehat{v}).$$

Proof. If follows from the definition. ■

Notice that all the sets $\widehat{B^i} = \bigcup \{(i,q) : B_q \in \mathcal{B}^i\}$ are partnerships in $\widehat{v}$ and that $\widehat{v}\left(\widehat{B^i}\right) = v(\{i\})$ for all $i \in U$. Observe also that $\widehat{\mathcal{B}}$ is the natural coalition structure induced on $[N, \mathcal{B}]$ when every $i \in N$ is replaced in each $B_q \in \mathcal{B}^i$ by her associated split player (i,q). Moreover, (1) assures that $\widehat{\mathcal{B}} \in \mathbf{B}_0^{[N,\mathcal{B}]}$ is a coalition structure, so that $\eta^{\widehat{\omega},\widehat{\mathcal{B}}}$ is well-defined.

Let us fix $\mathcal{B} \in \mathbf{B}^N$. Every player $i \in N$ has a representative in each coalition $B_q \in \mathcal{B}^i$, given by (i,q). Suppose that these representatives form a queue outside a room in such a way that all representatives associated with every B_q are together. These queues can be represented by the members $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$. We suppose that all the orderings of coalitions $B_q \in \mathcal{B}$ are equally likely and that players in every B_q are ordered as follows. One player in B_q is picked up at a time and she is placed in the front of the queue which is partially formed. Once a player is picked up she is not picked up any more and the probability of picking up a player (i,q) is given by the split weight $\omega_{(i,q)}$ divided by the weights $\omega_{(j,q)}$ associated with the players (j,q) who are not yet in the queue. After forming the queue the representatives proceed to enter in the room. When a representative of a player enters in the room a coalition forms if all the representatives of that player have entered in the room. Moreover this coalition, say S, is formed by the players whose representatives are all in the room. When the last representative of player i enters in the room she will be given her marginal contribution to the coalition S, that is, $v(S) - v(S \backslash \{i\})$. The expected marginal contribution of player i will be (see Proposition 3 below) her *weighted bounded configuration value* associated with ω and $\mathcal{B}$.

Let us formalize this interpretation.

Given $\mathcal{B} \in \mathbf{B}^N$, for every $B_q \in \mathcal{B}$ we denote by $\Sigma(B_q)$, as in the previous section, the set of orders of B_q and by $\mathcal{A}_{\mathcal{B}}(N)$ the set of tuples $(\sigma_1, \ldots, \sigma_m)$ such that for every $l = 1, ..., m$ it holds that

(1) $\sigma_l \in \Sigma(B_q)$ for some $B_q \in \mathcal{B}$, and

(2) $l' \neq l$ and $\sigma_{l'} \in \Sigma(B_q)$ implies $\sigma_l \notin \Sigma(B_q)$.

Now, every element $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ induces an order (with repetition) in which players in every $B_q \in \mathcal{B}$ appear successively. We denote this order with repetition by R_α and by $R_\alpha^{B_q}$ the order $\sigma_l \in \Sigma(B_q)$.

Consider the probability distribution $Q^{\omega,\mathcal{B}}$ on $\mathcal{A}_{\mathcal{B}}(N)$ such that

$$Q^{\omega,\mathcal{B}}(\alpha) = \frac{1}{|\mathcal{B}|!} \cdot \prod_{B_q \in \mathcal{B}} Q_q^{\omega}\left(R_{\alpha}^{B_q}\right),$$

where Q_q^{ω} is the probability distribution on $\Sigma(B_q)$ such that $Q_q^{\omega}(\sigma_l) = \prod_{t=1}^{b} \frac{\omega_{(t,q)}}{\sum_{s=1}^{t} \omega_{(s,q)}}$ when $B_q = \{1, \ldots, b\}$ and $\sigma_l = (1, \ldots, b) \in \Sigma(B_q)$.

Given $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ and $i \in N$ denote by $P_{\alpha}[i]$ the set of players whose last position in R_{α} is before the last position of player i, including player i. Notice that this definition generalizes that of $\mathcal{A}_{\mathcal{B}}(N)$ defined in the previous section, so it share the same name. Given $(\mathcal{B}, v) \in \mathbf{BG}^N$, for every $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ and $i \in N$, the *marginal contribution* of player i in order R_{α} is defined by $C_i(v, R_{\alpha}) = v(P_{\alpha}[i]) - v(P_{\alpha}[i] \setminus \{i\})$.

Proposition 3. *The weighted bounded configuration value of player i in $\mathbf{G}^N$ associated with ω is*

$$\phi_i^{\omega}(\mathcal{B}, v) = \sum_{\alpha \in \mathcal{A}_{\mathcal{B}}(N)} Q^{\omega,\mathcal{B}}(\alpha) \cdot C_i(v, R_{\alpha}).$$

Proof. Let ϕ^{ω} and $(\mathcal{B}, v) \in \mathbf{BG}^N$. There is a natural bijection between $\mathcal{A}_{\mathcal{B}}(N)$ and $\mathcal{A}_{\widehat{\mathcal{B}}}([N, \mathcal{B}])$ that associates each $\alpha = (\sigma_1, ..., \sigma_m) \in \mathcal{A}_{\mathcal{B}}(N)$ with $\widehat{\alpha} \in \mathcal{A}_{\widehat{\mathcal{B}}}([N, \mathcal{B}])$ just replacing player $i \in \sigma_l \in \Sigma(B_q)$ by (i, q). Furthermore, for every $i \in N$ it holds that $C_i(v, R_{\alpha}) = \sum_{B_q \in \mathcal{B}^i} C_{(i,q)}(\widehat{v}, R_{\widehat{\alpha}})$ and $Q^{\omega,\mathcal{B}}(\alpha) = Q^{\widehat{\omega},\widehat{\mathcal{B}}}(\widehat{\alpha})$. Therefore, we obtain the required result. ■

Remark 1. *When the weights are natural numbers then the weighted bounded configuration value can also be calculated as follows. The weights can be seen as the number of representatives that players put in the queue outside the room. Suppose for example that $\{1, 2, 3\} = N \subseteq A$, $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = 1$, $B_1 = \{1, 2\}$, $B_2 = \{2, 3\}$, and $\omega_{(2,1)} = 2$, $\omega_{(2,2)} = 1$. Let $\mathcal{B}_1 = \{B_1, B_2\}$, so that $(2, 1)$ and $(2, 2)$ represent player 2, respectively, in B_1 and in B_2. So players 1, 2 and 3 have, respectively, ω_1, ω_2 and ω_3 representatives in the coalitions they belong to, and $\omega_{(2,1)}$ and $\omega_{(2,2)}$ tell us how ω_2 is divided in the two coalitions player 2 belongs to. Therefore, player 1 has one representative in coalition B_1, player 3 has one representative in B_2, and player 2 has two representatives in B_1 and one in B_2. We denote by $2, 2'$ the two representatives of player 2 in B_1 and maintain the denomination of the player for the other representatives. If we consider the orders of all the representatives in such a way that those associated with the same coalition are together, that is, $122'23$, $12'223$, $122'32$, ..., if we suppose that these orders are equally likely and that the sets form as explained in the Introduction for the dual configuration value, the expected marginal contribution of a player $i \in \{1, 2, 3\}$ is her value according to ϕ^{ω}. For the first game*

v presented in the Introduction, $\phi^{\omega}(\mathcal{B}_1, v) = \left(\frac{2}{3}, 3\frac{5}{6}, \frac{1}{2}\right)$. For the unanimity game u^N, $\phi^{\omega}\left(\mathcal{B}_1, u^N\right) = \left(\frac{1}{6}, \frac{7}{12}, \frac{1}{4}\right)$.

These results can be proved as in Kalai and Samet's Theorem 9 (1987), taking into account Proposition 3.

4. Characterization of the weighted bounded configuration values

In this section we characterize the family formed by the values ϕ^{ω} by means of the following properties. We denote by ψ a solution on $\mathbf{BG}$ and $N \subseteq A$.

The first five properties are adaptations to this framework of the properties that characterize the weighted coalition values with intercoalitional symmetry, except the Null Player Property.

Efficiency. For every $(\mathcal{B}, v) \in \mathbf{BG}^N$ it holds that

$$\sum_{i \in N} \psi_i(\mathcal{B}, v) = v(N).$$

Linearity. For every $(\mathcal{B}, v_1), (\mathcal{B}, v_2) \in \mathbf{BG}^N$ and $\lambda, \mu \in \mathbb{R}$ it holds that

$$\psi(\mathcal{B}, \lambda v_1 + \mu v_2) = \lambda \psi(\mathcal{B}, v_1) + \mu \psi(\mathcal{B}, v_2).$$

Coalition Configuration Positivity. If $v \in \mathbf{G}^N$ is monotonic and $C_i(v, R_\alpha) > 0$ for some $\alpha \in \mathcal{A}_{\mathcal{B}}(N)$ then $\psi_i(\mathcal{B}, v) > 0$.

Intercoalitional Symmetry. Let $(\mathcal{B}, v) \in \mathbf{BG}_0^N$. If $B_p, B_q \in \mathcal{B}$ are such that for every $\mathcal{C} \subseteq \mathcal{B} \setminus \{B_p, B_q\}$ it holds $v\left(B_p \cup \bigcup_{B_r \in \mathcal{C}} B_r\right) = v\left(B_q \cup \bigcup_{B_r \in \mathcal{C}} B_r\right)$, then

$$\sum_{i \in B_p} \psi_i(\mathcal{B}, v) = \sum_{i \in B_q} \psi_i(\mathcal{B}, v).$$

Intracoalitional Partnership. Let $(\mathcal{B}, v) \in \mathbf{BG}^N$. If $T \subseteq N$ is a partnership in v and $B_q \in \mathcal{B}$ is such that $B_q \cap T \neq \emptyset$ and $|\mathcal{B}^i| = 1$ for every $i \in B_q \cap T$, then for every $i \in B_q \cap T$ it holds that

$$\psi_i\left(\mathcal{B}, u^T\right) \sum_{j \in B_q \cap T} \psi_j(\mathcal{B}, v) = \psi_i(\mathcal{B}, v) \sum_{j \in B_q \cap T} \psi_j\left(\mathcal{B}, u^T\right).$$

The next axiom requires the solution to be independent of the null players with no relevant role in a coalition structure.

Null Players Out. Let $(\mathcal{B}, v) \in \mathbf{BG}_0^N$. If $i \in N$ is a null player in v, then

$$\psi_j\left(\mathcal{B}_{-i}, v_{N\setminus\{i\}}\right) = \psi_j\left(\mathcal{B}, v\right)$$

for every $j \in N \setminus \{i\}$.

For the next axiom we need two notations. Recall $N^{-i,S}$ denotes the set $(N \setminus \{i\}) \cup S$.

Given $(\mathcal{B}, v) \in \mathbf{BG}^N$, $i \in N$ and $|\mathcal{B}^i| > 1$, the game $v^{*i} \in \mathbf{G}^{N^{-i,\widehat{B^i}}}$, where $\widehat{B^i} = \{(i,q) : B_q \in \mathcal{B}^i\}$, is defined by

$$v^{*i}(T) = \begin{cases} v(T \cap N) & \text{if } \widehat{B^i} \nsubseteq T \\ v((T \cap N) \cup \{i\}) & \text{if } \widehat{B^i} \subseteq T. \end{cases}$$

In this game player i has been replaced by $\widehat{B^i}$, being the proper subsets of $\widehat{B^i}$ powerless, that is, being $\widehat{B^i}$ a partnership.

Given $\mathcal{B} \in \mathbf{B}^N$, $i \in N$ and $|\mathcal{B}^i| > 1$, we write

$$\mathcal{B}^{*i} = \left(\mathcal{B} \setminus \mathcal{B}^i\right) \cup \left\{B_q^{-i,\{(i,q)\}}\right\}_{B_q \in \mathcal{B}^i}.$$

That is, player i is replaced by her representatives in the coalitions (of the coalition configuration) she belongs to.

The next axiom states that if a player belongs to several coalitions of a coalition configuration, then she can be substituted by her representatives associated with this coalition configuration without changing the payoff of the other players.

Merger. Let $(\mathcal{B}, v) \in \mathbf{BG}^N$ and $i \in N$ such that $|\mathcal{B}^i| > 1$. Then,

$$\psi_j\left(\mathcal{B}, v\right) = \psi_j\left(\mathcal{B}^{*i}, v^{*i}\right)$$

for every $j \in N \setminus \{i\}$.

The dual configuration value (Albizuri et al., 2006) satisfies all the above properties[4]. Moreover, the weighted bounded configuration value also satisfies them (see Proposition 4 below). Hence, we need an additional axiom to decide which of these two values is more suitable in a given situation.

[4]See Theorem 5.1 in Albizuri et al. (2006) for Efficiency, Linearity and Intercoalitional Symmetry (called Coalitional Symmetry). Coalition Configuration Positivity, Intracoalitional Parnership, Null Players Out and Merger follow easily from the definition.

Example 4. *Assume there are one player (say, player 2) and two others (say, players $(2,1)$ and $(2,2)$), who may be representatives of player 2. Under (3), either player 2 belongs to a unique coalition, or, if she does belong two more than one coalition, then neither $(2,1)$ nor $(2,2)$ are active players. On the other hand, we do not rule out the possibility that the three of them be simultaneously active players. In that case, player 2 should be indistinguishable from players $(2,1)$ and $(2,2)$ together.*

For example, let $\{1,2,3\} \subseteq U$ and $\mathcal{C} = \{C_1, C_2\} \in \mathbf{B}^{\{1,2,3\}}$ with $C_1 = \{1,2\}$ and $C_2 = \{2,3\}$. So, players $(2,1)$ and $(2,2)$ are the representatives of player 2 when the coalition configuration is $\mathcal{C}$.

Let $N = \{1,2,3,(2,1),(2,2)\}$. This is a feasible set of active players since conditions (2) and (3) allow player 2 and her representatives to be all together, as for example with the coalition configuration $\mathcal{B} = \{\{1\}, \{2,3,(2,1),(2,2)\}\}$. This case may be interpreted as that player 2 has been replicated and $\{(2,1),(2,2)\}$ are the representatives of the replica.

Let u^N be the unanimity game on N, so that $\{2,(2,1),(2,2)\}$ is a partnership in u^N. The dual configuration value (Albizuri et al., 2006) assigns:

$$\phi_2^* \left(\mathcal{B}, u^N\right) = \frac{1}{8} \neq \frac{1}{4} = \phi_{(2,1)}^* \left(\mathcal{B}, u^N\right) + \phi_{(2,2)}^* \left(\mathcal{B}, u^N\right).$$

On the other hand, a weighted bounded configuration value assigns

$$\begin{aligned} \phi_2^\omega \left(\mathcal{B}, u^N\right) &= \frac{1}{2} \frac{\omega_2}{\omega_2 + \omega_3 + \omega_{(2,1)} + \omega_{(2,2)}} \\ &= \frac{1}{2} \frac{\omega_{(2,1)} + \omega_{(2,2)}}{\omega_2 + \omega_3 + \omega_{(2,1)} + \omega_{(2,2)}} \\ &= \phi_{(2,1)}^\omega \left(\mathcal{B}, u^N\right) + \phi_{(2,2)}^\omega \left(\mathcal{B}, u^N\right). \end{aligned}$$

We consider that $\phi_2^\omega \left(\mathcal{B}, u^N\right) = \phi_{(2,1)}^\omega \left(\mathcal{B}, u^N\right) + \phi_{(2,2)}^\omega \left(\mathcal{B}, u^N\right)$ is a natural requirement for a coalition configuration value, since players $(2,1)$ and $(2,2)$ are representatives of player 2, and they belong to a common partnership, being therefore indistinguishable in this game.

In the following axiom we formalize the general situation described in Example 4. Consider a player and the players who represent her (with respect to some coalition configuration) all of them forming a partnership in a game, and therefore being indistinguishable in this game. If we take another coalition configuration in which such a player and her representatives belong to the same coalition, the axiom requires the solution to give the same value to the player and such representatives.

Partnership Additivity. Let $\mathcal{B} \in \mathbf{B}^N$, $\mathcal{C} \in \mathbf{B}^{N'}$, and $i \in N \cap N' \cap U$ such that $|\mathcal{C}^i| > 1$, $(i,q) \in N$ and $\mathcal{B}^i = \mathcal{B}^{(i,q)}$ for all $C_q \in \mathcal{C}^i$, and $v \in \mathbf{G}^N$ such that

$\bigcup_{C_q \in \mathcal{C}^i} \{(i,q)\} \cup \{i\}$ is a partnership in v. Then

$$\psi_i(\mathcal{B}, v) = \sum_{C_q \in \mathcal{C}^i} \psi_{(i,q)}(\mathcal{B}, v).$$

Recall that $\left|\mathcal{B}^{(i,q)}\right| = 1$ for all $B_q \in \mathcal{B}^i$ because $(i,q) \in U^*$. Hence, players i and (i,q) belong to a unique coalition in $\mathcal{B}$, that is, $\mathcal{B}^i = \mathcal{B}^{(i,q)} = \{B_q\}$.

Proposition 4. *The weighted bounded configuration value ϕ^ω satisfies Efficiency, Linearity, Coalition Configuration Positivity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger and Partnership Additivity.*

Proof. By definition, it is clear that ϕ^ω satisfies Efficiency, Linearity and Coalition Configuration Positivity. If $\mathcal{B}$ is a coalition structure then the mapping $\psi(v) = \phi^\omega(\mathcal{B}, v)$ is a weighted coalition structure value and therefore ψ satisfies Intercoalitional Symmetry and Null Players Out. Intracoalitional Partnership and Merger also follow from Levy and McLean (1989).

We now prove that ϕ^ω satisfies Partnership Additivity. Let $\mathcal{B} \in \mathbf{B}^N$, $\mathcal{C} \in \mathbf{B}^{N'}$, $i \in N \cap N' \cap U$ such that $|\mathcal{C}^i| > 1$, $(i,q) \in N$ and $\mathcal{B}^i = \mathcal{B}^{(i,q)} = \{B_q\}$ for all $B_q \in (B')^i$, and $v \in \mathbf{G}^N$ such that $T = \{(i,q) : B_q \in \mathcal{B}^i\} \cup \{i\}$ is a partnership in v. By the Intracoalitional Partnership Axiom it holds that

$$\phi_i^\omega(\mathcal{B}, v) = \frac{\phi_i^\omega\left(\mathcal{B}, u^T\right)}{\sum_{j \in B_q \cap T} \phi_j^\omega\left(\mathcal{B}, u^T\right)} \sum_{j \in B_q \cap T} \phi_j^\omega(\mathcal{B}, v),$$

where the denominator is not zero by Coalition Configuration Positivity. Moreover, $B_q \cap T = T$ and hence

$$\phi_i^\omega(\mathcal{B}, v) = \frac{\phi_i^\omega\left(\mathcal{B}, u^T\right)}{\sum_{j \in T} \phi_j^\omega\left(\mathcal{B}, u^T\right)} \sum_{j \in T} \phi_j^\omega(\mathcal{B}, v).$$

Taking into account Proposition 3 and (1) we obtain

$$\phi_i^\omega(\mathcal{B}, v) = \frac{\omega_i}{\sum_{j \in T} \omega_j} \sum_{j \in T} \phi_j^\omega(\mathcal{B}, v). \tag{5}$$

Analogously, for every $B_q \in \mathcal{B}^i$ it holds that

$$\phi_{(i,q)}^\omega(\mathcal{B}, v) = \frac{\omega_k}{\sum_{j \in T} \omega_j} \sum_{j \in T} \phi_j^\omega(\mathcal{B}, v). \tag{6}$$

By (4) we know that $\omega_i = \sum_{k \in T\setminus\{i\}} \omega_k$, which, joint with (5) and (6), implies

$$\phi_i^\omega(\mathcal{B}, v) = \sum_{k \in T\setminus\{i\}} \phi_k^\omega(\mathcal{B}, v) = \sum_{B_q \in \mathcal{B}^i} \phi_{(i,q)}^\omega(\mathcal{B}, v),$$

that is, ϕ^ω satisfies Partnership Additivity. ■

Theorem 5. *A solution ψ on* **BG** *satisfies Efficiency, Linearity, Coalition Configuration Positivity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger and Partnership Additivity if and only if there exists ω such that $\psi = \phi^\omega$.*

Proof. Proposition 4 is the if part of this Theorem. So it remains to prove that if a solution ψ on **BG** satisfies the above axioms then there exists ω such that $\psi = \phi^\omega$. Let $(\mathcal{B}, v) \in \mathbf{BG}^N$ with $N \subseteq A$ so that $\widehat{\mathcal{B}} \in \mathbf{B}_0^{[N,\mathcal{B}]}$ is a coalition structure. The mapping $\psi^{\widehat{\mathcal{B}}}(\widehat{v}) = \psi\left(\widehat{\mathcal{B}}, \widehat{v}\right)$ satisfies Efficiency, Linearity, the Null Player Axiom, $\widehat{\mathcal{B}}$-Positivity, Intercoalitional Symmetry and Intracoalitional Partnership. Therefore, by Theorem 1 there exists $\lambda\left(\widehat{\mathcal{B}}\right) \in \mathbb{R}_+^{[N,\mathcal{B}]}$ such that

$$\psi^{\widehat{\mathcal{B}}}(v) = \psi\left(\widehat{\mathcal{B}}, \widehat{v}\right) = \eta^{\lambda(\widehat{\mathcal{B}}),\widehat{\mathcal{B}}}(\widehat{v}). \tag{7}$$

Moreover, Levy and McLean (1989, Lemma 1B, p. 244) prove that $\lambda\left(\widehat{\mathcal{B}}\right)_{(i,q)}$ is proportional to $\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[\mathcal{B},v]}\right)$ for all $(i,q) \in \widehat{B_q} \in \widehat{\mathcal{B}}$. Let $B_q \in \mathcal{B}$ and $i, j \in B_q$. By Intracoalitional Partnership,

$$\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right) \sum_{(k,q)\in\widehat{B_q}} \psi_{(k,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right) = \psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N\mathcal{B}]}\right) \sum_{(k,q)\in B_q} \psi_{(k,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right).$$

Under Coalition Configuration Positivity, we can rewrite the above equality as

$$\frac{\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right)}{\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right)} = \frac{\sum_{(k,q)\in\widehat{B_q}} \psi_{(k,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right)}{\sum_{(k,q)\in\widehat{B_q}} \psi_{(k,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right)}.$$

Since we have the same equality for player (j,q), it follows that

$$\frac{\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right)}{\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right)} = \frac{\psi_{(j,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right)}{\psi_{(j,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right)},$$

that is,

$$\frac{\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)}{\psi_{(j,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)}=\frac{\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{\widehat{B_q}}\right)}{\psi_{(j,q)}\left(\widehat{\mathcal{B}},u^{\widehat{B_q}}\right)}=\frac{\psi_{(i,q)}\left(\left\{\widehat{B_q}\right\},u^{\widehat{B_q}}\right)}{\psi_{(j,q)}\left(\left\{\widehat{B_q}\right\},u^{\widehat{B_q}}\right)}, \tag{8}$$

where we have taken into account Null Players Out in the second equality. Now, we take into account again the Intracoalitional Partnership axiom, but now with $(\{A\},u^A)\in\mathbf{BG}^A$, coalition $A\in\{A\}$ and $T=\widehat{B_q}$. Then

$$\psi_{(i,q)}\left(\{A\},u^{\widehat{B_q}}\right)\sum_{(k,q)\in\widehat{B_q}}\psi_{(i,k)}\left(\{A\},u^A\right)=\psi_{(i,q)}\left(\{A\},u^A\right)\sum_{(k,q)\in\widehat{B_q}}\psi_{(k,q)}\left(\{A\},u^{\widehat{B_q}}\right).$$

Since we have the same equality for player (j,q), reasoning as above we have that

$$\frac{\psi_{(i,q)}\left(\{A\},u^A\right)}{\psi_{(j,q)}\left(\{A\},u^A\right)}=\frac{\psi_{(i,q)}\left(\{A\},u^{\widehat{B_q}}\right)}{\psi_{(j,q)}\left(\{A\},u^{\widehat{B_q}}\right)}=\frac{\psi_{(i,q)}\left(\left\{\widehat{B_q}\right\},u^{\widehat{B_q}}\right)}{\psi_{(j,q)}\left(\left\{\widehat{B_q}\right\},u^{\widehat{B_q}}\right)}. \tag{9}$$

Equalities (8) and (9) imply that

$$\frac{\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)}{\psi_{(i,q)}\left(\{A\},u^A\right)}=\frac{\psi_{(j,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)}{\psi_{(j,q)}\left(\{A\},u^A\right)}.$$

If we denote by c_q this ratio, we have that $\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^N\right)=c_q\cdot\psi_{(i,q)}\left(\{A\},u^A\right)$ for every $i\in B_q$. Let $\omega_k=\psi_k\left(\{A\},u^A\right)$ for every $k\in A$. Hence, $\lambda^N\left(\widehat{\mathcal{B}}\right)_{(i,q)}$ is proportional to $c_q\cdot\omega_{(i,q)}$ for every $i\in B_q$. Taking into account (1) and (7) we obtain

$$\psi\left(\widehat{\mathcal{B}},\widehat{v}\right)=\eta^{\omega_{[N,\mathcal{B}]},\widehat{\mathcal{B}}}\left(\widehat{v}\right) \tag{10}$$

for every $v\in\mathbf{G}^N$.

By Merger and Efficiency, for every $i\in N$,

$$\psi_i\left(\mathcal{B},v\right)=\sum_{B_q\in\mathcal{B}^i}\psi_{(i,q)}\left(\widehat{\mathcal{B}},\widehat{v}\right).$$

By (10) we have that $\psi\left(\widehat{\mathcal{B}},\widehat{v}\right)=\eta^{\widehat{\omega},\widehat{\mathcal{B}}}\left(\widehat{v}\right)$, and therefore

$$\psi_i\left(\mathcal{B},v\right)=\sum_{B_q\in\mathcal{B}^i}\eta^{\widehat{\omega},\widehat{\mathcal{B}}}_{(i,q)}\left(\widehat{v}\right).$$

We still need to prove that ω satisfies (4) so that the above equality implies $\psi = \phi^\omega$. Equality (4) is satisfied since by Partnership Additivity it follows that

$$\omega_i = \psi_i\left(\{A\}, u^A\right) = \sum_{B_q \in \mathcal{B}^i} \psi_{(i,q)}\left(\{A\}, u^A\right) = \sum_{B_q \in \mathcal{B}^i} \omega_{(i,q)}$$

for all $N \subseteq A$, $i \in U \cap N$ and $\mathcal{B} \in \mathbf{B}^N$. ■

The eight properties used in Theorem 5 are independent. We describe eight reasonable solutions. Each of them satisfies all the properties but one. Fix ω satisfying (4), as in the definition of some ϕ^ω.

- For any $\delta \in (0,1)$, the solution $\phi^{\delta,\omega}$ defined as $\phi^{\delta,\omega}(\mathcal{B}, v) = \delta\phi^\omega(\mathcal{B}, v)$ satisfies all the axioms but Efficiency. Notice that these solutions are reasonable expected payoff measures in monotonic games when there is a fixed discounting factor δ due to some unavoidable process of bargaining.

- For any $v \in \mathbf{G}^N$, let $C(v)$ denote the *carrier* of v, i.e. the set of non-null players. Then, solution $p^{0,\omega}$ defined as

$$p_i^{0,\omega}(\mathcal{B}, v) = v(N) \cdot \phi_i^\omega\left(\mathcal{B}, u^{C(v)}\right)$$

  for all $i \in N$, satisfies all the axioms but Linearity. This solution shares $v(N)$ equally among those coalitions in $\mathcal{B}$ with at least one non-null player and, inside them, proportionally among non-null players.

- Let $\omega(\varpi) \in \mathbb{R}_+^A$ be defined as in Example 2 for some $\varpi \in \mathbb{R}_+^U$ with all its coordinates different. For each $\alpha > 0$, let ϖ^α be defined as $\varpi_i^\alpha = (\varpi_i)^\alpha$ (ϖ_i raised to the power of α) for all $i \in U$. The value $\phi^{\infty,\varpi}$ defined as

$$\phi^{\infty,\varpi}(\mathcal{B}, v) = \lim_{\alpha \to \infty} \phi^{\omega(\varpi^\alpha)}(\mathcal{B}, v)$$

  satisfies all the axioms but Coalition Configuration Positivity. These values are priority rules where the highest priority goes to those players with the greatest coordinate in ϖ.

- Given $\mathcal{B} \in \mathbf{B}^N$, let $\omega^{\mathcal{B}} \in \mathbb{R}_+^N$ be defined as $\omega_i^{\mathcal{B}} = |\mathcal{B}^i|$ for all $i \in N \cap U$ and $\omega_{(i,q)}^{\mathcal{B}} = 1$ for all $(i,q) \in U^*$. The solution $Sh^e(\mathcal{B}, v) = Sh^{\omega^{\mathcal{B}}}(v)$, where $Sh^{\omega^{\mathcal{B}}}$ is the weighted Shapley value (Kalai and Samet, 1987), satisfies all the axioms but Intercoalitional Symmetry.

- Given $\alpha > 0$ and $T \subseteq A$, let $\mu(T, \alpha) \in \mathbb{R}_+^A$ be defined as $\mu(T,\alpha)_i = 1$ for all $i \in U$ and

$$\mu(T,\alpha)_{(i,q)} = \begin{cases} \frac{1}{|\mathcal{B}^i|} & \text{if } i \in T \text{ or } \{(i,r) : B_r \in \mathcal{B}^i\} \subseteq T \\ \alpha & \text{otherwise} \end{cases}$$

for all $(i,q) \in U^*$. The linear extension of the rule f^α defined as

$$f_i^\alpha\left(\mathcal{B}, u^T\right) = \sum_{B_q \in \mathcal{B}^i} \eta_{(i,q)}^{\mu(T,\alpha),\widehat{\mathcal{B}}}\left(\widehat{u^T}\right)$$

for all $T \subseteq N \subseteq A$ and $i \in N$, satisfies all the properties but Intracoalitional Partnership. This solution treats differently players in U^* that are isolated.

- Let $\omega' \in \mathbb{R}_+^A$ defined as in Example 1 and let $\omega(\varpi) \in \mathbb{R}_+^A$ be defined as in Example 2 for some $\varpi \in \mathbb{R}_+^U$ with all its coordinates different. The solution given by
$$\phi^\varpi(\mathcal{B}, v) = \begin{cases} \phi^{\omega(\varpi)}(\mathcal{B}, v) & \text{if } \mathcal{B} = \{A\} \\ \phi^{\omega'}(\mathcal{B}, v) & \text{otherwise} \end{cases}$$
satisfies all the axioms but Null Players Out. Notice, however, that it does satisfy the Null Player Axiom.

- For each $\mathcal{B} \in \mathbf{B}^N$, let $\mu^\mathcal{B} \in \mathbb{R}_+^A$ defined by $\mu_i^\mathcal{B} = 1$ for all $i \in U$ and $\mu_{(i,q)}^\mathcal{B} = \frac{|C_q \cap B_p|+1}{\sum_{C_r \in \mathcal{C}^i} |C_r \cap B_p| + |\mathcal{C}^i|}$ for all $(i,q) \in [N', \mathcal{C}] \subseteq U^*$ with $(i,q) \in B_q \in \mathcal{B}$. The linear extension of the solution g defined as
$$g_i\left(\mathcal{B}, u^T\right) = \sum_{B_q \in \mathcal{B}^i} \eta_{(i,q)}^{\mu^\mathcal{B},\widehat{\mathcal{B}}}\left(\widehat{u^T}\right)$$
for all $T \subseteq A$ and $i \in N$, satisfies all the axioms but Merger.

- The dual configuration value (Albizuri et al., 2006) satisfies all the axioms but Partnership Additivity.

Since the weighted bounded coalitional values generalize the Levy and McLean values for coalition configurations, we can consider it as a new property. Formally, this property can be defined as follows:

Extension of the Levy-McLean value. There exists $\omega \in \mathbb{R}^A$ such that

$$\psi(\mathcal{B}, v) = \eta^{\omega_N,\mathcal{B}}(v)$$

for all $(\mathcal{B}, v) \in \mathbf{BG}_0^N$.

If we substitute Linearity, Coalition Configuration Positivity, Intercoalitonal Symmetry, Intracoalitional Partnership and Null Players Out by this new property in Theorem 5, we obtain a new characterization of ϕ^ω.

Theorem 6. *A solution ψ on* **BG** *satisfies Efficiency, Merger, Partnership Additivity and Extension of the Levy-McLean value if and only if there exists ω such that $\psi = \phi^{\omega}$.*

Proof. It is straightforward to prove that if ψ is the bounded equally split value then it satisfies Extension of the Levy-McLean value. Moreover, it follows from Proposition 4 that it also satisfies the other properties.

Assume now that ψ satisfies these properties. We can use Extension of the Levy-McLean to get (10) in the proof of Theorem 5. Following the same reasoning as in that proof, we deduce the result by using Efficiency, Merger and Partnership Additivity. ■

These properties are independent. Solution $\phi^{1,\omega}$ defined as $\phi_i^{1,\omega}(\mathcal{B}, v) = \frac{1}{|\mathcal{B}^i|}\phi_i^{\omega}(\mathcal{B}, v)$ for all $i \in N$ satisfies all the properties but Efficiency. Solution g satisfies all the properties but Merger. The dual configuration value satisfies all the properties but Partnership Additivity. Solution Sh^e satisfies all the properties but Extension of the Levy-McLean value.

Remark 2. *The dual configuration value (Albizuri et al., 2006) is the dual version of an alternative value in which a coalition forms when the* first *(instead of the last) representative of a player enters the room. We can also define such a dual version of the weighted bounded configuration value and characterize it with the appropriate changes in the properties.*

5. The symmetric bounded equally split value

In this section we focus on a specific weighted bounded configuration value. It is the value obtained when the weights are given as in Example 1, i.e. all the players in U have weight 1 and they spread equally their weight over the coalitions they belong to. That is, $\omega_i = 1$ for all $i \in U$ and $\omega_{(i,q)} = \frac{1}{|\mathcal{B}^i|}$ for all $(i, q) \in U^*$.

We denote this value by $\phi^{\mathbf{e}}$ and we call it the *symmetric bounded equally split value.*

This value satisfies intracoalitional anonymity when a coalition structure with sets of players contained in U is formed. Formally this property is stated as follows:

Intracoalitional Anonymity. Let $(\mathcal{B}, v) \in \mathbf{BG}_0^N$ with $N \subseteq U$. If π is a permutation of N such that $\pi B_q = B_q$ for every $B_q \in \mathcal{B}$, then for every $i \in N$ it holds that

$$\psi_i (\mathcal{B}, \pi v) = \psi_{\pi i} (\mathcal{B}, v),$$

where $\pi v \in \mathbf{G}^N$ is defined as $(\pi v)(S) = v(\pi S)$ for all $S \subseteq N$.

The symmetric bounded equally split value also satisfies the following variation of Partnership Additivity.

Equally Partnership Additivity. Let $\mathcal{B} \in \mathbf{B}^N$, $\mathcal{C} \in \mathbf{B}^{N'}$ and $i \in N \cap N' \cap U$ such that $|\mathcal{C}^i| > 1$, $(i,q) \in N$ and $\mathcal{B}^i = \mathcal{B}^{(i,q)}$ for all $C_q \in \mathcal{C}^i$, and $v \in \mathbf{G}^N$ such that $\bigcup_{C_q \in \mathcal{C}^i} \{(i,q)\} \cup \{i\}$ is a partnership in v. Then

$$\psi_i(\mathcal{B}, v) = \sum_{C_q \in \mathcal{C}^i} \psi_{(i,q)}(\mathcal{B}, v)$$

and

$$\psi_{(i,q)}(\mathcal{B}, v) = \psi_{(i,p)}(\mathcal{B}, v)$$

for all $C_q, C_p \in \mathcal{C}^i$.

This axiom not only requires i and $\{(i,q) : C_q \in \mathcal{C}^i\}$ to obtain the same value according to ψ, but also all players in $\{(i,q) : C_q \in \mathcal{C}^i\}$ to obtain the same value. Observe that Equally Partnership Additivity implies Partnership Additivity.

If we remove Coalition Configuration Positivity, add Intracoalitional Anonymity and substitute Partnership Additivity by Equally Partnership Additivity in Theorem 5, we obtain a characterization of $\phi^{\mathbf{e}}$.

Theorem 7. *A solution ψ on $\mathbf{BG}$ satisfies Efficiency, Linearity, Intercoalitional Symmetry, Intracoalitional Partnership, Null Players Out, Merger, Equally Partnership Additivity, and Intracoalitional Anonymity if and only if ψ is the symmetric bounded equally split value.*

Proof. It is straightforward to prove that if ψ is the symmetric bounded equally split value then it satisfies Equally Partnership Additivity and also Intracoalitional Anonymity. Moreover, it follows from Proposition 4 that it also satisfies the other properties.

For the converse, we prove that in presence of these properties, Coalition Configuration Positivity is redundant. In particular, we first prove that $\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right) > 0$ for all $N \subseteq A$, $\mathcal{B} \in \mathbf{B}^N$ and $(i,q) \in [N, \mathcal{B}]$. This is enough to apply the same reasoning as in the proof of Theorem 5, because, from Lemma 1B in Levy and McLean (1989, p. 244), $\widehat{\mathcal{B}}$-Positivity is only needed for the unanimity game $u^{[N,\mathcal{B}]}$.

We proceed in two steps:

Step 1: Let $N = A$ and $\mathcal{B} = \{A\}$. Denote $B_1 = A$, so that $\widehat{U} = \{(i,1) : i \in U\}$. Since $\left|\widehat{U}\right| = |U|$, Efficiency and Intracoalitional Anonymity imply that $\psi_{(i,1)}\left(\left\{\widehat{U}\right\}, \widehat{u^U}\right) = \frac{1}{|U|} = \psi_i\left(\{U\}, u^U\right)$ for all $i \in U$.

Given $i \in U$, by Intracoalitional Partnership,

$$\psi_i\left(\{A\}, u^U\right) \sum_{k \in U} \psi_k\left(\{A\}, u^A\right) = \psi_i\left(\{A\}, u^A\right) \sum_{k \in U} \psi_k\left(\{A\}, u^U\right)$$

and therefore

$$\sum_{k\in U}\psi_k\left(\{A\},u^A\right)=|U|\,\psi_i\left(\{A\},u^A\right).$$

Hence,

$$\psi_i\left(\{A\},u^A\right)=\psi_j\left(\{A\},u^A\right)\tag{11}$$

for all $i,j\in U$.

Given $i\in U$, by Equally Partnership Additivity,

$$\psi_i\left(\{A\},u^A\right)=\sum_{C_q\in\mathcal{C}^i}\psi_{(i,q)}\left(\{A\},u^A\right)\text{ and }\psi_{(i,q)}\left(\{A\},u^A\right)=\psi_{(i,p)}\left(\{A\},u^A\right)$$

for all $\mathcal{C}\in\mathbf{B}^N$ such that $i\in N$, all $C_q,C_p\in\mathcal{C}^i$ and all $N\subseteq U$. Hence,

$$\psi_{(i,q)}\left(\{A\},u^A\right)=\frac{\psi_i\left(\{A\},u^A\right)}{|\mathcal{C}^i|}\tag{12}$$

for all $\mathcal{C}\in\mathbf{B}^N$ such that $i\in N$, all $C_q\in\mathcal{C}^i$ and all $N\subseteq U$. Hence, since $u^A(A)=1>0$, taking into account expression (11) and Efficiency, we have $\psi_i\left(\{A\},u^A\right)>0$ for all $i\in A$.

Step 2: Let $N\subseteq A$, $\mathcal{B}\in\mathbf{B}^N$ and $(i,q)\in[N,\mathcal{B}]$. We need to prove that $\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)>0$. By Intracoalitional Partnership,

$$\psi_{(i,q)}\left(\{A\},u^{\widehat{B_q}}\right)\sum_{(j,q)\in\widehat{B_q}}\psi_{(j,q)}\left(\{A\},u^A\right)=\psi_{(i,q)}\left(\{A\},u^A\right)\sum_{(j,q)\in\widehat{B_q}}\psi_{(j,q)}\left(\{A\},u^{\widehat{B_q}}\right).$$

By Null Players Out and Efficiency,

$$\psi_{(i,q)}\left(\{A\},u^{\widehat{B_q}}\right)\sum_{(j,q)\in\widehat{B_q}}\psi_{(j,q)}\left(\{A\},u^A\right)=\psi_{(i,q)}\left(\{A\},u^A\right).$$

By Step 1, $\psi_{(i,q)}\left(\{A\},u^{\widehat{B_q}}\right)>0$.

Applying Null Players Out twice,

$$\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{\widehat{B_q}}\right)=\psi_{(i,q)}\left(\left\{\widehat{B_q}\right\},u^{\widehat{B_q}}\right)=\psi_{(i,q)}\left(\{A\},u^{\widehat{B_q}}\right)>0.\tag{13}$$

Taking into account again Intracoalitional Partnership,

$$\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{\widehat{B_q}}\right)\sum_{(j,q)\in\widehat{B_q}}\psi_{(j,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)=\psi_{(i,q)}\left(\widehat{\mathcal{B}},u^{[N,\mathcal{B}]}\right)\sum_{(j,q)\in\widehat{B_q}}\psi_{(j,q)}\left(\widehat{\mathcal{B}},u^{\widehat{B_q}}\right)$$

By Null Players Out and Efficiency,

$$\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right) \sum_{(j,q)\in\widehat{B_q}} \psi_{(j,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right) = \psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right).$$

Moreover, Intercoalitional Symmetry implies that $\sum_{(j,q)\in\widehat{B_q}} \psi_{(j,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right) = \frac{1}{\left|\widehat{\mathcal{B}}\right|}$, and therefore, the equality above turns into

$$\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right) = \frac{\psi_i\left(\widehat{\mathcal{B}}, u^{\widehat{B_q}}\right)}{\left|\widehat{\mathcal{B}}\right|}.$$

Under (13), $\psi_{(i,q)}\left(\widehat{\mathcal{B}}, u^{[N,\mathcal{B}]}\right) > 0$ and Step 2 is finished.

Therefore, we can proceed as in the proof of Theorem 5 in order to prove that $\psi = \phi^{\omega}$ with $\omega_i = \psi_i\left(\{A\}, u^A\right)$ for every $i \in A$. Now, it follows from (11) and (12) that $\psi = \phi^{\mathbf{e}}$. ■

These properties are independent. For the independence of Linearity and Efficiency, we can use the respective solutions described for the independence of ϕ^{ω} with ω defined as in Example 1. For the rest of properties, we can use the respective solutions described for the independence of the weighted bounded configuration value.

As in the previous section, we can study the extension of known coalitional solutions to coalition configurations. In this case, it is relevant to study the Owen value (1977), which is the only symmetric Levy-McLean value. In our case, however, there exist two types of players: the non-split players and the split player. Given this, our generalization will require, apart from generalizing a Levy-McLean value, to also generalize the the Owen value for non-split players. We call this property Standard for Coalition Structures.

Standard for Coalition Structures. There exists $\omega \in \mathbb{R}^A$ such that

$$\psi(\mathcal{B}, v) = \eta^{\omega_N, \mathcal{B}}(v)$$

for all $(\mathcal{B}, v) \in \mathbf{BG}_0^N$, and, when $N \subseteq U$, $\psi(\mathcal{B}, v)$ coincides with the Owen value of $(\mathcal{B}, v)$.

Notice that this property only requires coincidence with the Owen value when $N \subseteq U$. In our context, it makes sense to treat players in U and players in U^* in a different way, because split players in U^*, as potential representatives, are qualitatively different than non-split players in U.

If we add Standard for Coalition Structures in Theorem 6 and substitute Partnership Additivity by Equally Partnership Additivity, we obtain a new characterization of $\phi^{\mathbf{e}}$.

Theorem 8. *A solution ψ on* **BG** *satisfies Efficiency, Merger, Equally Partnership Additivity and Standard for Coalition Structures if and only if $\psi = \phi^{\mathbf{e}}$.*

Proof. It is straightforward to prove that if ψ is the bounded equally split value then it satisfies Standard for Coalition Structures. Moreover, it follows from Theorem 6 and Theorem 7 that it also satisfies the other properties.

Assume now that ψ satisfies these properties. Since Standard for Coalitions Structures implies Extension of the Levy-McLean value, from Theorem 6 there exists ω such that $\psi = \phi^{\omega}$. By Standard for Coalitions Structures, we have $\psi_i\left(\{U\}, u^U\right) = \frac{1}{|U|}$ for all $i \in U$. On the other hand, $\phi_i^{\omega}\left(\{U\}, u^U\right) = \frac{\omega_i}{\sum_{j \in U} \omega_j}$. Hence, $\omega_i = \omega_0$ for all $i \in U$. We now use Equally Partnership Additivity as in the proof of Theorem 7 to conclude the result. ■

These properties are independent. Solution $\phi^{1,\mathbf{e}}$ defined as $\phi_i^{1,\mathbf{e}}(\mathcal{B}, v) = \frac{1}{|\mathcal{B}^i|}\phi_i^{\mathbf{e}}(\mathcal{B}, v)$ for all $i \in N$ satisfies all the properties but Efficiency. Solution g satisfies all the properties but Merger. The dual configuration value satisfies all the properties but Equally Partnership Additivity. The weighted bounded configuration value with ω defined as in Example 2 for some $\varpi \in \mathbb{R}_+^U$ with all its coordinates different satisfies all the properties but Standard for Coalition Structures.

6. CONCLUSIONS

In this paper, we define and characterize a family of values for cooperative games in which a player can participate in more than one coalition. As opposed to other values in the literature, we assume that a player splits her weight (identified as effort or size) when she participates in two or more coalitions.

Some subtle difficulties arise when formalizing this approach, due to the presence of split players that represent others. We proceed by defining a set of potential players that do not allow representative players to split again. This approach allows us to maintain the original definitions of the other solutions in the literature, so that they can be compared to the new one.

In particular, our family is a generalization of the weighted coalition structure values with intercoalitional symmetry studied in Levy and McLean (1989), which also generalize the Owen value (1977) and the Shapley value (1953). Our family constitutes an alternative to the values defined by Albizuri et. al (2006), where players do not split their weights when they participate in more than one coalition. Furthermore, we also present and characterize the only symmetric member of this family.

In the next table we summarize our results and compare the properties satisfied by the configuration value ϕ and the dual configuration value ϕ^* presented in Albizuri et al. (2006), and the weighted bounded configuration values ϕ^{ω} and the symmetric bounded equally split value $\phi^{\mathbf{e}}$ presented in this paper.

Property	ϕ	ϕ^*	ϕ^ω	$\phi^{\mathbf{e}}$
Linearity	OK*	OK*	OK*	OK*
Efficiency	OK*	OK*	OK*+	OK*+
Coalition Configuration Positivity	OK	OK	OK*	OK
Intercoalitional Symmetry	OK*	OK*	OK*	OK*
Intracoalitional Partnership	OK	OK	OK*	OK*
Null Player Axiom	OK*	OK*	OK	OK
Null Players Out	OK	OK	OK*	OK*
Merger (as defined in this paper)	OK	OK	OK*+	OK*+
Extension of the Levy-McLean value	OK	OK	OK+	OK+
Partnership Additivity	no	no	OK*+	OK
Equally Partnership Additivity	no	no	no	OK*+
Intracoalitional Anonymity	OK	OK	no	OK*
Standard for Coalition Structures	OK	OK	no	OK*+
Anonymity (as defined in Albizuri et al. (2006))	OK*	OK*	no	no
Merger (as defined in Albizuri et al. (2006))	OK*	no	no	OK
Merger* (as defined in Albizuri et al. (2006))	no	OK*	OK	OK

Symbol "*" (resp. "+") means that this property, together with the others with "*" (resp. "+") in the same column, provides a characterization result.

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