# On the effect of taxation in the online sports betting market*

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**Abstract**

We analyze the effect of taxation in the online sport betting market. A relevant characteristic of this market is its negligible marginal cost on bet volume. Taxation can be on gross profit (Gross Profit Tax) or on volume (General Betting Duty). We model the two most popular online sport betting bets: fixed-odds and spread, as compared with another traditional sport betting: parimutuel. We characterize the odds and the bookmaker's payoff in (strong) subgame perfect equilibrium for each of the three types of bets under both taxation schemes. The results show that taxation on gross profit maximizes the utilitarian social welfare. Moreover, the three types of bets are equivalent when the market is symmetric.

**Keywords**: Taxation; Online betting market; Sport betting; Bookmaker.

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# 1 Introduction

A remarkable feature of online betting (which includes sports, casino and card games such as poker) is that their operators require little more than an internet web page to enter a new market. As opposed to offline operators, they do not need to open physical selling points. Under an unregulated market, the cost of offering a bet is inelastic with respect to the bet volume, i.e. the total sum of betting stakes. This is because online betting users can bet from any internet terminal, even at home. As opposed, offline betting users need to be physically at a selling point.

Things may be different in a regulated market. Over the last years, many European countries have been regulating their online betting and gaming sector. However, this regulation has not been done in a uniform way throughout the different countries.

In general, the basic taxation scheme is based on two types of taxes: the General Betting Duty (GBD) is levied as a proportion of betting stakes; whereas the Gross Profits Tax (GPT) is levied as a proportion of the net revenue of the operators.

Some examples: the United Kingdom applied a 6.75% tax on GBD until October 2001, when it was replaced by a 15% tax in GPT (National Audit Office, 2005). Italy applies a 2%-5% tax on GBD (Ficom Leisure, 2011) for general sport betting and a 20% tax on GPT for spread bets (PwC, 2011). France applies a 8.5% tax on GBD (Global Betting and Gaming Consultancy, 2011) since 2010. In Germany, tax rates largely depend on the respective federal state, and they vary between 20% and 80% on GPT plus a 5% federal tax on GBD Hofmann and Spitz (2015). In 2011, Spanish authorities approved a law[1] that applies a 25% tax on GPT for some types of sports bets and a 22% tax on GBD for others, plus a 0.1% tax on GBD. In table 1 we summarize the data.

[1] Ley 13/2011, de 27 de mayo, de regulación del juego (in Spanish). Boletín Oficial del Estado 127, ref. BOE-A-2011-9280. Available at `https://www.boe.es/buscar/pdf/2011/BOE-A-2011-9280-consolidado.pdf`

Table 1: Taxation schemes for online sports betting

| Country | General Betting Duty (GBD) | Gross Profits Tax (GPT) |
|---|---|---|
| UK | 6.75% (until 2001) | 15% (since 2001) |
| Italy | 2-5% (general) | 20% (spread) |
| France | 8.5% | - |
| Germany | 5% | 20-80% |
| Spain | 22% (parimutuel)+0.1% (all) | 25% (fixed odds and spread) |

In the Spanish case, GBD has been the taxation scheme in the most traditional offline sport betting (*la quiniela*), which takes a parimutuel structure.

In a parimutuel market, a winning bet pays off a proportional share of the total stake on all outcomes. However, the most popular online sport operators are specialized in another two markets: Fixed-odds and spread. In a fixed-odd market, the operator sets the odds for each possible outcome of the match, and the bettors decide whether they accept or not these odds. In a spread market, the operator acts as an intermediator among the users, who bargain the odds.

For sport matches, a bet of 1 monetary unit on a particular team yields a return of $\frac{1}{\pi}$ monetary units in case the team wins the match, and 0 otherwise. In this context, an odd $\pi \in (0, 1)$ is defined as the probability assigned by the market. Notice that any risk-neutral bettor would find it profitable to bet at odd $\pi$ when her private probability estimation is higher than $\pi$.

In parimutuel and spread bets, the operator's profit comes from a commission on either the amount at risk or the winning amount (typically around 5% in online spread operators). In fixed-odds bets, the operator's profit comes from the odds, which should sum up more than 100%[2] for all the possible outcomes of the sport match[3].

[2]In case the odds summed up less than 100%, it would be possible, by betting an appropriate amount of money on each possible match outcome, to win a positive amount irrespectively of the final match outcome.

[3]The sum of the odds, called *overround*, provides a way to measure the operator ad-

In this paper, we model the three types of market in a general setting. The regulator decides on the general taxing scheme (either GBD or GPT) and the operators decide on their commission (parimutuel and spread operators) or odds (fixed-odds operators). We assume that the spread betting commission is applied to the winning bets (as it is typical in online spread operators), whereas commission in parimutuel betting applies to the total amount (as in the Spanish regulation).

We show that, from the online bettor's point of view, it is preferable a GPT scheme, in the following sense: In equilibrium, the odds are not affected by the taxation under GPT; whereas a GBD scheme would reduce the odds and hence the bettors' utility.

These results agree with the ones presented by Smith (2000) and Paton et al. (2002, 2001), whom analyse the effect of the different taxation schemes in Australian, UK and USA betting markets. These results, however, are more focused on offline betting operators and government revenue. Moreover, they take into account the marginal cost of each bet. As opposed, we assume that these marginal costs are negligible.

There are other works that focus on parimutuel markets: Ottaviani and Sørensen (2009) provide a model that explains the empirical evidence of underdogs overbet. These authors argue that this bias may be due to privately informed bettors. As opposed, we prove (Corollary 3.1) that the spread bet operator would get a higher profit if the underdog wins the game.

Other works concentrate on fixed-odd markets. For example, Bag and Saha (2011, 2016) study the externalities due to bribery in sports; and Levitt (2004) argues that the operators may achieve higher profits by an accurate prediction of the match outcome.

As far as we know, no similar research has been addressed for spread markets.

The rest of the paper is organized as follows. In Section 2 we describe the model. In Section 3, we characterize the equilibrium payoffs in each

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vantage.

of the markets. In Section 4 we provide the main results. In Section 5, we study the symmetric case. In Section 6, we present some concluding remarks. Technical proofs are deferred to the Appendix.

## 2 The model

Two teams (*home* and *away*) play a competitive sport match; the match being drawn is not a possibility.

There are three types of agents in the model: A continuum set $\mathcal{B}$ of *bettors* are interested in betting, but only if the odds are attractive; a finite set $K$ of *bookmakers* that offer bets; and a *regulator* (Government) that decides on taxes.

We assume that bettors are risk neutral and try to maximize their expected profit. Each bettor $i \in \mathcal{B}$ is characterized by her individual belief (i.e. the probability) $x_i$ that the home team wins ($1 - x_i$ is the probability that the away team wins); $x_i$ is distributed following an absolutely continuous cdf with probability density function $f$ and full support over $(0, 1)$. The bookmakers, on the other side, also want to maximize their own profit. We consider two possibilities:

**Risk-adverse case:** Bookmakers do not have any belief on the true probability for the home team to win. Hence, it is not possible to estimate an expected profit for them. Instead, we assume that each bookmaker tries to maximize her monetary profit under the worst possible outcome of the match.[4]

**Risk-neutral case:** Bookmakers have a precise common estimation of the true probability $q \in (0, 1)$ for the home team to win. This estimation may arise from their own expertise on the sport discipline plus a detailed study of the match, or by a previous sampling among users with

[4]There are other posible decision criteria, as for example the Hurwicz's rule (see Section 6). We study the maximin case (maximizing profits under the worst case scenario) due to its simplicity.

the most accurate bet record, or both. In this case, we assume that each bookmaker is risk-neutral and tries to maximize her expected final monetary profit.

In each case, there are three possible types of bookmakers: Fixed-odd bookmakers, spread bookmakers and parimutuel bookmakers. Fixed-odd bookmakers decide odds $\pi^H, \pi^A \in [0, 1]$ such that any bettor that bets on the home (away) team receives $\frac{1}{\pi^H} - 1$ ($\frac{1}{\pi^A} - 1$) in case of home (away) win, and $-1$ in case of away (home) win. Spread bookmakers decide a commission $c$ on the profit of any winning bettor. Parimutuel bookmakers decide a commission $c$ on the stake of any bettor.

The third type of agent is the regulator, or Government, that looks for the social welfare via taxes. We consider that the regulator has the same attitude towards risk as the bookmaker, i.e. risk-adverse when the bookmakers are risk-adverse, and risk-neutral when the bookmakers are risk-neutral. As a way to measure social welfare, we consider two criteria: the total tax income and the utilitarian social welfare function. Our aim is to estimate the optimal tax (GBD and/or GPT) in order to maximize each of these two criteria.

## 2.1 The non-cooperative game

Assume the regulator announces a tax, that could be a percentage $\upsilon$ on volume (GBD), a percentage $\rho$ on gross profit (GPT), or both. The (non-cooperative) game has two steps:

**Step 1** Each bookmaker $k \in K$ observes $\upsilon$ and $\rho$ and announces her odds (fixed-odds) or commission (spread/parimutuel). Let $s_k$ denote this choice and let $s_K = (s_k)_{k \in K}$.

**Step 2** Each bettor $i \in \mathcal{B}$ observes $s_K$, and chooses whether to participate or not, and, in the former case, with which bookmaker and in which team she bets on. Let $s_i(s_K)$ denote this choice.

For each $L \subseteq K$, let $s_L = (s_k)_{k \in L}$ denote a strategy profile for bookmakers in $L$. Analogously, for each $\mathcal{C} \subseteq \mathcal{B}$, let $s_{\mathcal{C}}(\cdot) = (s_i(\cdot))_{i \in \mathcal{C}}$ denote a strategy profile for bettors in $\mathcal{C}$.

Following Neyman (2002), we assume that, for any bettors' strategy profile, the set $\mathcal{C}$ of bettors that give any particular signal is always Borel-measurable[5], and we denote its volume as $\|\mathcal{C}\|$.

For any set $S$, we denote as $\mathbb{R}^S$ the Euclidean space where the coordinates are indexed by the elements of $S$. Given an admissible strategy profile $s = (s_K, s_{\mathcal{B}}(\cdot))$, we denote as $u(s) \in \mathbb{R}^{K \cup \mathcal{B}}$, or simply $u$, the final payoff allocation of the noncooperative game.

## 2.2 The equilibrium concept

We will work with the standard concept of subgame perfect equilibrium and a natural extension of it, named bettor-strong subgame perfect equilibrium. Notice that the only proper subgames arise in Step 2.

**Definition 2.1** *A strategy profile $s = (s_K, s_{\mathcal{B}}(\cdot))$ is a* subgame perfect equilibrium *if two conditions hold:*

1. *For each $i \in \mathcal{B}$, each bookmakers' strategy profile $\widetilde{s}_K$ and all bettor $i$'s strategy $\widetilde{s}_i(\cdot)$,*

$$u_i\left(\widetilde{s}_K, \widetilde{s}_i\left(\widetilde{s}_K\right), s_{\mathcal{B}\setminus\{i\}}\left(\widetilde{s}_K\right)\right) \leq u_i\left(\widetilde{s}_K, s_{\mathcal{B}}\left(\widetilde{s}_k\right)\right).$$

2. *For each $k \in K$ and all bookmaker $k$'s strategy $\widetilde{s}_k$,*

$$u_k\left(\widetilde{s}_k, s_{K\setminus\{k\}}, s_{\mathcal{B}}\left(\widetilde{s}_k, s_{K\setminus\{k\}}\right)\right) \leq u_k\left(s_K, s_{\mathcal{B}}\left(s_k\right)\right).$$

The first part of the definition states that no bettor has incentives to deviate in Step 2, even if the bookmakers did. The second part states that no bookmaker has incentives to deviate in Step 1.

[5]This is done in order to avoid meaningless strategies such as, for example, to bet iff $x_i$ is a rational number.

Subgame perfect equilibria is a standard solution concept. However, we will also focus on a refinement of it. Notice that one of the assumptions is that each bettor has the same amount of money to bet. But this is obviously not a realistic assumption. Another interpretation is that the bettors are in fact minimal bet stakes, or coins, willing to be spend by the actual users, each of them owning many coins. Hence, it is obvious that different coins can coordinate their strategies, being held by the same user. Our next definition of equilibrium allows to capture this coordination. It also covers situations where the bettors increase their stakes when the bookmaker improves the odds (or decreases the commission).

**Definition 2.2** *A strategy profile* $s = (s_K, s_{\mathcal{B}}(\cdot))$ *is a* bettor-strong subgame perfect equilibrium *if two conditions hold:*

1. *For each* $\mathcal{C} \subseteq \mathcal{B}$*, all bookmakers' strategy* $\widetilde{s}_K$ *and all strategy profile* $\widetilde{s}_{\mathcal{C}}(\cdot)$*,*
$$u_i\left(\widetilde{s}_K, \widetilde{s}_{\mathcal{C}}\left(\widetilde{s}_K\right), s_{\mathcal{B}\setminus\mathcal{C}}\left(\widetilde{s}_K\right)\right) \leq u_i\left(\widetilde{s}_K, s_{\mathcal{B}}\left(\widetilde{s}_k\right)\right)$$
*for all* $i \in \mathcal{C}$*.*

2. *For each* $k \in K$ *and all bookmaker* $k$*'s strategy* $\widetilde{s}_k$*,*
$$u_k\left(\widetilde{s}_k, s_{K\setminus\{k\}}, s_{\mathcal{B}}\left(\widetilde{s}_k, s_{K\setminus\{k\}}\right)\right) \leq u_k\left(s_K, s_{\mathcal{B}}\left(s_k\right)\right).$$

The first part of the definition states that no coalition of bettors has incentives coordinate in order to deviate in Step 2, even if the bookmakers did. The second part states that no bookmaker has incentives to deviate in Step 1.

# 3 Characterization of equilibria

In this section, we study the equilibrium payoff in each of the three bet markets and each attitude towards risk. We distinguish two possible scenarios: The monopolistic market and the competitive market. We say that the market is monopolistic when there exists a unique bookmaker. Remarkably, the results change drastically when we add a second one. In particular, the market becomes competitive with two bookmakers. There are no further changes in payoffs when adding a third, fourth, an so on. Hence, we define competitive market as that in which there are more than one bookmaker.

The monopolistic market does not only cover situations where there is an actual monopoly. The licensees in a particular country are offering bets continuously and the non-cooperative game that we model can be seen as just a particular instance of a game that is repeatedly played. As it is well-know from the theory of repeated games (Aumann and Shapley, 1994; Rubinstein, 1994; Joosten et al., 2003), almost any individual rational payoff is supported by subgame perfect equilibria. Hence, the bookmakers can eventually cooperate, even without forming a cartel, and ending up offering the bets of the monopolistic market.

## 3.1 Fixed-odds bookmakers

In the fixed-odd case, each bookmaker $k \in K$ chooses odds $\pi_k^H$ and $\pi_k^A$, i.e. $s_k = \left(\pi_k^H, \pi_k^A\right) \in [0,1] \times [0,1]$. Each bettor $i \in \mathcal{B}$ observes the odds and chooses $s_i(s_K) \in \{D\} \cup K \times \{H, A\}$ with the following interpretation:

- If $s_i(s_K) = D$, bettor $i$ declines to bet (abstains) and her final payoff is zero.
- If $s_i(s_K) = (k, H)$, bettor $i$ bets for the home team at odd $\pi_k^H$ and her final payoff is

$$u_i = \left(\frac{1}{\pi_k^H} - 1\right) x_i + (-1)(1 - x_i).$$

- If $s_i(s_K) = (k, A)$, bettor $i$ bets for the away team at odd $\pi_k^A$ and her final payoff is

$$u_i = (-1) x_i + \left(\frac{1}{\pi_k^A} - 1\right)(1 - x_i).$$

For each $k \in K$, let

$$\mathcal{B}_k^H = \{i \in \mathcal{B} : s_i(s_K) = (k, H)\}$$
$$\mathcal{B}_k^A = \{i \in \mathcal{B} : s_i(s_K) = (k, A)\}$$

and let $h_k = ||\mathcal{B}_k^H||$ and $a_k = ||\mathcal{B}_k^A||$ be their respective volumes. Then, bookmaker $k$'s final payoff is

$$\begin{aligned} u_k &= (1-\rho)\min\left\{(1-\upsilon)(h_k+a_k) - \frac{1}{\pi_k^H}h_k, (1-\upsilon)(h_k+a_k) - \frac{1}{\pi_k^A}a_k\right\} \\ &= (1-\rho)\left((1-\upsilon)(h_k+a_k) - \max\left\{\frac{1}{\pi_k^H}h_k, \frac{1}{\pi_k^A}a_k\right\}\right) \end{aligned} \tag{1}$$

in the risk-adverse case and

$$\begin{aligned} u_k &= (1-\rho)\left((1-\upsilon)(h_k+a_k) - \frac{q}{\pi_k^H}h_k - \frac{1-q}{\pi_k^A}a_k\right) \\ &= (1-\rho)\left(1-\upsilon-\frac{q}{\pi_k^H}\right)h_k + (1-\rho)\left(1-\upsilon-\frac{1-q}{\pi_k^A}\right)a_k \end{aligned} \tag{2}$$

in the risk-neutral case.

The next result characterizes the (bettor-strong) subgame perfect equilibrium in the monopolistic case:

**Proposition 3.1** *Given $\upsilon$ and $\rho$, there exists a (bettor-strong) subgame perfect equilibrium in the fixed-odds monopolistic market. In equilibrium, each bettor $i \in \mathcal{B}$ bets for the home team if $x_i > \pi^H$, for the away team if $x_i < 1 - \pi^A$, and declines to bet otherwise, where $\pi^H$ and $\pi^A$ are characterized by the following maximization problems:*

**Risk-adverse case:**

$$\max\left(1-\upsilon-\frac{1}{\pi^H+\pi^A}\right)\left(\int_{\pi^H}^{1} f(t)\,dt + \int_0^{1-\pi^A} f(t)\,dt\right) \tag{3}$$

*subject to*

$$\frac{1}{\pi^H}\int_{\pi^H}^{1} f(t)\,dt = \frac{1}{\pi^A}\int_0^{1-\pi^A} f(t)\,dt \tag{4}$$
$$\pi^H, \pi^A \in [0,1], \pi^H + \pi^A \geq 1.$$

**Risk-neutral case:**

$$\pi^H \in \arg\max_{\pi\in(0,1]} \left(1-\upsilon-\frac{q}{\pi}\right)\int_{\pi}^{1} f(t)\,dt \tag{5}$$

$$\pi^A \in \arg\max_{\pi\in(0,1]} \left(1-\upsilon-\frac{1-q}{\pi}\right)\int_{0}^{1-\pi} f(t)\,dt. \tag{6}$$

**Proof.** See Appendix. ■

From the previous result, we see that a bookmaker looks to balance the positive effect of a large volume (given by $\int_{\pi^H}^1 f(t)\,dt$ and $\int_0^{1-\pi^A} f(t)\,dt$) against the negative effect of a big prize (given by $\frac{1}{\pi^H}\int_{\pi^H}^1 f(t)\,dt = \frac{1}{\pi^A}\int_0^{1-\pi^A} f(t)\,dt$ in the risk-adverse case, and by $\frac{q}{\pi}$ and $\frac{1-q}{\pi}$ in the risk-neutral case). A large volume is obtained by setting low $\pi^H$ and low $\pi^A$. A low prize is obtained by setting high $\pi^H$ and high $\pi^A$.

The effect of $\rho$ (tax on profit) is irrelevant for the maximization problem. Hence, the optimal $\pi_k^H$ and $\pi_k^A$ are independent of the chosen $\rho$. A different issue happens with $\upsilon$, which gives less weight to the positive effect of a large volume. This suggests that the bookmaker would set a higher $\pi_k^H$ (and a higher $\pi_k^A$) a larger $\upsilon$ is, which means that the utility of bettors is reduced.

The next result characterizes the (bettor-strong) subgame perfect equilibrium in the competitive case:

**Proposition 3.2** *Given $\upsilon$ and $\rho$, there exists a (bettor-strong) subgame perfect equilibrium in the fixed-odds competitive market. In equilibrium, the final payoff for each bookmaker is zero. The optimal odds in equilibrium, $\pi^H$ and $\pi^A$, are proposed by at least two bookmakers, who clear the market, and are characterized as follows:*

**Risk-adverse case:** *Equation (4) and* $\pi^H + \pi^A = \min\left\{2, \frac{1}{1-\upsilon}\right\}$.

**Risk-neutral case:** $\pi^H = \max\left\{1, \frac{q}{1-\upsilon}\right\}$ *and* $\pi^A = \max\left\{1, \frac{1-q}{1-\upsilon}\right\}$.

*In equilibrium, each bettor $i \in \mathcal{B}$ bets for the home team if $x_i > \pi^H$, for the away team if $x_i < 1-\pi^A$, and declines to bet otherwise.*

**Proof.** See Appendix. ■

Again, the effect of $\rho$ (tax on profit) is irrelevant. The optimal $\pi^H$ and $\pi^A$ are independent of the chosen $\rho$. As opposed, a higher $\upsilon$ increases the overround $\pi^H + \pi^A$, which means that the utility of bettors is reduced.

## 3.2 Spread bookmakers

In the spread case, each bookmaker $k \in K$ chooses commission $c_k \in (0,1)$, i.e. $s_k = c_k \in (0,1)$. Each bettor $i \in \mathcal{B}$ observes $s_K$ and chooses $s_i(s_K) \in \{D\} \cup K \times \{H, A\} \times (0,1)$ with the following interpretation:

- If $s_i(s_K) = D$, bettor $i$ declines to bet and her final utility is zero.
- If $s_i(s_K) = (k, H, \pi^H)$, bettor $i$ declares that she wants to bet in $k$ for the home team at odd at most $\pi^H$.
- If $s_i(s_K) = (k, A, \pi^A)$, bettor $i$ declares that she wants to bet in $k$ for the away team at odd at most $\pi^A$.

Each bookmaker $k \in K$ matches $(k, H, \pi^H)$-bettors with $(k, A, \pi^A)$-bettors that satisfy $\pi^H \geq 1 - \pi^A$ with odds $\pi_k$, $1 - \pi_k$ such that: $\pi_k \leq \pi^H$ and $1 - \pi_k \leq \pi^A$. The matching is done in such a way that each $\pi_k$ volume of $(k, H, \pi^H)$-bettors is matched with a $1-\pi_k$ volume of $(k, A, \pi^A)$-bettors. The reason is that, in case home team wins, a $1-\pi_k$ volume of money is transferred from $(k, A, \pi^A)$-bettors to $(k, H, \pi^H)$-bettors, so that each $(k, H, \pi^H)$-bettor receives a gross winning (profit + bet):

$$\frac{1-\pi_k}{\pi_k} + 1 = \frac{1}{\pi_k} \geq \frac{1}{\pi^H}$$

hence granting their request to bet for the home team at odd at least $\pi^H$.

Analogously, in case away team wins, a $\pi_k$ volume of money is transferred from $(H, \pi^H)$-bettors to $(A, \pi^A)$-bettors, so that each $(A, \pi^A)$-bettor receives a gross winning (profit + bet):

$$\frac{\pi_k}{1-\pi_k} + 1 = \frac{1}{1-\pi_k} \geq \frac{1}{\pi^A}$$

hence granting their request to bet for the away team at odd at least $\pi^A$.

Hence, $\pi_k$ is chosen so that

$$\begin{aligned}(1-\pi_k)\left\|\mathcal{B}_k^H \cup \overline{\mathcal{B}}_k^H\right\| &\geq \pi_k \left\|\mathcal{B}_k^A\right\| \\ \pi_k \left\|\mathcal{B}_k^A \cup \overline{\mathcal{B}}_k^A\right\| &\geq (1-\pi_k)\left\|\mathcal{B}_k^H\right\|\end{aligned}$$

where

$$\begin{aligned}\mathcal{B}_k^H &= \left\{i \in \mathcal{B} : s_i = \left(k, H, \pi^H\right), \pi^H > \pi_k\right\} \\ \overline{\mathcal{B}}_k^H &= \left\{i \in \mathcal{B} : s_i = (k, H, \pi_k)\right\} \\ \mathcal{B}_k^A &= \left\{i \in \mathcal{B} : s_i = \left(k, A, \pi^A\right), \pi^A > 1-\pi_k\right\} \\ \overline{\mathcal{B}}_k^A &= \left\{i \in \mathcal{B} : s_i = (k, A, 1-\pi_k)\right\}.\end{aligned}$$

If $s_i = \left(k, H, \pi^H\right)$ with $\pi^H > \pi_k$, bettor $i$ bets in $k$ for the home team at odd $\pi$ and her final payoff is

$$u_i = (1-c_k)\left(\frac{1}{\pi_k} - 1\right)x_i + (-1)(1-x_i). \tag{7}$$

If $s_i = \left(k, A, \pi^A\right)$ with $\pi^A > 1-\pi_k$, bettor $i$ bets in $k$ for the away team at odd $1-\pi_k$ and her final payoff is

$$u_i = (-1)x_i + (1-c_k)\left(\frac{1}{1-\pi_k} - 1\right)(1-x_i). \tag{8}$$

If $s_i = D$, or $s_i = \left(k, H, \pi_k^H\right)$ with $\pi^H < \pi_k$, or $s_i = \left(k, A, \pi^A\right)$ with $\pi^A < 1-\pi_k$, bettor $i$ does not bet and her final payoff is zero.

When $s_i = (k, H, \pi_k)$ or $s_i = (k, A, 1-\pi_k)$, we have two cases:

**Case 1**: $\frac{\left\|\mathcal{B}_k^H \cup \overline{\mathcal{B}}_k^H\right\|}{\pi_k} \leq \frac{\left\|\mathcal{B}_k^A \cup \overline{\mathcal{B}}_k^A\right\|}{1-\pi_k}$. If $s_i = (k, H, \pi_k)$, then bettor $i$ bets for the home team and her final payoff is (7). If $s_i = (k, A, 1-\pi_k)$, then bettor $i$ bets in $k$ for the away team with probability $p_A = \frac{1-\pi_k}{\pi_k}\frac{\left\|\mathcal{B}_k^H \cup \overline{\mathcal{B}}_h^H\right\|}{\left\|\overline{\mathcal{B}}_k^A\right\|} - \frac{\left\|\mathcal{B}_k^A\right\|}{\left\|\overline{\mathcal{B}}_k^A\right\|}$ and her final payoff is

$$u_i = \left[(-1)x_i + (1-c_k)\left(\frac{1}{1-\pi_k} - 1\right)(1-x_i)\right]p_A.$$

**Case 2**: $\frac{\left\|\mathcal{B}_k^H \cup \overline{\mathcal{B}}_k^H\right\|}{\pi_k} \geq \frac{\left\|\mathcal{B}_k^A \cup \overline{\mathcal{B}}_k^A\right\|}{1-\pi_k}$. If $s_i = (k, A, 1-\pi_k)$, then bettor $i$ bets in $k$ for the away team and her final payoff is (8). If $s_i = (k, H, \pi_k)$, then bettor $i$ bets in $k$ for the home team with probability $p_H = \frac{\pi_k}{1-\pi_k} \frac{\left\|\mathcal{B}_k^A \cup \overline{\mathcal{B}}_k^A\right\|}{\left\|\overline{\mathcal{B}}_k^H\right\|} - \frac{\left\|\mathcal{B}_k^H\right\|}{\left\|\overline{\mathcal{B}}_k^H\right\|}$ and her final payoff is

$$u_i = \left[(1-c_k)\left(\frac{1}{\pi_k} - 1\right) x_i + (-1)(1-x_i)\right] p_H.$$

We describe this protocol in the following examples:

**Example 3.1** *Assume* $f(x) = 1$ *for all* $i \in \mathcal{B}$, $\|\mathcal{B}\| = 1$ *and* $K = \{k\}$ *and each bettor* $i \in \mathcal{B}$ *announces* $(k, H, x_i)$ *if* $x_i > 0.5$ *and* $(k, A, 1-x_i)$ *if* $x_i < 0.5$. *Under these bets,* $\pi_k = 0.5$ *clears the market, so that the ratio of* $H$*-bettors to* $A$*-bettors should be* 1. *Moreover,* $\left\|\mathcal{B}_k^H\right\| = \left\|\mathcal{B}_k^A\right\| = 0.5$ *and* $\left\|\overline{\mathcal{B}}_k^H\right\| = \left\|\overline{\mathcal{B}}_k^A\right\| = 0$. *Hence, there exists no excess of* $H$*-bettors nor* $A$*-bettors. All bettors will be matched. In particular, the whole* 0.5 *volume of* $(k, H, x_i)$*-bettors matches the* 0.5 *volume of* $(k, A, x_i)$*-bettors.*

**Example 3.2** *Assume* $\|\mathcal{B}\| = 1$ *and* $K = \{k\}$ *and the bets are* $D$, $(k, H, 0.4)$, $(k, H, 0.6)$, $(k, H, 0.8)$, $(k, A, 0.2)$, $(k, A, 0.4)$, *and* $(k, A, 0.6)$ *with volumes* 0.2, 0.1, 0.3, 0.1, 0.1, 0.1, *and* 0.1, *respectively, as shown in the first two columns of Table 2. Under these bets,* $\pi_k = 0.6$ *clears the market, so that the ratio of* $H$*-bettors to* $A$*-bettors should be* $\frac{0.6}{1-0.6} = \frac{3}{2}$. *Moreover,* $\left\|\mathcal{B}_k^H\right\| = 0.1$, $\left\|\overline{\mathcal{B}}_k^H\right\| = 0.3$, $\left\|\mathcal{B}_k^A\right\| = 0.1$, *and* $\left\|\overline{\mathcal{B}}_k^A\right\| = 0.1$. *Since* $\frac{\left\|\mathcal{B}_k^H \cup \overline{\mathcal{B}}_k^H\right\|}{\pi_k} = \frac{0.4}{0.6} > \frac{0.2}{0.4} = \frac{\left\|\mathcal{B}_k^A \cup \overline{\mathcal{B}}_k^A\right\|}{1-\pi_k}$, *we are in Case 2 and there exists an excess of* $H$*-bettors that will not be matched. In particular, the whole* 0.1 *volume of* $(k, H, 0.8)$*-bettors matches a* $\frac{0.2}{3}$ *volume of* $(k, A, 0.6)$*-bettors; a* 0.05 *volume of* $(k, H, 0.6)$*-bettors matches the remaining* $\frac{0.1}{3}$ *volume of* $(k, A, 0.6)$*-bettors; finally, a* 0.15 *volume of* $(k, H, 0.6)$*-bettors matches the remaining* 0.1 *volume of* $(k, A, 0.4)$*-bettors. The remaining* 0.1 *volume of* $(k, H, 0.6)$*-bettors, the* 0.1 *volume of* $(k, A, 0.2)$*-bettors, and the* 0.1 *volume of* $(k, H, 0.4)$*-bettors remain unmatched.*

We now compute the bookmaker's payoff. Analogously to the previous subsection, we denote $h_k = \left\|\mathcal{B}_k^H\right\|$, $\overline{h}_k = \left\|\overline{\mathcal{B}}_k^H\right\|$, $a_k = \left\|\mathcal{B}_k^A\right\|$, and $\overline{a}_k = \left\|\overline{\mathcal{B}}_k^A\right\|$.

Table 2: Example of a spread market

| Bet | Volume | Matched |
|---|---|---|
| $D$ | 0.2 | No (abstain) |
| $(H, 0.4)$ | 0.1 | No |
| $(H, 0.6)$ | 0.3 | 67% |
| $(H, 0.8)$ | 0.1 | 100% |
| $(A, 0.2)$ | 0.1 | No |
| $(A, 0.4)$ | 0.1 | 100% |
| $(A, 0.6)$ | 0.1 | 100% |
| Total | 1 | 50% |

Now, in case the home team wins, the monetary transfer from $(k, A)$-bettors to $(k, H)$-bettors is

$$\begin{aligned}\alpha_k &= \left(\frac{1}{\pi_k} - 1\right)\left(h_k + \min\{1, p_H\}\,\overline{h}_k\right) \\ &= \frac{1-\pi_k}{\pi_k}\min\left\{h_k + \overline{h}_k, h_k + p_H\overline{h}_k\right\} \\ &= \min\left\{\frac{1-\pi_k}{\pi_k}(h_k + \overline{h}_k), a_k + \overline{a}_k\right\}.\end{aligned}$$

Analogously, in case the away team wins, the monetary transfer from $(k, H)$-bettors to $(k, A)$-bettors is

$$\beta_k = \min\left\{\frac{\pi_k}{1-\pi_k}(a + \overline{a}_k), h_k + \overline{h}_k\right\}.$$

Then, the total volume is $\alpha_k + \beta_k$ and the final payoff for bookmaker $k$ is

$$u_k = (1-\rho)\left(\min\{\alpha_k, \beta_k\}c_k - (\alpha_k + \beta_k)\,\upsilon\right)$$

in the risk-adverse case and

$$u_k = (1-\rho)\left(\left(q\alpha_k + (1-q)\beta_k\right)\right)c - (\alpha_k + \beta_k)\,\upsilon)$$

in the risk-neutral case.

The next result characterizes the bet volume in Step 2 for the spread bets case:

**Lemma 3.1** *Given $c \in [0,1]$, the bet volume and odds that clear the market in equilibrium in the spread bets market are characterized by:*

$$\gamma = \frac{1}{\pi} \int_{\frac{\pi}{1-(1-\pi)c}}^{1} f(t)\, dt = \frac{1}{1-\pi} \int_{0}^{\frac{(1-c)\pi}{1-\pi c}} f(t)\, dt \tag{9}$$

$$\pi \in [0,1].$$

**Proof.** See Appendix. ■

It follows from Lemma 3.1 that, as opposed to fixed-odds, the spread bookmaker are not indifferent to which team will win the match. In fact, the bookmaker would always prefer the underdog (non-favorite) to win the match, as next result shows:

**Corollary 3.1** *Let $\pi$, $1-\pi$ be the odds that clear the market for some spread bookmaker with nonzero bet volume. If $\pi > \frac{1}{2}$, then the bookmaker's ex-post payoff is bigger when the away team wins. If $\pi < \frac{1}{2}$, then the bookmaker's ex-post payoff is bigger when the home team wins. If $\pi = \frac{1}{2}$, then the spread bookmaker's ex-post payoff is independent of which team wins.*

**Proof.** See Appendix. ■

Intuitively, the explanation for this result is the following: The spread bookmaker has only one degree of freedom to modulate the actual thresholds that determine the bets. She can make the $H$ and $A$ bets volumes simultaneously larger or smaller, but not individually in order to equalize both scenarios. The worst-case scenario arises when the favorite team wins. Since commission is applied to prizes, when the favorite team wins, the bet volume is not high enough to compensate the low prize for winning a bet.

The next result characterizes the (bettor-strong) subgame perfect equilibria in the monopolistic case:

**Proposition 3.3** *Given $\upsilon$ and $\rho$, there exists a (bettor-strong) subgame perfect equilibrium with undominated strategies*[6] *in the spread bets monopolistic*

[6] Undominated strategies are required in order to avoid meaningless equilibria of the form "everybody chooses $D$". This refinement is not needed for the bettor-strong subgame pefect equilibrium.

*game. Moreover, the commission and odds in equilibrium are characterized by the following maximization problems:*

**Risk-adverse case:** $\max_{c\in[0,1]} \left(\min\{1-\pi,\pi\}c - \upsilon\right)\gamma$.

**Risk-neutral case:** $\max_{c\in[0,1]} \left((q+\pi-2q\pi)c - \upsilon\right)\gamma$.

*subject (in both cases) to (9). In equilibrium, each bettor $i \in \mathcal{B}$ bets for the home team if $x_i > \frac{\pi}{1-(1-\pi)c}$, for the away team if $x_i < \frac{(1-c)\pi}{1-c\pi}$, and declines to bet otherwise.*

**Proof.** See Appendix. ■

As $c$ increases, the percentage of winners profits increase too, but this winner profit decreases because less bettors participate. Hence, the bookmaker looks to balance the positive effect of a big commission (hence big percentage of winnings) against the negative effect on the winnings (which decreases with $c$).

Like fixed-odds bookmakers, the effect of $\rho$ (tax on profit) is irrelevant for the maximization problem. Hence, the optimal $c$ is independent of the chosen $\rho$. Again, a different issue happens with $\upsilon$, which penalizes the effect of a large volume. Hence, like fixed-odds, the bookmaker would set a higher $c$, which means that the utility of the bettors is reduced.

The next result characterizes the (bettor-strong) subgame perfect equilibria in the competitive case:

**Proposition 3.4** *Given $\upsilon$ and $\rho$, there exists a (bettor-strong) subgame perfect equilibrium with undominated strategies[7] in the spread bets competitive game. In equilibrium, the final payoff for each bookmaker is zero. The minimal commission, $c$, clears the market and is characterized as follows:*

**Risk-adverse case:** $c = \min\left\{1, \frac{\upsilon}{\min\{1-\pi,\pi\}}\right\}$

**Risk-neutral case:** $c = \min\left\{1, \frac{\upsilon}{q+\pi-2q\pi}\right\}$

[7] Again, undominated strategies are not required for the bettor-strong subgame equilibrium.

*subject (in both cases) to (9). In equilibrium,* $c = \min_{k \in K} c_k$*, each bettor* $i \in \mathcal{B}$ *bets in* $k^* \in \arg\min_{k \in K} c_k$ *for the home team if* $x_i > \frac{\pi}{1-(1-\pi)c}$*, for the away team if* $x_i < \frac{(1-c)\pi}{1-c\pi}$*, and declines to bet otherwise.*

**Proof.** See Appendix. ■

Proposition 3.4 requires either bettor-strong subgame perfect equilibria, or subgame perfect equilibria with undominated strategies. There are multiple subgame perfect equilibria with dominated strategies. For example, assume w.l.o.g. $1 \in K$. Then, for any $c \in [0,1]$, let $\pi \in [0,1]$ given by (9), and consider the following strategy profile: $c_1 = c$ and $c_k = 0$ otherwise, and:

- if $\widetilde{c}_1 = c$, then $s_i(\widetilde{c}_K) = (1, H, \pi)$ for all $i \in \mathcal{B}$ such that $x_i > \frac{\pi}{1-(1-\pi)c}$, $s_i(\widetilde{c}_K) = (1, A, 1-\pi)$ for all $i \in \mathcal{B}$ such that $x_i < \frac{(1-c)\pi}{1-c\pi}$, and $s_i(\widetilde{c}_K) = D$ otherwise;
- if $\widetilde{c}_1 \neq c$, then $s_i(\widetilde{c}_K) = D$ for all $i \in \mathcal{B}$.

This is a subgame perfect equilibrium. In words, it says that all bettors will bet in 1 when $c_1 = c$, even if it has not the lowest commission. If bookmaker 1 deviates, then all bettors will abstain. Hence, any $c \in [0,1]$ is supported in a subgame perfect equilibrium.

## 3.3 Parimutuel bookmakers

In the parimutuel case, each bookmaker $k \in K$ chooses commission $c_k \in (0,1)$, i.e. $s_k = c_k \in (0,1)$. Each bettor $i \in \mathcal{B}$ observes $s_K$ and (simultaneously) chooses $s_i(c) \in \{D\} \cup K \times \{H, A\}$ with the following interpretation. Let $\mathcal{B}_k^H = \{i \in \mathcal{B} : s_i = (k, H)\}$ and $\mathcal{B}_k^A = \{i \in \mathcal{B} : s_i = (k, A)\}$:

- If $s_i(s_K) = D$, bettor $i$ declines to bet and her final payoff is zero.
- If $\left\|\mathcal{B}_k^H\right\| = 0$ or $\left\|\mathcal{B}_k^A\right\| = 0$, bets are canceled for bookmaker $k$. The final payoff is zero for bookmaker $k$ and bettors in $\mathcal{B}_k^H \cup \mathcal{B}_k^A$.

- If $s_i(s_K) = (k, H)$, bettor $i$ declares that she wants to bet for the home team in $k$. If $\left\|\mathcal{B}_k^H\right\| > 0$ and $\left\|\mathcal{B}_k^A\right\| > 0$, her final payoff is:

$$u_i = \frac{\left\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\right\|}{\left\|\mathcal{B}_k^H\right\|}(1 - c_k)\, x_i - 1.$$

- If $s_i(s_K) = (k, A)$, bettor $i$ declares that she wants to bet for the away team in $k$. If $\left\|\mathcal{B}_k^H\right\| > 0$ and $\left\|\mathcal{B}_k^A\right\| > 0$, her final payoff is:

$$u_i = \frac{\left\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\right\|}{\left\|\mathcal{B}_k^A\right\|}(1 - c_k)(1 - x_i) - 1.$$

If $\left\|\mathcal{B}_k^H\right\| > 0$ and $\left\|\mathcal{B}_k^A\right\| > 0$, bookmaker $k$'s final payoff is

$$\begin{aligned} u_k &= (1-\rho)\left(\left\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\right\| c_k - \left\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\right\| \upsilon\right) \\ &= (1-\rho)(c_k - \upsilon)\left\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\right\|. \end{aligned}$$

Notice that attitude towards risk is irrelevant in the parimutuel case.

The next result characterizes the (bettor-strong) subgame perfect equilibria in the monopolistic case:

**Proposition 3.5** *Given $\upsilon$ and $\rho$, there exists a unique bettor-strong subgame perfect equilibrium in the parimutuel monopolistic market, where each bettor $i \in \mathcal{B}$ bets for the home team if $x_i > \pi^H$ and for the away team if $x_i < \pi^A$ for some thresholds $\pi^H, \pi^A \in [0,1]$. Moreover, the commission and thresholds in equilibrium are characterized by the maximization problem*

$$\max_{c \in \left[0, \frac{1}{2}\right]} (c - \upsilon)\left(\int_{\pi^H}^{1} f(t)\, dt + \int_{0}^{1-\pi^A} f(t)\, dt\right) \tag{10}$$

*subject to*

$$\begin{aligned} \frac{1}{\pi^H}\int_{\pi^H}^{1} f(t)\, dt &= \frac{1}{\pi^A}\int_{0}^{1-\pi^A} f(t)\, dt & (11) \\ \pi^H + \pi^A &= \frac{1}{1-c} & (12) \\ \pi^H, \pi^A &\in [0,1]. \end{aligned}$$

**Proof.** See Appendix. ■

Proposition 3.5 uses bettor-strong subgame perfect equilibria. There are multiple subgame perfect equilibria, but they will involve an unreasonable coordination among bettors. For example, assume $K = \{k\}$. Then, for any $c^* \in \left[0, \frac{1}{2}\right]$, consider the following strategy profile: $s_k = c^*$ and $s_i(\widetilde{c}_k) = D$ for all $i \in \mathcal{B}$ when $\widetilde{c}_k \neq c^*$; when $\widetilde{c}_k = c^*$, $s_i(\widetilde{c}_k)$ is defined as in Proposition 3.5. This is a subgame perfect equilibrium. Hence, any $c \in \left[0, \frac{1}{2}\right]$ is supported in a subgame perfect equilibrium.

The next result characterizes the bettor-strong subgame perfect equilibria in the competitive case:

**Proposition 3.6** *Given $\upsilon$ and $\rho$, there exists a unique bettor-strong subgame perfect equilibrium payoff allocation in the parimutuel competitive market. In equilibrium, the minimum commission is $\upsilon$, offered by at least two bookmakers, each bookmaker receives zero and each bettor $i \in \mathcal{B}$ bets for the home team if $x_i > \pi^H$ and for the away team if $x_i < \pi^A$ where $\pi^H, \pi^A \in [0,1]$ are characterized by (11) and*

$$\pi^H + \pi^A = \min\left\{2, \frac{1}{1-\upsilon}\right\}. \tag{13}$$

**Proof.** See Appendix. ■

Next result follows from Proposition 3.1, Proposition 3.2, Proposition 3.5 and Proposition 3.6:

**Proposition 3.7** *For any $\upsilon$ and $\rho$, risk-adverse fixed-odds and parimutuel yield the same payoff allocation in bettor-subgame perfect subgame equilibrium.*

**Proof.** See Appendix. ■

# 4 Effect of taxation

We can now state our main results. These results hold for each of the three types of bookmakers: fixed-odds, spread, and parimutuel. The first proposition is implied by the results presented in the previous section. It follows from the fact that $\rho$ does not play any role in the characterization of the equilibria.

**Proposition 4.1** *In a monopolistic market, tax on profit ($\rho$) leaves odds, commissions and bettors' utilities unaffected, and decreases linearly the bookmaker's payoff. The maximum tax income is achieved for $\rho = 1$.*

**Proof.** See Appendix. ■

The second part of Proposition 4.1 simply says that the maximum tax income is achieved when the monopolistic bookmaker is a state-owned company.

As opposed, the role of $\upsilon$ will depend on the particular distribution on the bettors. In general, one may expect that an increase in $\upsilon$ would decrease the bet volume. Hence, the maximum utilitarian social welfare should be achieved for $\upsilon = 0$. We will check it in a particular example after presenting the main result, which describes the effect of taxation in competitive markets.

**Theorem 4.1** *In a competitive market:*

- *a) Taxes on profit ($\rho$) leave odds, commissions, tax income, and utilities unaffected.*
- *b) Taxes on volume ($\upsilon$) increase odds and commission, and reduces the utility of bettors. The utility of bookmakers remains unaffected.*
- *c) Maximum utilitarian social welfare is achieved for $\upsilon = 0$.*
- *d) Maximum tax income is achieved for some $\upsilon \in \left(0, \frac{1}{2}\right)$ in the risk-adverse case, and $\upsilon \in (0, \max\{q, 1-q\})$ in the risk-neutral case.*

**Proof.** See Appendix. ■

Theorem 4.1 provides a range of values where the tax income maximizer can be. The exact value of the maximizing $\upsilon$ will depend on the distribution of bettors given by $f$. On the other hand, we have no complete counterpart for Proposition 4.1 in the monopolistic case, but we can still figure out how it behaviours for some particular function $f$ and (for the risk-neutral case) probability $q$.

For the risk-neutral case, a natural choice for $q$ is the one that agrees with $f$ in the sense that odds $q, 1-q$ will clear the market with maximum bet volume. Next lemma characterizes this $q$, that we denote as $q^*$.

**Lemma 4.1** *There exists a unique $q^*$ such that odds $q^*, 1-q^*$ maximize the bet volume, and it is characterized by*

$$q^* = \int_{q^*}^{1} f(t)dt.$$

**Proof.** See Appendix. ■

For example, when $f$ is symmetric (i.e. $f(x) = f(1-x)$ for all $x \in (0,1)$) it is clear that $q^* = \frac{1}{2}$. When $f(x) = 2x$ for all $x \in (0,1)$, then $q^* = \frac{\sqrt{5}-1}{2} \approx 0.618$.

As a paradigmatic case, assume the allocation of bettors follows the linear distribution $f(x) = 2x$. This distribution represents a match where the home team is favourite. Despite its simplicity, it is nontrivial to prove the results in Proposition 4.1 for the monopolistic market in this particular example. However, a simulation analysis[8] shows the following:

- Taxes on volume ($\upsilon$) increase odds and commission, and reduce the utility of both bettors and bookmaker.
- Maximum utilitarian social welfare is achieved for $\upsilon = 0$.
- In the monopolistic case, maximum tax income is achieved for $\rho = 1$ and $\upsilon = 0$.

The maximum tax income is described in Table 3.

Apart from the risk-adverse spread case, where the bookmaker has no capability to adjust both equilibrium odds, the maximum tax income is similar in all the other markets.

[8]Tested on a sampling of 1000 instances of $\upsilon$ uniformly distributed on $[0,1]$ in each market.

Table 3: Effect of taxation when $f(x) = 2x$ and $q = \frac{\sqrt{5}-1}{2}$.

| Bookmaker | Market | Risk | Max. income | Maximizer |
|---|---|---|---|---|
| Fixed odds | Monopoly | Adverse | 0.143087 | $\rho = 100\%, \upsilon = 0\%$ |
| Fixed odds | Monopoly | Neutral | 0.143852 | $\rho = 100\%, \upsilon = 0\%$ |
| Spread | Monopoly | Adverse | 0.110593 | $\rho = 100\%, \upsilon = 0\%$ |
| Spread | Monopoly | Neutral | 0.143828 | $\rho = 100\%, \upsilon = 0\%$ |
| Parimutuel | Monopoly | - | 0.143087 | $\rho = 100\%, \upsilon = 0\%$ |
| Fixed odds | Competition | Adverse | 0.143087 | $\upsilon = 25\%$ |
| Fixed odds | Competition | Neutral | 0.140669 | $\upsilon = 24\%$ |
| Spread | Competition | Adverse | 0.110531 | $\upsilon = 19\%$ |
| Spread | Competition | Neutral | 0.143828 | $\upsilon = 25\%$ |
| Parimutuel | Competition | - | 0.143087 | $\upsilon = 25\%$ |

# 5 Effect of taxation in the symmetric case

In this section, we study the effect of taxation in the symmetric case, i.e. when $q = \frac{1}{2}$ and $f$ is symmetric:

$$f(x) = f(1 - x)$$

for all $x \in (0, 1)$.

This case covers situations where there is no favourite team in the sport match, or when there exists a favourite but it has a handicap that makes the match even. Such handicap bets are quite common in online betting, and allow the bookmakers to assure that the volume of bets between home and away teams are balanced. In our model, this is particularly relevant for the spread bets bookmaker, since it makes her indifferent of who is the winning team (Corollary 3.1).

The next result characterizes the equilibrium payoffs and states that fixed odds, spread bets and parimutuel are equivalent in the symmetric case.

**Proposition 5.1** *Assume $f(x) = f(1 - x)$ for all $x \in (0, 1)$. Then, fixed-odds, spreads and parimutuel yield the same payoff allocation in equilibrium* *for both risk-adverse case and risk-neutral case with* $q = \frac{1}{2}$*. The optimal odds in the fixed-odds market are* $\pi^H = \pi^A = \pi^*$*, the optimal commission in the spread market is* $c = 2 - \frac{1}{\pi^*}$*, and the optimal commission in the parimutuel market is* $c = 1 - \frac{1}{2\pi^*}$*, where* $\pi^*$ *is given as follows:*

*a) Monopolistic case:* $\pi^* = \arg\max_{\pi \in [\frac{1}{2},1]} \left(2(1-\upsilon) - \frac{1}{\pi}\right) \int_\pi^1 f(t)dt$.

*b) Competitive case:* $\pi^* = \min\left\{1, \frac{1}{2(1-\upsilon)}\right\}$.

**Proof.** See Appendix. ■

The previous result allows us to analyse the effect of $\rho$ and $\upsilon$ on the odds/commissions and the bookmakers' payoffs for a particular cdf. It is still too general for a characterization of the $\upsilon$ that maximizes the tax income. In order to study a relevant example, consider the symmetric beta distribution. Symmetry means that shape parameters $\alpha, \beta$ coincide, $\alpha = \beta$. Hence, the symmetric probability density function is given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{\int_0^1 t^{\alpha-1}(1-t)^{\alpha-1}dt}$$

for some $\alpha \in (0, \infty)$. The beta distribution is a suitable model for a random allocation of percentages (see Evans et al. (2000); Ferrari and Cribari-Neto (2004) and references herein). Hence, it is justifiable to use it for estimating the distribution of bettors' odds. Moreover, the family of symmetric beta distributions is rich enough to cover a wide range of symmetric distributions, including the uniform distribution ($\alpha = 1$), unimodal distributions ($\alpha > 1$) with a unique central peak, and bimodal distributions ($\alpha < 1$) with two peaks at 0 and 1, respectively. The interpretation is that $\alpha > 1$ describes a society where bettors agree that the chances of home win is around $\frac{1}{2}$, and $\alpha > 1$ a society where bettors are divided half-half between those that believe that the home team is favorite, and those that believe that away team is favorite.

As a paradigmatic case, next proposition shows the effect of taxes when $\alpha = 1$, i.e. the uniform distribution $f(x) = 1$ for all $x \in (0, 1)$.

**Proposition 5.2** *Assume* $f(x) = 1$ *for all* $x \in (0, 1)$ *and* $q = \frac{1}{2}$*. Then,*

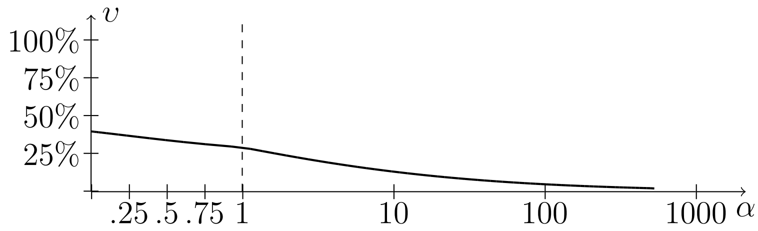


Figure 1: Tax on volume ($\upsilon$) that maximizes tax income in the symmetric ($\alpha = \beta$) competitive market. Scale is linear on $\alpha \in (0, 1)$ and logarithmic on $\alpha \in (1, \infty)$.

*a) Taxes on volume ($\upsilon$) increase odds and commission, and reduces the utility of bettors. In a monopolistic market, they also decrease the utility of the bookmaker when $\rho < 1$.*

*b) Maximum utilitarian welfare is achieved for $\upsilon = 0$.*

*c) Maximum tax income is achieved as follows:*

*c1) In the competitive case, by $\rho = 1$ and $\upsilon = 0$.*

*c2) In the monopolistic case, by $\upsilon = 2 - \frac{\sqrt{2}}{2} \approx 29.3\%$.*

**Proof.** See Appendix. ■

For arbitrary $\alpha \in (0, \infty)$, a simulation analysis shows that Proposition 5.2, parts a), b) and c1), hold in general, and the maximizing $\upsilon$ in the competitive case (Proposition 5.2, part c2)) decreases with $\alpha$. The $\upsilon$ that maximizes tax income in the competitive case is represented in Fig. 1.

The interpretation is that the more agreement among bettors that the probability of local is around 0.5, the smaller the optimal tax is. Reciprocally, when bettors disagree half-half on who the favorite team is, it is easier to extract the utility surplus via taxes.

# 6 Concluding remarks

In this paper, we model three different online betting markets: those given by fixed-odds, spread bets, and parimutuel, respectively. This allows us to analyse the effect of two different tax schemes: On volume (GBD) and on profit (GPT). In all these markets, odds (fixed-odds) and commission (spread bets and parimutuel) are unaffected by GPT but they are by GBD. Hence, it should be expected that odds and commission to depend on the particular regulation. For example, Paddy Power Betfair, which includes one of the largest Internet spread betting companies, charges a different commission for spread bets on each country. This commission is 5% in the United Kingdom, Ireland, Italy, Gibraltar and Malta; 7% in Albany, Armenia, Croatia, Monaco, Serbia, Montenegro and Slovakia; and 6.5% in the rest of the countries, including Spain. Moreover, the company is restricted in Belgium, Greece, Germany[9], Turkey, Israel, France and Portugal, among other countries.

As opposed to other approaches in the literature, we do not need to assume the existence of an actual probability for the home (or away) team to win the match. Instead, the bettors are characterized by their subjective beliefs on this probability. An alternative interpretation is that each bettor is characterized by the the odd at which she is willing to bet, which includes the individual surplus of the act of betting itself. In this sense, a natural extension of the model, which does not change the results, is to assume that there are two subsets of bettors: one of them willing to bet for the home team, another willing to bet for the away team, and both characterized by the minimum odd they will bet.

As for the bookmakers, we cover two situations: either they are risk-adverse and play a maximin strategy (i.e. they maximize profits under the worst match outcome scenario), or they are risk-neutral because of a precise common estimation of the true probability of the match outcome. Assuming

[9]Betfair is only restricted in Germany for spread bets.

there is no such precise estimation, a more general decision criterium than maximin would be the Hurwicz's rule, which uses a weighted average between both match outcomes. Checking the implications of using the Hurwicz's rule is an open question. My own feeling is that the general results remain with a more elaborate characterization of the bet volume in equilibrium (as given by (4)).

Another extension is to consider bettors betting on more than one event simultaneously. Of course, bettors decisions will become more elaborate when they have a limited budget and several matches to choose. Competition among different matches may arise. In fact, this situation is already partially covered by the model, because: 1) distribution $f$ may depend on the existence of other potential matches, and 2) bettors' strategies are not affected when they have no budget restrictions, so that they are able to bet in all the matches they find profitable.

# Appendix

**Proof of Proposition 3.1.** Assume w.l.o.g. $\|\mathcal{B}\| = 1$ and $K = \{k\}$. In Step 2 of the game, it is optimal for any bettor $i \in \mathcal{B}$ with $x_i > \pi_k^H$ to announce $s_i\left(\pi_k^H, \pi_k^A\right) = (k, H)$. Analogously, it is optimal for any bettor $i \in \mathcal{B}$ with $1 - x_i > \pi_k^A$ to announce $s_i\left(\pi_k^H, \pi_k^A\right) = (k, A)$. Hence, $h_k = \int_{\pi_k^H}^1 f(t)\, dt$ and $a_k = \int_0^{1-\pi_k^A} f(t)\, dt$.

**Risk-adverse case:** Equality (4) comes from the fact that the bookmaker wants to minimize $\max\left\{\frac{1}{\pi_k^H} h_k, \frac{1}{\pi_k^A} a_k\right\}$, that is the maximum bettors' winnings when either the home team ($\frac{1}{\pi_k^H} h_k$) or the away team ($\frac{1}{\pi_k^A} a_k$) wins. In equilibrium both amounts should be equal, since otherwise (say, $\frac{1}{\pi_k^H} h_k < \frac{1}{\pi_k^A} a_k$) there would be a profitable deviation by the bookmaker (she could reduce $\pi_k^H$ slightly so that $h_k$ increases without affecting her worst case scenario). For a fixed $\pi_k^A$, there exists a unique $\pi_k^H$ that satisfies (4). To see why, notice that $\phi(\pi) = \frac{1}{\pi} \int_\pi^1 f(t)\, dt$ is a continuous and strictly decreasing function on $\pi \in (0, 1)$ with

$\phi\left(0^{+}\right)=+\infty$ and $\phi\left(1^{-}\right)=0^{+}$, whereas $\psi\left(\pi\right)=\frac{1}{1-\pi}\int_{0}^{\pi}f\left(t\right)dt$ is a continuous and strictly increasing function on $\pi\in(0,1)$ with $\psi\left(0^{+}\right)=0^{+}$ and $\psi\left(1^{-}\right)=+\infty$. Hence, for each $\pi_{k}^{H}$, there exists a unique $\pi_{k}^{A}$ with $\phi\left(\pi_{k}^{H}\right)=\psi\left(1-\pi_{k}^{A}\right)$. Moreover, the larger $\pi_{k}^{H}$ is, the larger $\pi_{k}^{A}$ is. Let $\Pi_{k}^{A}:(0,1)\rightarrow(0,1)$ be the function that assigns to each $\pi_{k}^{H}$ its corresponding $\pi_{k}^{A}$. This function is well-defined, strictly increasing, and it satisfies $\Pi_{k}^{A}\left(0^{+}\right)=0^{+}$ and $\Pi_{k}^{A}\left(1^{-}\right)=1^{-}$. Notice that the bet volume is given by $h_{k}+a_{k}$. Since tax $\upsilon$ applies on volume, and $\rho$ applies on profit, the bookmaker would maximize

$$(1-\rho)\left\{(1-\upsilon)\left(h_{k}+a_{k}\right)-\frac{1}{\pi_{k}^{H}}h_{k}\right\}. \tag{14}$$

Under (4),

$$h_{k}+a_{k}=h_{k}+\frac{\pi_{k}^{A}}{\pi_{k}^{H}}h_{k}=\frac{\pi_{k}^{H}+\pi_{k}^{A}}{\pi_{k}^{H}}h_{k}$$

which implies $\frac{1}{\pi_{k}^{H}}h_{k}=\frac{1}{\pi_{k}^{H}+\pi_{k}^{A}}(h_{k}+a_{k})$. Hence, (14) can be rewritten as

$$(1-\rho)\left(1-\upsilon-\frac{1}{\pi_{k}^{H}+\pi_{k}^{A}}\right)\left(h_{k}+a_{k}\right)$$

which, apart from the irrelevant effect of $1-\rho$, coincides with the desired maximizing function. Furthermore, it is straightforward to check that there exists at least one maximizing $\pi_{k}^{H}\in(0,1)$.

**Risk-neutral case:** The bookmaker looks to maximize (2), which has two terms. The first term only depends on $\pi_{k}^{H}$ and the second one only on $\pi_{k}^{A}$. Hence, the maximal odds are obtained independently, yielding (5) and (6). To see that there exists at least one maximum for (5) (the reasoning for (6) is analogous), notice $\phi(\pi)=\left(1-\upsilon-\frac{q}{\pi}\right)\int_{\pi}^{1}f\left(t\right)dt$ is a continuous function on $(0,1]$ satisfying $\phi(0^{+})=-\infty$ and $\phi(1)=0$. Since $\phi(0^{+})=-\infty$, we deduce that there exists some $\pi_{0}$ such that $\phi(\pi)\leq0$ for all $\pi\in(0,\pi_{0}]$. Existence of a maximum in $[\pi_{0},1]$ is then guaranteed by the Weierstrass extreme value proposition.

■

**Proof of Proposition 3.2.** Assume we are in a subgame perfect equilibrium. Each bookmaker $k \in K$ can assure a final payoff of at least zero by setting $\pi_k^H = \pi_k^A = 1$. Hence, no bookmaker receives a negative payoff. We will prove that each bookmaker receives zero. Assume, on the contrary, that there exists some bookmaker with a positive final payoff. W.l.o.g., we take $1 \in K$ and $u_1 > 0$. Let $\pi_{\min}^H = \min_{k\in K} \pi_k^H$ and $\pi_{\min}^A = \min_{k\in K} \pi_k^A$. Taking into account the optimal strategy of the bettors, any $k \in K$ with $\pi_k^H > \pi_{\min}^H$ implies $h_k = 0$. Analogously, $\pi_k^A > \pi_{\min}^A$ implies $a_k = 0$. Since bookmaker 1 gets a positive payoff, we deduce that either $\pi_1^H = \pi_{\min}^H$ or $\pi_1^A = \pi_{\min}^A$ or both. Any other bookmaker $k \in K \setminus \{1\}$ with $u_k = 0$ would find it profitable to change her bets to $\pi_k^A = \pi_1^A - \epsilon$ and $\pi_k^H = \pi_1^H - \epsilon$, for $\epsilon$ small enough, in order to get a positive payoff. Hence, $u_k > 0$ for all $k \in K$, which implies $h_k+a_k > 0$ for all $k \in K$. This implies that it is profitable for bookmaker 1 to change her odds to $\widetilde{\pi}_1^A = \pi_1^A-\epsilon$ and $\widetilde{\pi}_1^H = \pi_1^H-\epsilon$, in order to attract to herself all the $(k,\cdot)$-bettors and change her final payoff to at least $\sum_{k\in K} u_k - O(\epsilon)$. For for $\epsilon$ small enough, this payoff is strictly larger than $u_1$. This contradiction shows that no bookmaker can get a positive payoff. On the other hand, for any $k \in K$, there exists $l \in K \setminus \{k\}$ with $h_l + a_l > 0$, because otherwise bookmaker $k$ would get a positive payoff by setting monopolistic odds.

The rest of the proof depends on the bookmakers' risk attitude:

**Risk-adverse case:** By an analogous argument to that used in the proof of Proposition 3.1 to prove equation (4), we deduce

$$\frac{1}{\pi_k^H} h_k = \frac{1}{\pi_k^A} a_k \tag{15}$$

for all $k \in K$. Hence, $\pi_k^H = \pi_{\min}^H$ and $\pi_k^A = \pi_{\min}^A$ for all $k \in K$ with $h_k + a_k > 0$. There are then at least two bookmakers that propose the optimal odds in equilibrium, $\pi^H = \pi_{\min}^H$ and $\pi^A = \pi_{\min}^A$, and clear the market. Let $h = \sum_{k\in K} h_k$ and $a = \sum_{k\in K} a_k$. Then,

$$0 = \sum_{k\in K} u_k = (1-\rho)\left((1-\upsilon)(h+a) - \frac{1}{\pi^H} h\right)$$

which implies

$$(1-\upsilon)(h+a) = \frac{h}{\pi^H}. \tag{16}$$

Analogously,

$$(1-\upsilon)(h+a) = \frac{a}{\pi^A}. \tag{17}$$

From (16) and (17), we deduce $\frac{h}{\pi^H} = \frac{a}{\pi^A}$, which is equivalent to equation (4), and $\left(\pi^H + \pi^A\right)(1-\upsilon) = 1$, which is equivalent to $\pi^H + \pi^A = \frac{1}{1-\upsilon}$.

We now prove the existence of a (bettor-strong) subgame perfect equilibrium. Take $\pi^H, \pi^A$ satisfying equation (4) and $x^H + x^A = \frac{1}{1-\upsilon}$. Consider the following strategy profile: For all $k \in K$, $s_k = \left(\pi^H, \pi^A\right)$. Given $\widetilde{s}_K = \left(\widetilde{\pi}_k^H, \widetilde{\pi}_k^A\right)_{k \in K}$, let $s_{\mathcal{B}}$ be defined as follows. For all $i \in K$, $s_i(\widetilde{s}_K) = D$ if $1 - \widetilde{\pi}_{\min}^A < x_i < \widetilde{\pi}_{\min}^H$, $s_i(\widetilde{s}_K) = (k, H)$ if $x_i \geq \widetilde{\pi}_k^H = \widetilde{\pi}_{\min}^H$, and $s_i(\widetilde{s}_K) = (k, A)$ if $x_i \leq 1 - \widetilde{\pi}_k^A = 1 - \widetilde{\pi}_{\min}^A$. In case of more than one minimizing $k$, bettor $i$ chooses the first one in a predefined order. It is straightforward to check that this strategy profile constitutes a bettor-strong subgame perfect equilibrium.

**Risk-neutral case:** For all $k \in K$, we know that $\pi_k^H = \pi_{\min}^H$ if $h_k > 0$ and $\pi_k^A = \pi_{\min}^A$ if $a_k > 0$. Each bookmaker $k \in K$ looks to maximize (2), which sums up zero and has two terms. The first term only depends on $\pi_k^H$ and the second one only on $\pi_k^A$. Hence, the maximal odds are obtained independently, which implies both terms should be zero in equilibrium, or otherwise the bookmaker could increase the odd in the negative term in order to turn the correspondent volume into zero. This implies that the optimal odds satisfy $1 - \upsilon - \frac{q}{\pi^H} = 1 - \upsilon - \frac{1-q}{\pi^A} = 0$, yielding $\pi^H = \frac{q}{1-\upsilon}$ and $\pi^A = \frac{1-q}{1-\upsilon}$.

In order to prove the existence of a (bettor-strong) subgame perfect equilibrium, we take the same strategy profile as in the risk-adverse case, but with $\pi_k^H = \frac{q}{1-\upsilon}$ and $\pi_k^A = \frac{1-q}{1-\upsilon}$ for all $k \in K$. It is straightforward to check that this strategy profile constitutes a bettor-strong subgame perfect equilibrium.

■

**Proof of Lemma 3.1.** Bettors will always choose $k \in K$ with minimum $\pi_k$ so that their expected utility is maximized. We can then focus on one $k \in K$ that minimizes $\pi_k$. Assume w.l.o.g. $\|\mathcal{B}_k\| = 1$. For simplicity, we write $c$ and $\pi$ instead of $c_k$ and $\pi_k$, respectively. In Step 2 of the game, it is optimal for any bettor $i \in \mathcal{B}$ to bet for the home team at odds $\pi$ when $\pi$ is such that

$$(1-c)\left(\frac{1}{\pi}-1\right)x_i + (-1)(1-x_i) > 0$$

which is equivalent to:

$$x_i > \frac{\pi}{1-(1-\pi)c}.$$

Analogously, it is optimal for any bettor $i \in \mathcal{B}$ to bet for the away team at odd $1-\pi$ when $\pi$ is such that

$$(-1)x_i + (1-c)\left(\frac{1}{1-\pi}-1\right)(1-x_i) > 0$$

which is equivalent to:

$$x_i < \frac{(1-c)\pi}{1-c\pi}.$$

Then, the unique odd $\pi^c$ that clears the market is characterized as in (9) by

$$\gamma^c = \frac{1}{\pi^c}\int_{\frac{\pi^c}{1-(1-\pi^c)c}}^{1} f(t)\,dt = \frac{1}{1-\pi^c}\int_0^{\frac{(1-c)\pi^c}{1-c\pi^c}} f(t)\,dt. \tag{18}$$

To see that $\pi^c$ exists and it is unique, let $\phi, \psi : (0,1) \longrightarrow \mathbb{R}$ be two functions defined as $\phi(\pi) = \frac{1}{\pi}\int_{\frac{\pi}{1-(1-\pi)c}}^{1} f(t)\,dt$ and $\psi(\pi) = \frac{1}{1-\pi}\int_0^{\frac{(1-c)\pi}{1-c\pi}} f(t)\,dt$ for all $\pi \in (0,1)$, respectively. It is clear that $\phi$ is continuous and strictly decreasing with $\phi(0^+) = \infty$ and $\phi(1^-) = 0^+$, and that $\psi$ is continuous and strictly increasing with $\psi(0^+) = 0^+$ and $\psi(1^-) = \infty$. Hence, there exists a unique $\pi^c$ satisfying $\phi(\pi^c) = \psi(\pi^c)$. ■

**Proof of Corollary 3.1.** From Lemma 3.1, $h = \int_{\frac{\pi}{1-(1-\pi)c}}^{1} f(t)\,dt$ is the volume of bettors that bet for the home team, whereas $a = \int_0^{\frac{(1-c)\pi}{1-c\pi}} f(t)\,dt$ is the volume of bettors that bet for the away team. No other bettor has positive probability of betting. When the home team wins, the gross profit of the bookmaker is $\left(\frac{1}{\pi}-1\right)ch$. When the away team wins, the gross profit of the bookmaker is $\left(\frac{1}{1-\pi}-1\right)ca$. Moreover, we have $\frac{1}{\pi}h=\frac{1}{1-\pi}a$. Hence, the bookmakers prefers the home (away) team to win when $h<a$ ($h>a$). Equivalently, the bookmaker prefers the home (away) team to win when $\frac{h}{a}<1$ ($\frac{h}{a}>1$). Since $\frac{h}{a}=\frac{\pi}{1-\pi}$, it only happens when $\frac{\pi}{1-\pi}<1$ ($\frac{\pi}{1-\pi}>1$), i.e. $\pi<\frac{1}{2}$ ($\pi>\frac{1}{2}$). The result for $\pi=\frac{1}{2}$ is straightforward. ■

**Proof of Proposition 3.3.** Assume w.l.o.g. $\|\mathcal{B}\|=1$ and $K=\{k\}$. For simplicity, and since there is a unique bookmaker, we write $c$ and $\pi$ instead of $c_k$ and $\pi_k$, respectively. Under Lemma 3.1, for each $c\in[0,1]$, odds $\pi^c$ and $1-\pi^c$ that clear the market are characterized by (9). Notice that, with this $\pi^c$, ratios $\frac{\left\|\mathcal{B}_k^H\cup\overline{\mathcal{B}}_k^H\right\|}{\pi^c}$ and $\frac{\left\|\mathcal{B}_k^A\cup\overline{\mathcal{B}}_k^A\right\|}{1-\pi^c}$ coincide, so that the set of bettors whose probability of betting is in $(0,1)$ has volume zero. Moreover, a subgame perfect equilibrium strategy with undominated strategies is characterized as follows:

- Each bettor $i\in\mathcal{B}$ with $x_i>\frac{\pi^c}{1-(1-\pi^c)c}$ chooses $s_i(c)=(k,H,\pi^c)$.
- Each bettor $i\in\mathcal{B}$ with $x_i<\frac{(1-c)\pi^c}{1-c\pi^c}$ chooses $s_i(c)=(k,A,1-\pi^c)$.
- Any other bettor $i\in\mathcal{B}$ chooses $s_i(c)=D$.

This strategy profile induces $\pi=\pi^c$, and it is a strong equilibrium because no set of bettors can modify $\pi$ in its own benefit, and any other equilibria will also satisfy $\pi=\pi^c$. In general,

$$\mathcal{B}_k^H\cup\overline{\mathcal{B}}_k^H=\overline{\mathcal{B}}_k^H=\left\{i\in\mathcal{B}:x_i\geq\frac{\pi^c}{1-(1-\pi^c)\,c}\right\} \tag{19}$$

$$\mathcal{B}_k^A\cup\overline{\mathcal{B}}_k^A=\overline{\mathcal{B}}_k^A=\left\{i\in\mathcal{B}:x_i\leq\frac{(1-c)\,\pi^c}{1-c\pi^c}\right\} \tag{20}$$

$$\frac{h_k+\overline{h}_k}{a_k+\overline{a}_k}=\frac{\overline{h}_k}{\overline{a}_k}=\frac{\pi^c}{1-\pi^c}. \tag{21}$$

The (bettor-strong) subgame perfect equilibrium is then completely characterized in Step 1 by the maximization of the bookmaker's payoff:

**Risk-adverse case:** The bookmaker's payoff is

$$\max_{c\in[0,1]}(1-\rho)\left(\min\{\alpha,\beta\}c-(\alpha+\beta)\upsilon\right)$$

where

$$\alpha=\min\left\{\frac{1-\pi^c}{\pi^c}\left(h+\overline{h}\right),a+\overline{a}\right\}$$
$$\overset{(21)}{=}\frac{1-\pi^c}{\pi^c}\left(h+\overline{h}\right)\overset{(19)}{=}\frac{1-\pi^c}{\pi^c}\int_{\frac{\pi^c}{1-(1-\pi^c)c}}^{1}f(t)dt\overset{(18)}{=}(1-\pi^c)\gamma^c.$$

Analogously,

$$\beta\overset{(21)(20)}{=}\frac{\pi^c}{1-\pi^c}\int_0^{\frac{\pi^c(1-c)}{1-c\pi^c}}f(t)dt\overset{(18)}{=}\pi^c\gamma^c$$

from where the maximization problem becomes

$$\max_{c\in[0,1]}(1-\rho)\left(\min\{1-\pi^c,\pi^c\}c-\upsilon\right)\gamma^c.$$

Moreover, term $1-\rho$ is unnecesary since it does not depend on $c$.

**Risk-neutral case:** The bookmaker's payoff is

$$\max_{c\in[0,1]}(1-\rho)\left(\left(q\alpha+(1-q)\beta\right)c-(\alpha+\beta)\upsilon\right)$$

where $\alpha$ and $\beta$ are defined as in the risk-adverse case. Since $1-\rho$ does not depend on $c$, it is unnecesary and hence the maximization problem becomes

$$\max_{c\in[0,1]}\left(\left(q(1-\pi^c)\gamma^c+(1-q)\pi^c\gamma^c\right)c-\left((1-\pi^c)\gamma^c+\pi^c\gamma^c\right)\upsilon\right)$$
$$=\max_{c\in[0,1]}\left(\left(q+\pi^c-2q\pi^c\right)c-\upsilon\right)\gamma^c.$$

■

**Proof of Proposition 3.4.** The bettors' profiles in Step 2 are the same as in the proof of Proposition 3.3. Moreover, the chosen bookmakers will be among those with minimum commission. Assume, on the contrary, that some positive volume $\mathcal{B}_k$ of bettors choose a bookmaker $k\in K$ with non-minimum commission. Then,

- If the equilibrium is bettor-strong, bettors in $\mathcal{B}_k$ would improve by choosing a commission-minimizing bookmaker, which is a contradiction.
- If the equilibirum uses undominated strategies, then any bettor in $\mathcal{B}_k$ would improve by choosing a commission-minimizing bookmaker with positive volume of bettors. Such a bookmaker exists, because otherwise some positive volume of bettors would abstain. Their strategy would be dominated by another one that chooses a commission-minimizing bookmaker.

Following a similar reasoning as that on the proof of Proposition 3.2, no bookmaker $k$ can get a positive final payoff because otherwise another bookmaker $k'$ would improve by undercutting $c_k$. Since each bookmaker gets zero in equilibrium, following the same reasoning as in the proof of Proposition 3.3, the minimal commission $c = \min_{k\in K} c_k$ is characterized as follows:

**Risk-adverse case:** $0 = u_k = (1-\rho)\left(\min\{1-\pi,\pi\}c - v\right)\gamma$

**Risk-neutral case:** $0 = u_k = (1-\rho)\left((q+\pi-2q\pi)c - v\right)\gamma$

where $c$, $\pi$ and $\gamma$ satisfy (9). Equality arises when $c = \frac{v}{\min\{1-\pi,\pi\}}$ in the risk-adverse case and when $c = \frac{v}{q+\pi-2q\pi}$ in the risk-neutral case. They will determine the minimal commission unless larger than 1, in which case $c = 1$ would suffice because in that case (9) would imply $\gamma = 0$. Moreover, this minimal commission should be offered by at least two bookmakers. If, on the contrary, only one bookmaker $k$ offers it, she could improve increasing it. ■

**Proof of Proposition 3.5.** Assume w.l.o.g. $K = \{k\}$. For $c_k$ small enough, no strategy profile by the bettors inducing $\left\|\mathcal{B}_k^H\right\| = \left\|\mathcal{B}_k^A\right\| = 0$ can be part of a bettor-strong subgame perfect equilibrium in Step 2. The reason is that we can always find $\epsilon > 0$ such that bettors in $\{i \in \mathcal{B} : x_i < \epsilon \text{ or } x_i > 1-\epsilon\}$ would find it profitable to bet. We can then assume that $\left\|\mathcal{B}_k^H\right\|, \left\|\mathcal{B}_k^A\right\| > 0$. In Step 2, an equilibrium profile $s$ is characterized by $u_i \geq 0$ for all $i \in \mathcal{B}$. That is:

- $s_i(c_k) = H$ iff $\frac{\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\|}{\|\mathcal{B}_k^H\|}(1-c_k)\,x_i \geq 1$. Analogously, $i \in \mathcal{B}_k^H$ iff $x_i \geq \frac{1}{(1-c_k)} \frac{\|\mathcal{B}_k^H\|}{\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\|}$, which implies that $\pi^H = \frac{1}{(1-c_k)} \frac{\|\mathcal{B}_k^H\|}{\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\|}$ satisfies $\mathcal{B}_k^H = \{i \in \mathcal{B} : x_i \in [\pi^H, 1]\}$.

- $s_i(c_k) = A$ iff $\frac{\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\|}{\|\mathcal{B}_k^A\|}(1-c_k)(1-x_i) \geq 1$. Analogously, $i \in \mathcal{B}_k^A$ iff $x_i \leq 1 - \frac{1}{(1-c_k)} \frac{\|\mathcal{B}_k^A\|}{\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\|}$, which implies that $\pi^A = \frac{1}{(1-c_k)} \frac{\|\mathcal{B}_k^A\|}{\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\|}$ satisfies $\mathcal{B}_k^A = \{i \in \mathcal{B} : x_i \in [0, 1-\pi^A]\}$.

Moreover, $\pi^H + \pi^A = \frac{1}{1-c_k}$. Hence, these $\pi^H$ and $\pi^A$ are characterized by:

$$\pi^H \int_0^{1-\pi^A} f(t)dt = \pi^A \int_{\pi^H}^1 f(t)dt$$
$$\pi^H + \pi^A = \frac{1}{1-c_k}$$

which are equivalent to (11) and (12). These conditions characterize the strong equilibrium because no subset of bettors can get advantage by changing their bets. In order to prove existence of $\pi^H$ and $\pi^A$, note first that these conditions are not possible when $c_k > \frac{1}{2}$. In that case, the only equilibrium is achieved with $s_i = D$ for all $i \in \mathcal{B}$, which gives the bookmaker a payoff of zero. In case $c_k = \frac{1}{2}$, the unique solution is $\pi^H = \pi^A = 1$, which again gives the bookmaker a payoff of zero. Assuming $c_k < \frac{1}{2}$, let $d = \frac{1}{1-c_k} \in (1,2)$. We define $\phi : (d-1, 1) \longrightarrow \mathbb{R}$ as $\phi(\pi) = \frac{1}{\pi}\int_0^{1-\pi} f(t)dt - \frac{1}{d-\pi}\int_{d-\pi}^1 f(t)dt$. It is straightforward to check that $\phi$ is continuous, strictly decreasing, and satisfies $\phi(1-d^-) > 0$, and $\phi(1^-) < 0$. Hence, there exists a unique $\pi = \pi^A$ such that $\phi(\pi) = 0$ and, moreover, the bookmaker gets a positive payoff. ■

**Proof of Proposition 3.6.** The bettors' subgame perfect equilibrium profiles in Step 2 are the same as in the proof of Proposition 3.5. No bookmaker $k \in K$ can get a negative final payoff because by offering $c_k = \frac{1}{2}$ she assures $\|\mathcal{B}_k^H \cup \mathcal{B}_k^A\| = 0$ and hence a final payoff of zero. Following a similar reasoning as that on the proof of Proposition 3.2 and Proposition 3.4, no bookmaker $k$ can get a positive final payoff because otherwise another bookmaker $k'$ would improve by undercutting $c_k$. This implies that some bookmaker offers a commission $\upsilon$ (or between $\frac{1}{2}$ and $\upsilon$ when $\frac{1}{2} < \upsilon$) $\min\left\{\frac{1}{2}, \upsilon\right\}$. Moreover, when $\upsilon < \frac{1}{2}$ at least two bookmakers should offer this commission. Otherwise, there would exists $k \in K$ with $c_k = 0 < \min_{l \in K \setminus \{k\}} c_l$ and bookmaker $k$ would obtain a positive payoff by setting $0 < \tilde{c}_k < \min_{l \in K \setminus \{k\}} c_l$. Following the same reasoning as in the proof of Proposition 3.5, thresholds $\pi^H, \pi^A$ are characterized by (11) and (13). ■

**Proof of Proposition 3.7.** Equation (12) can be rewritten as $c = 1 - \frac{1}{\pi^H + \pi^A}$, from where it is straightforward to check that (3) is equivalent to (10)-(12). Moreover, (4) is equivalent to (11) and hence the result holds for the monopolistic case. For the competitive case, Proposition 3.2) and Proposition 3.6 provide the same characterization for $\pi^H$ and $\pi^A$. ■

**Proof of Proposition 4.1.** Since $\rho$ does not play any role in the characterization of odds and commissions in equilibrium, it follows that it leaves them unaffected. The equilibrium strategies of the bettors only consider odds and commissions, and hence their utilities will also not be affected by $\rho$. By definition, $\rho$ decreases linearly the bookmaker's payoff. From these results, no $\rho < 1$ would maximize tax income, since any $\rho' = \rho - \epsilon$, with $0 < \epsilon < \rho$ would increase it. Hence, the maximum tax income is achieved for $\rho = 1$ and the bookmaker choosing optimal odds/commission. ■

**Proof of Theorem 4.1.**

a) Odds, commissions and utilities do not depend on $\rho$ because it does not play any active role in the characterization of the equilibria. Tax income is not affected because in the competitive market the bookmakers' profit is zero.

b) In all cases, the utility of bookmakers remains zero because the competitive market. For the fixed-odds case, it follows from (4) that the larger $\pi^H$ the larger $\pi^A$. Hence, $\pi^H + \pi^A = \min\left\{2, \frac{1}{1-\upsilon}\right\}$ implies that an increase in $\upsilon$ leads to an increase in both $\pi^H$ and $\pi^A$. Thus, the utility of bettors and the volume of bets decrease as $\upsilon$ increases. For the parimutuel case, the reasoning for bettors' utilities is the same as in the fixed-odds case. The commission that clears the market increases with

$\upsilon$ because it coincides with $\upsilon$ when $\upsilon < \frac{1}{2}$. For the spread case, $\pi$ in (9) does not depend on $\upsilon$, so either commission $c = \min\left\{1, \frac{\upsilon}{\min\{1-\pi,\pi\}}\right\}$ or $c = \min\left\{1, \frac{\upsilon}{q+\pi-2q\pi}\right\}$ strictly increases with $\upsilon$. As $c$ increases, the bettors' utilities decrease.

c) It follows from b) that the bet volume decreases with $\upsilon$. This decrease is strict for $\upsilon$ close to zero. Since the utilitarian social welfare strictly decreases with the bet volume, we conclude that the unique maximum is achieved at $\upsilon = 0$.

d) For parimutuel and fixed-odds with risk-adverse bookmakers, $\upsilon \geq \frac{1}{2}$ induces a zero bet volume and hence the market is empty, so the maximum tax income should be achieved for $\upsilon \in \left(0, \frac{1}{2}\right)$. For fixed-odds with risk-neutral bookmakers, $\upsilon \geq \max\{q, 1-q\}$ implies $1 \leq \frac{q}{1-\upsilon}$ and $1 \leq \frac{1-q}{1-\upsilon}$. By Proposition 3.2, these imply $\pi^H = \pi^A = 1$, so again the bet volume is zero and the market empty, so the maximum tax income should be achieved for $\upsilon \in (0, \max\{q, 1-q\})$. Assume now we are in a spread market. In the risk-adverse case, $\upsilon \geq \frac{1}{2}$ implies $\upsilon \geq \min\{1-\pi, \pi\}$ and hence $c = 1$, which induces a zero volume. Thus, the maximum tax income should be achieved for $\upsilon \in \left(0, \frac{1}{2}\right)$. In the risk-neutral case, the reasoning is analogous. It suffices to check that $\upsilon \geq \max\{q, 1-q\}$ implies $\upsilon \geq q+\pi-2q\pi$. Assume $\upsilon \geq \max\{q, 1-q\}$. We have two cases:

   – If $q \geq \frac{1}{2}$, then $1 - 2q \leq 0$ and thus $\pi - 2q\pi = (1-2q)\pi \leq 0$. Hence,
   $$\upsilon \geq \max\{q, 1-q\} = q \geq q + \pi - 2q\pi.$$

   – If $q \leq \frac{1}{2}$, then $1 - q \geq \frac{1}{2}$ and, taking into account that $\pi \leq \frac{1}{2}$,
   $$\begin{aligned}\upsilon &\geq \max\{q, 1-q\} = 1 - q \geq \frac{1}{2} \\ &= (1-2\pi)\frac{1}{2} + \pi \geq (1-2\pi)\, q + \pi = q + \pi - 2q\pi.\end{aligned}$$

■

**Proof of Lemma 4.1.** We need to find $q^*$ such that

$$\frac{1}{q^*}\int_{q^*}^{1} f(t)dt = \frac{1}{1-q^*}\int_{0}^{1-(1-q^*)} f(t)dt$$

equivalently,

$$\frac{1-q^*}{q^*}\int_{q^*}^{1} f(t)dt = 1 - \int_{q^*}^{1} f(t)dt$$

from where $q^* = \int_{q^*}^{1} f(t)dt$ is easily deduced. To see that $q^*$ is unique, notice that $\phi(q) = \int_{q}^{1} f(t)dt$ is a continuously decreasing function with $\phi(0) = 1$ and $\phi(1) = 0$, so there exists a unique $q^*$ such that $q^* = \phi(q^*)$. ■

**Proof of Proposition 5.1.** a) We focus first on the fixed-odds case characterized in Proposition 3.1. Under symmetry, (4) becomes

$$\frac{1}{\pi^H}\int_{\pi^H}^{1} f(t)\,dt = \frac{1}{\pi^A}\int_{\pi^A}^{1} f(t)\,dt.$$

This equality holds when $\pi^H = \pi^A$. Since $F(x) = \frac{1}{x}\int_{x}^{1} f(t)\,dt$ is a strictly decreasing function, we deduce that, for each $\pi^A$, there exists a unique $\pi^H$ that satisfies $F(\pi^H) = F(\pi^A)$. Hence, (4) is equivalent to $\pi^H = \pi^A$. The other restrictions are $\pi^H, \pi^A \in [0,1]$ and $\pi^H + \pi^A \geq 1$, which become $\pi^H = \pi^A \in \left[\frac{1}{2}, 1\right]$. The maximization problem given in Proposition 3.1 becomes

$$\max_{\pi\in\left[\frac{1}{2},1\right]} (1-\rho)\left(1-\upsilon-\frac{1}{2\pi}\right)\left(\int_{\pi}^{1} f(t)dt + \int_{\pi}^{1} f(t)dt\right)$$

$$= \max_{\pi\in\left[\frac{1}{2},1\right]} (1-\rho)\left(2(1-\upsilon)-\frac{1}{\pi}\right)\int_{\pi}^{1} f(t)dt$$

which coincides with the characterization of $\pi^H$ and $\pi^A$ in the risk-neutral case when $q = \frac{1}{2}$.

We now focus on the spread bets market characterized in Proposition 3.3. Under symmetry, (9) becomes

$$\frac{1}{\pi}\int_{\frac{\pi}{1-(1-\pi)c}}^{1} f(t)\,dt = \frac{1}{1-\pi}\int_{1-\frac{(1-c)\pi}{1-\pi c}}^{1} f(t)\,dt$$

equivalently,

$$\frac{1}{\pi}\int_{\frac{\pi}{1-(1-\pi)c}}^{1} f(t)\,dt = \frac{1}{1-\pi}\int_{\frac{1-\pi}{1-\pi c}}^{1} f(t)\,dt$$

or

$$G(\pi) = G(1-\pi)$$

where $G(x) = \frac{1}{x}\int_{\frac{x}{1-(1-x)c}}^{1} f(t)\,dt$. This equality holds when $\pi = \frac{1}{2}$. It is straightforward to check that $G$ is a strictly decreasing function on $[0,1]$, and so $\pi = \frac{1}{2}$ is the only solution to $G(\pi) = G(1-\pi)$. Hence, (9) is equivalent to $\pi = \frac{1}{2}$ and $\gamma = 2\int_{\frac{1}{2-c}}^{1} f(t)dt$. The maximization problem given in Proposition 3.3 becomes

$$\begin{aligned}
&\max_{c\in[0,1]} (1-\rho)\left(c\min\left\{1-\frac{1}{2},\frac{1}{2}\right\}-\upsilon\right)\gamma \\
&= \max_{c\in[0,1]} (1-\rho)\left(\frac{c}{2}-\upsilon\right)\gamma \\
&= \max_{c\in[0,1]} (1-\rho)\left(\frac{c}{2}-\upsilon\right)2\int_{\frac{1}{2-c}}^{1} f(t)dt
\end{aligned}$$

which coincides with the characterization of $c$ in the risk-neutral case when $q = \frac{1}{2}$. We now proceed by a change of variable: $\pi = \frac{1}{2-c}$, so that $c \in [0,1]$ is equivalent to $\pi \in \left[\frac{1}{2}, 1\right]$ and the maximization problem becomes

$$\begin{aligned}
&\max_{\pi\in\left[\frac{1}{2},1\right]} (1-\rho)\left(\frac{2-\frac{1}{\pi}}{2}-\upsilon\right)2\int_{\pi}^{1} f(t)dt \\
&= \max_{\pi\in\left[\frac{1}{2},1\right]} (1-\rho)\left(2(1-\upsilon)-\frac{1}{\pi}\right)\int_{\pi}^{1} f(t)dt.
\end{aligned}$$

For parimutuel market, we apply Proposition 3.5 and Corollary 3.7. It follows from (12) that $c = 1 - \frac{1}{\pi^H+\pi^A} = 1 - \frac{1}{2\pi^*}$ is the optimal commission.

b) For competitive fixed-odds and parimutuel markets, we deduce, analogously to case a), that $\pi^H = \pi^A$. From Proposition 3.2 and Proposition 3.6, in the risk-adverse case we have $\pi^H + \pi^A = \min\left\{2, \frac{1}{1-\upsilon}\right\}$, so $\pi^H = \pi^A = \min\left\{1, \frac{1}{2(1-\upsilon)}\right\}$. Moreover, these equalities also arise in the risk-neutral case with $q = \frac{1}{2}$.

For competitive spread markets, we deduce, analogously to case a), that $\pi = \frac{1}{2}$. From Proposition 3.4, the optimal commission is

$$c = \min\left\{1, \frac{\upsilon}{\min\{1-\pi,\pi\}}\right\} = \min\left\{1, \frac{\upsilon}{\min\left\{1-\frac{1}{2},\frac{1}{2}\right\}}\right\} = \min\{1, 2\upsilon\}$$

for the risk-adverse case and, taking $q = \frac{1}{2}$,

$$c = \min\left\{1, \frac{\upsilon}{q + \pi - 2q\pi}\right\} = \min\left\{1, \frac{\upsilon}{\frac{1}{2} + \frac{1}{2} - 2\frac{1}{2}\frac{1}{2}}\right\} = \min\{1, 2\upsilon\}$$

for the risk-neutral case. In both cases, the optimal commission is $c = \min\{1, 2\upsilon\}$. With the change of variable $c = 2 - \frac{1}{\pi^*}$, we get

$$\pi^* = \frac{1}{2 - c} = \frac{1}{2 - \min\{1, 2\upsilon\}} = \min\left\{1, \frac{1}{2(1-\upsilon)}\right\}.$$

■

**Proof of Proposition 5.2.** For the monopolistic case, by Proposition 5.1, odds and commission are given by

$$\arg\max_{\pi \in \left[\frac{1}{2}, 1\right]} \left(2\left(1 - \upsilon\right) - \frac{1}{\pi}\right)\left(1 - \pi\right) = \min\left\{1, \sqrt{\frac{1}{2(1-\upsilon)}}\right\}. \tag{22}$$

a) For the competitive case, the result follows from part b) in Proposition 4.1. For the monopolistic case, by (22) it is straightforward to check that, for $\upsilon < \frac{1}{2}$, an increase in $\upsilon$ increases the odds and commissions, and decreases the bettors' payoffs as well as the bookmaker's payoff when $\rho < 1$. For $\upsilon \geq \frac{1}{2}$, the bet market in empty and so any further increase is irrelevant.

b) For the competitive case, the result follows from part c) in Proposition 4.1. For the monopolistic case, (22) implies that an increment in $\upsilon$ decreases the bet volume, strictly for $\upsilon$ small, and hence the maximum welfare is achieved for $\upsilon = 0$.

c1) $\rho = 1$ follows from Proposition 4.1 and $\upsilon = 0$ follows from (22) by the same reasoning as in the proof of part b).

c2) By Theorem 4.1b we can assume $\upsilon \in \left(0, \frac{1}{2}\right)$. By Proposition 5.1b the bet volume is given by

$$\int_{\frac{1}{2(1-\upsilon)}}^{1} f(t)dt + \int_{0}^{1-\frac{1}{2(1-\upsilon)}} f(t)dt = 2 - \frac{1}{1-\upsilon}$$

and hence the tax income is $G(\upsilon) = \left(2 - \frac{1}{1-\upsilon}\right)\upsilon$. It is straightforward to check that $G$ is a concave function on $\left(0, \frac{1}{2}\right)$ with unique maximum in $\upsilon = 2 - \frac{\sqrt{2}}{2}$.

■

## References


Aumann, R. J. and Shapley, L. S. (1994). Long-term competition—a game-theoretic analysis. In Megiddo, N., editor, *Essays in Game Theory: In Honor of Michael Maschler*, pages 1–15. Springer, New York.

Bag, P. K. and Saha, B. (2011). Match-fixing under competitive odds. *Games and Economic Behavior*, 73(2):318–344.

Bag, P. K. and Saha, B. (2016). Match-fixing in a monopoly betting market. *Journal of Economics and Management Strategy.*

Evans, M., Hastings, N., and Peacock, B. (2000). Beta distribution. In *Statistical distributions*, chapter 5, pages 34–42. John Wiley and Son, New York.

Ferrari, S. and Cribari-Neto, F. (2004). Beta regression for modelling rates and proportions. *Journal of Applied Statistics*, 31(7):799–815.

Ficom Leisure (2011). Update on Italy's online betting and gaming market.

Global Betting and Gaming Consultancy (2011). It's all about the tax.

Hofmann, J. and Spitz, M. (2015). Germany. In Steward-Jones, H., editor, *Gambling 2016: A practical cross-border insight into gambling law*, chapter 11, pages 54–58. Global Legal Group.

Joosten, R., Brenner, T., and Witt, U. (2003). Games with frequency-dependent stage payoffs. *International Journal of Game Theory*, 31(4):609–620.

Levitt, S. D. (2004). Why are gambling markets organized so differently from financial markets? *Economic Journal*, 114(495):223–246.

National Audit Office (2005). Gambling duties. HM customs and excise. Report by the controller and Auditor General. HC 138 Session 2004-2005. URL `http://www.nao.org.uk/publications/0405/gambling_duties.aspx`.

Neyman, A. (2002). Values of games with infinitely many players. In Aumann, R. and Hart, S., editors, *Handbook of game theory with economic applications*, volume III, chapter 56, pages 2121–2167. Elsevier Science, Amsterdam.

Ottaviani, M. and Sørensen, P. N. (2009). Surprised by the parimutuel odds? *American Economic Review*, 99(5):2129–2134.

Paton, D., Siegel, D. S., and Williams, L. V. (2001). Gambling taxation: A comment. *Australian Economic Review*, 34(4):437–440.

Paton, D., Siegel, D. S., and Williams, L. V. (2002). A policy response to the e-commerce revolution: the case of betting taxation in the UK. *Economic Journal*, 112(480):296–314.

PwC (2011). Taxation and online sports betting in Germany: considering the relative merits of a tax on gross gaming revenue and a tax on stakes for the potential regulation of online sports betting stakes.

Rubinstein, A. (1994). Equilibrium in supergames. In Megiddo, N., editor, *Essays in Game Theory: In Honor of Michael Maschler*, chapter 2, pages 17–27. Springer, New York.

Smith, J. (2000). Gambling taxation: Public equity in the gambling business. *Australian Economic Review*, 33(2):120–144.