The Shapley value in minimum cost spanning tree problems*

Christian Trudeau[†] Juan Vidal-Puga[‡]

Published as Chapter 24 in *Handbook of the Shapley Value*[§]

1 Introduction

The Shapley value is a very appealing solution concept that is characterized by its reliance on contributions (Young, 1985) and satisfies many interesting properties in the general set of cooperative games, such as "additivity" (Shapley, 1953b) and "balanced contributions". A drawback is that the Shapley value payoff vector might not be stable in the sense of core selection: even for games for which the core is nonempty, the Shapley value might propose allocations giving some coalitions incentives to secede.

An interesting family of balanced games in which the Shapley value has nonetheless received considerable attention is the class of minimum cost spanning tree (*mcst*) problems, which model situations where a group of agents,

*Christian Trudeau acknowledges financial support by the Social Sciences and Humanities Research Council of Canada [grant number 435-2014-0140]. Juan Vidal-Puga acknowledges financial support by the Spanish Ministerio de Economía y Competitividad (ECO2014-52616-R), Ministerio de Economía, Industria y Competitividad (ECO2017-82241-R), and Xunta de Galicia (GRC 2015/014). The authors thank William Thomson for helpful comments.

[†]Department of Economics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario, Canada. Email: trudeauc@uwindsor.ca

[‡]Economics, Society and Territory (ECOSOT) and Departamento de Estatística e IO. Email: vidalpuga@uvigo.es

[§]DOI: 10.1201/9781351241410-24. E. Algaba, V. Fragnelli, J. Sánchez-Soriano (2019) Handbook of the Shapley Value. CRC Press, Taylor Francis Group, USA. C.Trudeau, J. Vidal-Puga (2019) The Shapley value in minimum cost spanning tree problems. Handbook of the Shapley Value, 537-559.

located at different nodes of a network, require a service provided by a source and do not care if they connect directly to the source or through other agents already connected. The cost of an edge between two nodes has to be paid when used, but cost remains invariant if more than one agent uses it to connect to the source.

As the cost of the efficient network connecting all agents to the source is easily found, we consider the problem of sharing the cost of that network among its users. There are many ways to define a TU-game from the mcst problem, depending on our assumptions on property rights and on the behavior of non-cooperating agents. The most commonly used game is the private mcst game, which limits the nodes that a coalition can use to those of its members.

Regardless of how the game is defined, a mcst problem typically induces a game with a large nonempty core and, moreover, it allows population monotonic allocations schemes (Norde et al., 2004). However, for the private mcst game, the Shapley value does not always belong to its core. This fact has led some authors to claim that the Shapley value is not a good solution concept in mcst problems (Sziklai et al., 2016).

Yet, even if we are interested in the private mcst game, we can use the Shapley values of some reasonable alternative cost games that belong to the core of the private mcst game. Moreover, those solutions rely on contributions and maintain the nice properties of "additivity", "balanced contributions", among others, in the context of minimum cost spanning tree problems.

This chapter surveys the growing literature on mcst games and it is organized as follows: In Section 2 we present the model and some basic definitions. In Section 3 we define three different cooperative cost games that can be associated to a minimum cost spanning tree problem. In Section 4 we describe the respective Shapley values associated to the previous cooperative cost games. In Section 5 we review their axiomatic characterizations. In Section 6 we study the weighted versions of the Shapley value and also compare it with other solution concepts such as the nucleolus. In Section 7 we comment on some studies of the Shapley value in other problems related to mcst. Finally, in Section 8 we conclude.

2 Definitions

We first define general cost games, before introducing minimum cost spanning tree problems.

2.1 Cooperative cost games

Let $\mathcal{N} = \{1, 2, \dots\}$ be a (countably infinite) set of potential agents, and let $N = \{1, \dots, n\}$ be a generic nonempty, finite set of $\mathcal{N}$.

A *(cost sharing) game* is a pair (N, C) where C is a *cost function* that assigns to each nonempty *coalition* $S \subseteq N$ the cost $C(S) \in \mathbb{R}_+$ that agents in S should pay in order to receive the service.

For any $S \subseteq N$, let $x(S) = \sum_{i \in S} x_i$. A *preimputation* is an allocation $x \in \mathbb{R}^N$ such that $x(N) = C(N)$. Given $S \subseteq N$ and $x \in \mathbb{R}^N$, we denote as $x_S \in \mathbb{R}^S$ the restriction of x to $\mathbb{R}^S$.

We define the set of stable allocations as $Core(C)$. Formally, an allocation x belongs to $Core(C)$ if it is a preimputation such that $x(S) \leq C(S)$ for all $S \subset N$.

2.2 Minimum cost spanning tree problems

We assume that the agents in N need to be connected to a source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i, j) of distinct elements of Z. In our context, any element (i, j) in Z^p represents the (undirected) edge between nodes i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with $N_0^p = (N_0)^p$ and c_e representing the cost of edge e. Given $E \subset N_0^p$, its associated cost is $c(E) = \sum_{e \in E} c_e$. For simplicity, we write c_{ij} instead of $c_{(i,j)}$ for all $i, j \in N_0$.

Since c assigns a cost to all edges e, we often abuse language and call c a *cost matrix*. Let Γ be the set of all cost matrices. A mcst problem is a pair (N_0, c). When there is no ambiguity, we identify a mcst problem (N_0, c) by its cost matrix c.

Given $l \in N_0$, a *cycle* p_{ll} is a set of $K \geq 3$ edges (i_{k-1}, i_k), with $k \in \{1, \dots, K\}$ and such that $i_0 = i_K = l$ and $i_1, \dots, i_{K-1}$ distinct and different than l. Given $l, m \in N_0$, a *path* ψ_{lm} between l and m is a set of K edges (i_{k-1}, i_k), with $k \in \{1, \dots, K\}$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $\Psi_{lm}(N_0)$ be the set of all paths between nodes l and m.

A *spanning tree* is a non-oriented graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges.

We call *mcst* a spanning tree that has a minimal cost. It can be obtained using a greedy algorithm, for example Prim (1957), Kruskal (1956), or Borůvka (1926) algorithms.

3 Associated cooperative cost games

Having established how to connect efficiently all agents to the source, we now examine how to share the cost of such connections. To derive from a mcst problem a cooperative game that represents the cost for each coalition to act alone, we need to determine the rules of the game, i.e., what exactly a coalition is allowed to do when it is alone. In the context of mcst problems, we consider three possibilities:

The private mcst game The cost assigned to coalition $S \subset N$ is computed by assuming that nodes in S should connect without using nodes $N \setminus S$, i.e., nodes outside S are unavailable.

The public mcst game The cost assigned to coalition $S \subset N$ is computed by assuming that agents in S may use edges in $N \setminus S$, paying the costs of the edges they use.

The optimistic mcst game The cost assigned to coalition $S \subset N$ is computed by assuming that nodes in $N \setminus S$ are already connected to the source, and hence agents in S just need to connect either to a node in $N \setminus S$ or to the source.

We analyze each of these possibilities one by one.

3.1 The private mcst game

The most common assumption in the literature is that a coalition only has access to the nodes of its members to connect to the source. In this approach, we assume that agents have property rights over their respective nodes, forcing a coalition to only use the nodes of its members. We thus call the resulting game the *private mcst game*.

Formally, let c_S be the restriction of the cost vector c to the coalition $S_0 \subseteq N_0$. Let $C(S,c)$ be the cost of the mcst of the problem (S_0, c_S). We say that C is the *stand-alone cost* for the private mcst game associated with c.

3.2 The public mcst game

An alternative approach is to suppose that there are no property rights on nodes: a coalition S can use the nodes of its neighbours in $N \setminus S$ to connect to the source if they desire so. We call the resulting game the *public mcst game*. It was first explicitly considered in Bogomolnaia and Moulin (2010), and also examined and contrasted with the private game in Trudeau (2013); Trudeau and Vidal-Puga (2017b).

We thus obtain the following characteristic cost function. For all $S \subseteq N$, we have

$$C^{Pub}(S,c) = \min_{T \subseteq N \setminus S} C(S \cup T, c).$$

It is thus obvious that for all $S \subset N$, $C^{Pub}(S,c) \leq C(S,c)$ and $C^{Pub}(N,c) = C(N,c)$.

3.3 The optimistic mcst game

We finally consider the case where agents are the last to move: others have connected to the source, and they only need to add themselves to the tree.[1] A coalition S can either connect to the source or to any node in $N \setminus S$. We call the resulting game the *optimistic mcst game*. It was first used in the context of mcst problems[2] by Bergantiños and Vidal-Puga (2007b).

Formally, let c^{+S} be the cost matrix on S_0 defined as $c^{+S}_{0i} = \min_{j \in N_0 \setminus S} c_{ij}$ and $c^{+S}_{ij} = c_{ij}$ otherwise, for all $i, j \in S$.

We thus obtain the following cost function. For all $S \subseteq N$, we have

$$C^{+}(S,c) = C\left(S, c^{+S}\right).$$

By contrast, the two other approaches are pessimistic, as the stand-alone cost of a coalition is computed under the assumption that the other agents are not connected.

It is obvious that for all $S \subset N$, $C^{+}(S,c) \leq C^{Pub}(S,c) \leq C(S,c)$ and $C^{+}(N,c) = C^{Pub}(N,c) = C(N,c)$.

[1]See Bergantiños and Lorenzo (2004) for a real-life example of this situation.

[2]Maniquet (2003) considered the same idea in the context of queueing problems.

S	$C(S,c)$	$C^{Pub}(S,c)$	$C^{+}(S,c)$
$\{1\}$	42	42	36
$\{2\}$	90	84	36
$\{3\}$	48	48	36
$\{1,2\}$	84	84	72
$\{1,3\}$	78	78	72
$\{2,3\}$	84	84	72
$\{1,2,3\}$	114	114	114

3.4 Example

Consider the mcst problem described in Figure 1, for which $N = \{1, 2, 3\}$ and the cost of each edge is indicated on it.

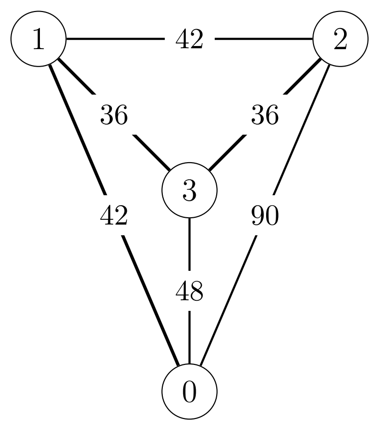

Figure 1: Minimum cost spanning tree problem.

There is a single mcst in this game: $t^* = \{(0,1),(1,3),(2,3)\}$. The functions C and C^{Pub} differ only for agent 2: in the private game she can only connect to the source by using edge $(0,2)$, at a cost of 90. In the public game, she can connect at a lower cost of 84 by using trees $\{(0,1),(1,2)\}$ or $\{(0,3),(2,3)\}$. The optimistic game is quite different, and only the grand coalition has the same cost as in other approaches. For example, agent 2 can now free-ride on the edges established by other agents and can connect by adding edge $(2,3)$ at a cost of 36.

4 The Shapley value

In what follows, a *solution* is a function that assigns to each mcst problem (N_0, c) a preimputation $y \in \mathbb{R}^N$. As in most cost sharing problems, the Shapley value is a natural candidate to share the cost in a mcst problem. We study three ways to do so, depending on the cooperative game associated to the mcst problem.

4.1 The Kar solution

The Shapley value of the private mcst game is known in the literature as the Kar solution, following the axiomatic analysis of the method in Kar (2002). Formally, we define the ***Kar solution*** as $y^K(c) = Sh(C(N, c))$. Ando (2012) shows that computing the Kar solution is #P-hard even if the edges are restricted to costs of 0 or 1.

The Shapley value of the public mcst game has not received much more attention. We define it here as $y^{K^{Pub}}(c) = Sh(C^{Pub}(N, c))$.

Whether we are applying it to the private or public version of the game, there is one major problem: even though the cores of the private and public mcst problems are always nonempty, the Shapley values might not be in them, a fact observed as early as in Bird (1976).

In our running example, we obtain $y^K(c) = (28, 55, 31)$ and $y^{K^{Pub}}(c) = (29, 53, 32)$. Notice that whether we are in the private or public game, the stand-alone cost for coalition $\{2, 3\}$ is 84. The (private) Kar solution assigns them a joint cost of 86, while the public version assigns them 85. Therefore, both are unstable.

Following this observation, researchers have proposed solutions that are in the core. We focus in this paper on the two solutions that are based on the Shapley value, the folk and the cycle-complete solutions. Other stable solutions are based on the network-building algorithms, and include the Bird solution (Bird, 1976), the Dutta-Kar solution (Dutta and Kar, 2004) and the obligation rules (Tijs et al., 2006; Lorenzo and Lorenzo-Freire, 2009; Bergantiños and Kar, 2010; Bergantiños et al., 2010, 2011).

4.2 The folk solution

The folk solution can be obtained by applying the Shapley value to two different situations.

The first one is by transforming the cost matrix into an *irreducible cost matrix*, which is such that no edge cost can be reduced without reducing the cost of the grand coalition to connect to the source Bird (1976). From any cost matrix c, we can define the *irreducible cost matrix* c^* as follows:

$$c^*_{ij} = \min_{\psi_{ij} \in \Psi_{ij}(N_0)} \max_{e \in \psi_{ij}} c_e \text{ for all } i, j \in N_0.$$

Interestingly, it is not difficult to verify that for any irreducible matrix c, $C(\cdot, c) = C^{Pub}(\cdot, c)$. That is, once we have transformed the cost matrix into its irreducible cost matrix, the property rights on the nodes are irrelevant. One can thus argue that using the irreducible matrix will yield solutions that are closer in spirit to the public approach than the private one Trudeau (2014a). Following Bergantiños and Vidal-Puga (2007a), the ***folk solution*** is defined as $y^f(c) = Sh(C^*(N, c))$, where $C^*(N, c)$ is the stand-alone game induced by the irreducible cost matrix c^*.

In our running example, we obtain the following irreducible matrix: Agent 3 is linked to the source with path $\{(0,1),(1,3)\}$ for which the most expensive edge costs 42, so $c^*_{03} = 42$. Agent 2 is linked to the source with path $\{(0,1),(1,3),(2,3)\}$ for which the most expensive edge costs 42, so $c^*_{02} = 42$. Agents 1 and 2 are linked to each other with the path $\{(1,3),(2,3)\}$ for which the most expensive edge costs 36, so $c^*_{12} = 36$. The costs of other edges stay unchanged. This results in $C^*(S, c) = 42$ if $|S| = 1$, $C^*(S, c) = 78$ if $|S| = 2$ and $C^*(N, c) = 114$. Thus, $y^f(c) = (38, 38, 38)$ due to the symmetry of the agents. See Figure 2.

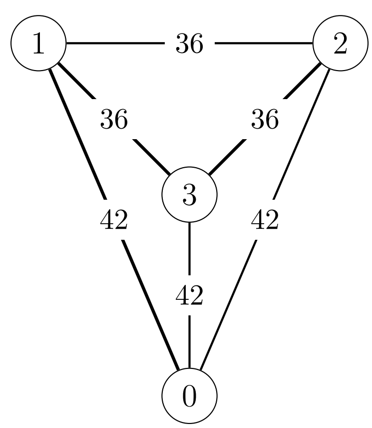

Figure 2: Irreducible matrix associated to Example 1.

Bird (1976) first studied $Core(C^*(N,c))$, and called it the *irreducible core*. Since C^* is a concave cost game (Proposition 3.3c in Bergantiños and Vidal-Puga (2007a)), its Shapley value belongs to the irreducible core. Finally, since $Core(C^*(N,c)) \subseteq Core(C^{Pub}(N,c)) \subseteq Core(C(N,c))$, we have the following result.

Theorem 1 (Bergantiños and Vidal-Puga (2007a)) *For all $c \in \Gamma$, $y^f(c)$ is in $Core(C(N,c))$.*

The folk solution is thus remarkably stable: it is always in the core, regardless of how we define the core.

The second definition of the folk rule using the Shapley value is through the optimistic version of the cost game. Bergantiños and Vidal-Puga (2007a) show that the folk rule is the Shapley value of the optimistic game, i.e., $y^f = Sh(C^+(N,c))$. This is due to the fact that the private mcst game associated with the irreducible cost vector is dual to the optimistic cost game, i.e., $C^*(S,c) + C^+(N \setminus S, c)$ is independent of S (Theorem 1 in Bergantiños and Vidal-Puga (2007b)).

Other definitions of the folk solution are possible. Bergantiños and Vidal-Puga (2007a) show that y^f can also be defined using Prim's algorithm (Prim, 1957) on the irreducible matrix. Feltkamp et al. (1994), using another network-building algorithm due to Kruskal (1956), were the first ones to define the folk rule with the name *Equal Remaining Obligations* (ERO) rule, renamed as P-value in Branzei et al. (2004). Equivalence between ERO and folk rules was first pointed out in Bergantiños and Vidal-Puga (2008); Bergantiños and Lorenzo-Freire (2008b). Bergantiños and Vidal-Puga (2011) use yet another network-building algorithm, due to Borůvka (1926).

Given the different definitions and names, y^f has been dubbed the *folk solution* by Bogomolnaia and Moulin (2010). We also use their term throughout the chapter.

4.3 The cycle-complete solution

As seen in the previous subsection, the folk solution proposes a stable allocation, but one in which we have introduced a lot of symmetry. While in the running example all agents pay the same amount, this is not a general result. It does, however, introduce a lot of symmetry by keeping only the information contained in the mcst. The idea of the cycle-complete solution, proposed in Trudeau (2012), is to try to keep more information from the original matrix while still proposing a stable allocation.

The method used is conceptually close to the one used to generate the folk solution, with changes made to the cost matrix, before taking the Shapley value of the corresponding (private) mcst game. Instead of looking at paths between pairs of edges, we look at cycles: for edge (i,j), we look at cycles that go through node i and node j. If there is one such cycle such that its most expensive edge is cheaper than a direct connection through edge (i,j), we assign this cost to edge (i,j).

From any cost matrix c, we formally define the cycle-complete cost matrix c^{**} as follows:

$$\begin{aligned} c^{**}_{ij} &= \max_{k \in N\setminus\{i,j\}} \left(c^{N\setminus\{k\}}\right)^*_{ij} \text{ for all } i,j \in N \\ c^{**}_{0i} &= \max_{k \in N\setminus\{i\}} \left(c^{N\setminus\{k\}}\right)^*_{0i} \text{ for all } i \in N \end{aligned}$$

where $\left(c^{N\setminus\{k\}}\right)^*$ indicates the matrix that we first restricted to agents in $N \setminus \{k\}$ before transforming into an irreducible matrix.

The ***cycle-complete solution*** is defined as $y^{cc}(c) = Sh(C^{**}(N,c))$, where $C^{**}(N,c)$ is the stand-alone game induced by c^{**}.

In our running example, the only change we make to the original cost matrix to obtain the cycle-complete matrix is to edge $(0,2)$. We can build cycle $\{(0,1),(1,2),(2,3),(0,3)\}$ that contains both 0 and 2 and for which the most expensive edge costs 48. Thus, $c^{**}_{02} = 48$.

In the private mcst game, the only change is to the stand-alone cost of agent 2, which goes from 90 to 48. Thus, $y^{cc}(c) = (35, 41, 38)$.

Trudeau (2012) shows that $C^{**}(N,c)$ is a concave game, and thus its Shapley value is in $Core(C^{**}(N,c))$. Since $Core(C^{**}(N,c)) \subseteq Core(C(N,c))$, the cycle-complete solution is in the core.[3]

Theorem 2 (Trudeau (2012)) *For all $c \in \Gamma$, $y^{cc}(c)$ is in $Core(C(N,c))$.*

In general, y^{cc} is not in $Core(C^{Pub}(N,c))$. Notice that the two approaches are incompatible, as the cycle-complete approach is about bargaining for the use of an outside edge, which the public game supposes is available for free.

[3]Trudeau and Vidal-Puga (2017b) show that if c is such that all edge costs are 0 or 1, then $Core(C^*(N,c)) = Core(C^{Pub}(N,c))$ and $Core(C^{**}(N,c)) = Core(C(N,c))$.

5 Axiomatic analysis

In this section we focus on the axiomatic characterization of the three methods defined in the previous section.[4] This means that we find properties, or axioms, that are satisfied by the solution and such that no other can satisfy them simultaneously. Throughout this section, $y(c)$ is a generic solution.

The first property requires that a solution propose a core allocation.

Core selection Let $c \in \Gamma$. Then, $y(c) \in Core(C(N, c))$.

We now define three new properties. One of them is stronger and the other two are weaker than core selection.

The stronger version of core selection requires that no agent be worse off when new agents join the society. Formally,

Population mononicity Let $c \in \Gamma$ and $S \subset N$. Then, for all $i \in S$, $y_i(c) \geq y_i(c_S)$.[5]

A weaker version of core selection is due to Branzei et al. (2004). It requires undominance in only some coalitions:

Upper bounded contribution Let $c \in \Gamma$ and $P \subset N_0$ such that, for all $i, j \in P$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, $y(P \cap N) \leq \min_{i \in P \cap N} c_{0i}$.

Obviously, we are interested in axioms that are related with mcst problems. Among the numerous characterizations of the Shapley value in the general transferable utility game case, the *balanced contributions property* proposed by Myerson (1980) is particularly interesting, since it is inspired by a property of edge deletion previously defined in Myerson (1977).

In the mcst problem context, we say that a solution satisfies equal treatment if variation in the cost of an edge affects equally both adjacent nodes. Formally,

Equal treatment Let $c, c' \in \Gamma$ be such that $c_{kl} = c'_{kl}$ for all $k, l \in N_0 \backslash \{i, j\}$. Then, $y_i(c) - y_i(c') = y_j(c) - y_j(c')$.

---

[4] A similar exercise was done in Trudeau (2013).

[5] We use $y(c_S)$ to designate the cost allocation to the mcst game involving only agents in S.

Equal treatment is, clearly, a fairness axiom. A weaker version that applies only when the change in the cost of the edge does not affect the total cost of the mcst problem is proposed by Trudeau (2014b):

Weak equal treatment Let $c, c' \in \Gamma$ be such that $c_{kl} = c'_{kl}$ for all $k, l \in N_0 \setminus \{i, j\}$ and $C(N, c) = C(N, c')$. Then, $y_i(c) - y_i(c') = y_j(c) - y_j(c')$.

For the next axiom, we need some additional notation. An edge (i, j) between agents $i, j \in N$ is *relevant* if $c_{ij} \leq \max\{c_{0i}, c_{0j}\}$. An edge is *strictly relevant* if $c_{ij} < \max\{c_{0i}, c_{0j}\}$, *irrelevant* if it is not relevant, and *weakly irrelevant* if it is not strictly relevant.

Let $\overline{\Gamma}$ be the set of elementary cost matrices with no irrelevant edges.

Notice that an irrelevant edge will never belong to an optimal tree. A path ψ_{ij} is an *irrelevant path* if it contains a weakly irrelevant edge. If all paths between i and j are irrelevant, then (one of) the efficient way(s) of connecting $\{i, j\}$ to the source is to connect them both directly to it. In other words, agent i does not help agent j connect to the source in a cheaper manner, and viceversa.

We say that an allocation satisfies *group independence* if we can partition agents in groups such that members of different groups only have irrelevant paths between them. Then, they never have any gain to cooperate with each other, even when considering the connection problem of subgroups of N. Formally,

Group independence Let $c \in \Gamma$ be such that there exists a partition $\mathcal{P}$ of N such that for all $i \in S$ and $j \in T$, and all S, T distinct in $\mathcal{P}$, we have that all $\psi_{ij} \in \Psi_{ij}(N_0)$ are irrelevant paths. Then, for all $i \in S \in \mathcal{P}$, $y_i(c) = y_i(c_S)$.

The next axiom is a stronger version of group independence, since it only requires the partition of N (which does not need to be unique) to be able to connect to the source independently.

Separability Let $c \in \Gamma$ be such that there exists a partition $\mathcal{P}$ of N such that $C(N, c) = \sum_{S \in \mathcal{P}} C(S, c)$. Then, for all $i \in S \in \mathcal{P}$, $y_i(c) = y_i(c_S)$.

Another version of group independence is when we build a partition of the set of followers of some node. Take, for example, the mcst depicted in Figure 3. Both nodes 2 and 3 always connect to the source through node 1.

They form two different branches. When these branches obtain no benefits by connecting with other agents and the costs inside them are not lower than the costs on the path from the source to the linking node, then we should be able to remove one of the branches in order to compute the allocation of the others.

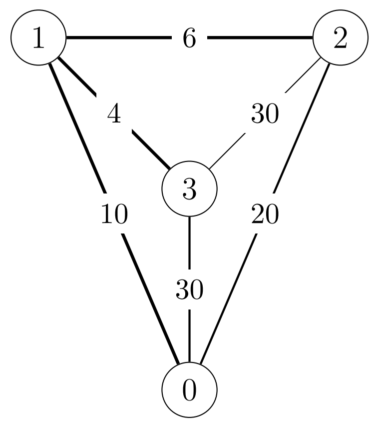

Figure 3: Nodes 2 and 3 constitute two branches of node 1.

Branch cutting Let $c \in \Gamma$, $S \subset N$ and $k \in N_0 \setminus S$. If:

- all the nodes in S are followers of node k,
- for all $i \in S$, $j \in N \setminus S$, $j \neq k$, c_{ij} is a weakly irrelevant edge, and
- for all $i, j \in S \cup \{k\}$, $c_{ij} \geq c_e$ for all e in a path from node k to the source in any optimal tree,

then

$$y_i(c) = \begin{cases} y_i(c'_S) & \text{if } i \in S \\ y_i(c_{N\setminus S}) & \text{if } i \in N \setminus S, i \neq k. \end{cases}$$

where $c'_{0i} = c_{ki}$ and $c'_{ij} = c_{ij}$ for all $i, j \in S$.

Notice that branch cutting does not say anything about the cost share of node k (in case $k \neq 0$). But, in this case, $y_k(c)$ can be deduced from budget balance once the other cost shares are known.

Theorem 3 (Kar (2002)) *The Kar rule is the unique solution which satisfies equal treatment, and group independence.*

Another relevant property of the Shapley value, in the context of cooperative game theory, is additivity. A natural definition of additivity in the context of mcst problems is to assume that $y(c+c') = y(c) + y(c')$, where $c + c'$ is defined in the natural way, i.e., $(c + c')_{ij} = c_{ij} + c'_{ij}$ for all $i, j \in N_0$.

However, no solution can satisfy this version of additivity in general. To see why, consider $N = \{1, 2\}$, $c_{12} = c'_{12} = 0$, $c_{01} = c'_{02} = 0$, and $c_{02} = c'_{01} = 1$. Then, $y_1(c) + y_2(c) = y_1(c') + y_2(c') = 0$, whereas $y_1(c + c') + y_2(c + c') = 1$.

The difficulty with this example is that there exists no tree that is optimal in all three problems c, c', and $c + c'$. We can define a weaker version of this property by requiring additivity only between mcst problems that share at least an optimal tree t such that, if we order the edges of t in non-decreasing cost, then we can obtain the same order in both problems.

Restricted additivity Let $c, c' \in \Gamma$ be such that there exists a common optimal tree $t^* \in \mathcal{T}^*(c) \cap \mathcal{T}^*(c')$ and an order π of the edges in t^* such that $c_{\pi_1} \leq c_{\pi_2} \leq \cdots \leq c_{\pi_n}$ and $c'_{\pi_1} \leq c'_{\pi_2} \leq \cdots \leq c'_{\pi_n}$. Then, $y(c + c') = y(c) + y(c')$.

A sufficient condition for such an optimal tree to exist is that both problems share a common ordering of the edges according to their cost.

Piece-wise linearity Let $c, c' \in \Gamma$ be such that there exists an ordering $e_1, e_2, \ldots$ of the edges such that $c_{e_1} \leq c_{e_2} \leq \ldots$ and $c'_{e_1} \leq c'_{e_2} \leq \ldots$. Then, for all $\alpha, \beta > 0$, $y(\alpha c + \beta c') = \alpha y(c) + \beta y(c')$.

The next property is due to Trudeau (2014b) and uses the fact that if we do not have irrelevant edges, then there always exists a mcst in which a single agent is connected to the source. We can then divide the problem, first sharing the cost of the unique connection to the source, before sharing the cost to connect the remaining agents to that source-connected agent.

Problem separation Let $\bar{c} \in \Gamma$ be such that $\hat{c}_{ij} = 0$ for all $i, j \in N$. Let $\tilde{c}, \dot{c} \in \Gamma$ be such that $\tilde{c}_{0i} = \dot{c}_{0i} = \max_{i \in N} \hat{c}_{i0}$ and $\dot{c}_{ij} = 0$ for all $i, j \in N$. Then, if $\hat{c} + \tilde{c} - \dot{c} \in \overline{\Gamma}$, $y(\hat{c} + \tilde{c} - \dot{c}) = y(\hat{c}) + y(\tilde{c}) - y(\dot{c})$.

In the preceding property, the problem of connecting a single agent to the source is represented by $\hat{c}$, while the problem of connecting the remaining agents to the source-connected agent is in $\tilde{c}$. Since we added a large source-connection cost in that second problem, it is removed by subtracting $\dot{c}$.

Trudeau (2014b) also proposes a weaker version of problem separation that applies only to problems for which there is no edge used in a mcst that is more expensive than the cheapest edge connecting an agent to the source.

Weak problem separation Let $\hat{c} \in \Gamma$ be such that $\hat{c}_{ij} = 0$ for all $i, j \in N$. Let $\tilde{c}, \dot{c} \in \Gamma$ be such that $\tilde{c}_{0i} = \dot{c}_{0i} = \max_{i \in N} \hat{c}_{0i}$ and $\dot{c}_{ij} = 0$ for all $i, j \in N$. Then, if $\hat{c} + \tilde{c} - \dot{c} \in \overline{\Gamma}$ and $c_e \leq \min_{i \in N} c_{0i}$ for all edge e in an optimal tree, $y(\hat{c} + \tilde{c} - \dot{c}) = y(\hat{c}) + y(\tilde{c}) - y(\dot{c})$.

The remaining properties are self-explanatory.

We require that agents that play the same role pay the same amount. Formally[6],

Symmetry Let $c \in \Gamma$ and $i, j \in N$ such that $c_{ik} = c_{jk}$ for all $k \in N_0 \setminus \{i, j\}$. Then, $y_i(c) = y_j(c)$.

The next property says the allocation does not depend on irrelevant edges.

Independence of irrelevant edges Let $c \in \Gamma$ and let $\overline{c} \in \Gamma$ be defined as $\overline{c}_{ij} = \min\{c_{ij}, \max\{c_{i0}, c_{j0}\}\}$ and $\overline{c}_{i0} = c_{i0}$ for all $i, j \in N$. Then, $y(c) = y(\overline{c})$.

Theorem 4 (Trudeau (2014b)) *The Kar rule is the only solution that satisfies weak equal treatment, group independence, piece-wise linearity, problem separation, symmetry, and independence of irrelevant edges.*

The Kar rule also satisfies other nice properties, such as *cost monotonicity* (Dutta and Kar, 2004), which states that an increase of the cost of an edge cannot benefit any adjacent agent. Formally,

Cost monotonicity Let $c, c' \in \Gamma$ be such that, for some $i \in N$ and $j \in N_0$, $c_{kl} = c'_{kl}$ for all $k, l \in N_0 \setminus \{i, j\}$, and $c_{ij} < c'_{ij}$. Then, $y_i(c) \leq y_i(c')$.

The property incentives efficiency, as it prevents nodes from benefitting by increasing their connection costs. In case sabotage of non-adjacent connection costs is possible, a stronger version of cost monotonicity is desirable. *Solidarity* states that an increase of the cost of an edge does not benefit *any* agent (and not only its adjacent ones). Formally,

[6] Some theorems below use, in the original article, the stronger property of anonymity, which requires that the allocation not depend on the name of the agents. All of them hold with symmetry.

Solidarity Let $c, c' \in \Gamma$ be such that $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$. Then, for all $i \in N$, $y_i(c) \leq y_i(c')$.

Despite the other nice properties, the Kar rule does satisfy neither solidarity nor, as shown in Dutta and Kar (2004) and in Subsection 4.1, stability.

The next theorem links the Kar rule and the folk rule:

Theorem 5 (Trudeau (2014b)) *A solution satisfies weak equal treatment, group independence, piece-wise linearity, weak problem separation, symmetry, and independence of irrelevant edges if and only if it is a convex combination of the Kar rule and the folk rule, i.e., there exists* $\alpha \in \mathbb{R}$ *such that* $y = \alpha y^K + (1 - \alpha) y^f$.

There exist several characterizations of the folk rule using the restricted additivity or piece-wise linearity:

Theorem 6 (Branzei et al. (2004)) *The folk rule is the only solution that satisfies upper bounded contribution, piece-wise linearity, and symmetry.*

Clearly, we can replace upper bounded contribution by core selection in this characterization.

Theorem 7 (Bergantiños and Vidal-Puga (2009)) *The folk rule is the only solution that satisfies separability, restricted additivity, and symmetry.*

Theorem 8 (Bergantiños et al. (2011)) *The folk rule is the only solution that satisfies core selection, restricted additivity, symmetry, and solidarity.*

The next axiom states that if all the nodes are close to each other and are at the same distance to the source, then any increase in the cost to the source should be shared equally among the agents. An example is depicted in Figure 4. All agents are equally far away from the source. So, an optimal tree should connect any of them to the source and then the others connect to the source through this one. The property of equal share of extra costs states that the cost allocation should be the same as before, and the extra cost (6 in our example) is shared equally among the agents, i.e., $y_i(c') = y_i(c) + 2$ for all i, where c is the cost matrix on the left and c' is the one on the right.

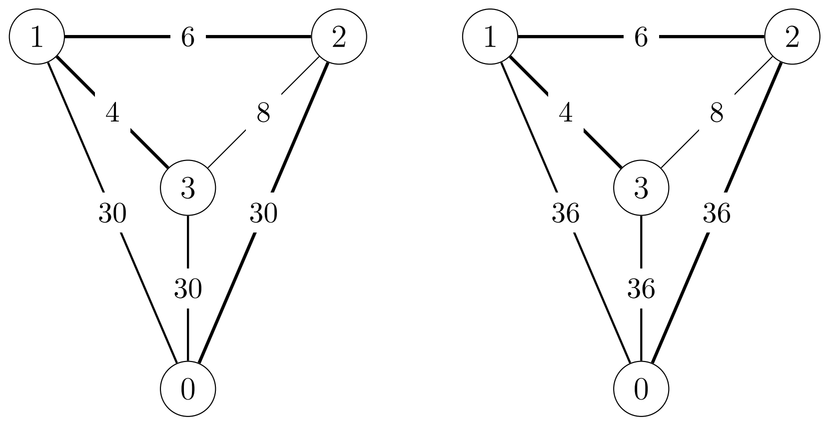

Figure 4: Example of equal share of extra costs.

Equal share of extra costs Let $c, c' \in \Gamma$ and $x_0, x'_0 \in \mathbb{R}$ be such that $c_{0i} = x_0 > x'_0 = c'_{0i} \geq c_{ij} = c'_{ij}$ for all $i, j \in N$. Then, for all $i \in N$, $y_i(c) = y_i(c') + \frac{x_0 - x'_0}{n}$.

The next property is a weaker version of equal share of extra costs:

Equal share of cost reduction Let $c, c' \in \Gamma$, $i \in N$ and $x \in [0, c_{0i}]$ such that $c_{0i} \leq c_{0j}$, $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ otherwise and, for all $j \in N \setminus \{i\}$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, for all $j \in N$, $y_j(c') = y_j(c) - \frac{x}{n}$.

An opposite viewpoint is that the cost reduction should be assigned solely to the agent that is closer to the source and can connect freely to all others.

Full share of cost reduction Let $c, c' \in \Gamma$, $i \in N$ and $x \in [0, c_{0i}]$ such that $c_{0i} \leq c_{0j}$, $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ otherwise and, for all $j \in N \setminus \{i\}$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, $y_i(c') = y_i(c) - x$ and $y_j(c') = y_j(c)$ for all $j \in N \setminus \{i\}$.

A compromise viewpoint from both previous properties is to make sure the fraction of the savings going to the cheapest source connector is the same in all such situations:

Constant share of cost reduction Let $c, c' \in \Gamma$, $i \in N$ and $x \in [0, c_{0i}]$ such that $c_{0i} \leq c_{0j}$, $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ otherwise and, for all $j \in$

$N \setminus \{i\}$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, there exists $\alpha \in \mathbb{R}$ such that $y_i(c') = y_i(c) - \frac{x}{n}(1 + (n-1)\alpha)$ and $y_j(c') = y_j(c) - \frac{x}{n}(1 - \alpha)$ for all $j \in N \setminus \{i\}$.

The following property, as equal share of extra costs, applies to problems where all the agents are equally far away from the source. It states that the cost sharing should be done in the same order as we find an optimal tree, i.e., by picking up one agent randomly, and connecting her to the source.

Decomposition Let $c \in \Gamma$ and $x_0 \in \mathbb{R}$ be such that $c_{0i} = x_0 \geq c_{ij}$ for all $i, j \in N$. Then, for all $i \in N$,

$$y_i(c) = \sum_{j \in N \setminus \{i\}} \frac{y_i(c^j)}{n} + y_i(\hat{c})$$

where $c^j_{0k} = c_{jk}$ and $c^j_{kl} = c_{kl}$ for all $k, l \in N \setminus \{j\}$, and $\hat{c}_{0j} = x_0$ and $c^j_{kl} = 0$ for all $k, l \in N$.

The following property also applies to problems where all the agents are far away from the source, but without requiring them to be equally far away nor any other similarity. It says that, in this case, agents should share the extra cost in the same way in both problems.

Constant share of extra costs Let $c, c' \in \Gamma$ such that $c_{0i} = \max_{j,k \in N_0} c_{jk}$ and $c'_{0i} = \max_{j,k \in N_0} c'_{jk}$ for all $i \in N$ and $x1$ be the cost matrix defined as $x1_{0i} = x$ and $x1_{ij} = 0$ for all $i, j \in N$, for some positive real number x.

Then, $y(c + x1) - y(c) = y(c' + x1) - y(c')$.

Theorem 9 (Bergantiños and Vidal-Puga (2007a)) *The folk rule is the only solution that satisfies population monotonicity, solidarity, and equal share of extra costs.*

Theorem 10 (Bergantiños and Kar (2010)) *The folk rule is the only solution that satisfies population monotonicity, symmetry, solidarity, and constant share of extra costs.*

Theorem 11 (Trudeau (2014a)) *The folk rule is the only solution that satisfies core selection, piece-wise linearity, branch cutting, decomposition, and equal share of cost reduction.*

The next theorem links the folk rule and the cycle-complete rule:

Theorem 12 (Trudeau (2014a)) *A solution satisfies core selection, piece-wise linearity, branch cutting, decomposition, and constant share of cost reduction if and only if it is a linear combination of the folk rule and the cycle-complete rule, i.e., there exists* $\alpha \in [0, 1]$ *such that* $y = \alpha y^f + (1 - \alpha) y^{cc}$.

A strengthening of constant share of cost reduction to give the savings to the agent with the cheap cost to the source yields a characterization of the cycle-complete solution.

Theorem 13 (Trudeau (2014a)) *The cycle-complete rule is the only solution that satisfies core selection, piece-wise linearity, branch cutting, decomposition, and full share of cost reduction.*

6 Correspondences with other concepts

We discuss how other solution concepts have been used in the mcst literature, and the links that have been found with the Shapley value.

6.1 Weighted Shapley values

The weighted versions of the Shapley value (Shapley, 1953a; Kalai and Samet, 1987) have also played a relevant role in mcst problems. Moreover, Bird (1976); Curiel (1997) also use the term weighted Shapley value to refer to restricted orders in the contribution vectors so that an optimal tree is constructed via Prim's algorithm following that order. Bird (1976) proves that this solution belongs to the irreducible core.

In what follows, we use the definition of weighted Shapley values first suggested by Shapley (1953a) and studied by Kalai and Samet (1987).

Bergantiños and Lorenzo-Freire (2008a,b) study the weighted Shapley values of the optimistic game introduced in Bergantiños and Vidal-Puga (2007b) and prove that they are obligation rules. Moreover, they characterize these rules using population monotonicity, solidarity, and weighted version of equal share of extra cost where the extra cost is divided proportionally to the weights of the agents.

Trudeau (2014c) obtains a family of weighted Shapley values when studying an extension of mcst problems in which some agents do not need to be connected to the source.

Gómez-Rúa and Vidal-Puga (2017) study mcst situations in which agents can merge in advance, paying their internal costs. They show that this situation can lead to inefficiencies and prove that the weighted Shapley value of the irreducible cost vector, with weights given by the size of the nodes, is immune to this manipulation. It also inherits most of the nice properties of the folk rule, such as population monotonicity, core selection, solidarity, and piece-wise linearity.

6.2 The core and the nucleolus

The *excess* of a coalition S in a TU game (N, v) with respect to a preimputation x is defined as $e(S, x, C) = C(S) - \sum_{i \in N} x_i$. The vector $\theta(x)$ is constructed by rearranging the 2^n excesses in (weakly) increasing order. If $x, y \in \mathbb{R}^N$ are two allocations, then $\theta(x) >_L \theta(y)$ means that $\theta(x)$ is lexicographically larger than $\theta(y)$. As usual, we write $\theta(x) \geq_L \theta(y)$ to indicate that either $\theta(x) >_L \theta(y)$ or $x = y$.

The *nucleolus* of the game C is the set

$$Nu(C) = \{x \in X : \theta(x) \geq_L \theta(y) \forall y \in X\}$$

where $X = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = C(N), x_i \geq v(\{i\}) \forall i \in N\right\}$ is the set of *imputations* (individually rational preimputations). When $X \neq \emptyset$, as it is the case for the TU games we study here, it is wellknown that $Nu(C)$ is a singleton, whose unique element we denote, with some abuse of notation, also as $Nu(C)$.

Let Π be the set of permutations of N. For all $\pi \in \Pi$, let $y^\pi \in Core(C)$ be the allocation that lexicographically maximizes the allocations with respect to the order given by the permutation. The permutation-weighted average of extreme points of the core is the average of these allocations: $\gamma(C) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^\pi(C)$. If the game is concave, γ is the Shapley value. It is also closely related to the selective value (Vidal-Puga, 2004) and the Alexia value (Tijs, 2005), the permutation-weighted average of leximals.

Consider the subset of mcst problems known as elementary[7] mcst problems: for any $i, j \in N_0$, $c_{ij} \in \{0, 1\}$. Let Γ^e be the set of elementary cost

[7]These cost games are named *information graph games* by Kuipers (1993).

matrices.

Theorem 14 (Trudeau and Vidal-Puga (2017a,b)) *For all $c \in \Gamma^e$, we have $y^{cc}(c) = Nu(C(N,c)) = \gamma(C(N,c))$.*

Theorem 15 (Trudeau and Vidal-Puga (2017a,b)) *For all $c \in \Gamma^e$, we have $y^f(c) = Nu(C^{Pub}(N,c)) = \gamma(C^{Pub}(N,c))$.*

7 The Shapley value in other related problems

Some of the different versions of the Shapley value have also been studied in different subclasses and extensions of mcst problems, but it is still a very unexplored field of research. In particular, the following literature focuses on the extensions of the folk solution in the private game case.

Dutta and Mishra (2012) and Bahel and Trudeau (2017) extend the folk rule to minimal cost arborescence problems, where the cost vector describing the cost of connecting each pair of nodes is not necessarily symmetric. An extension of the cycle-complete solution is offered in the latter.

Bergantiños and Gómez-Rúa (2010, 2015) extend the folk rule to mcst problems with groups, where agents are grouped by a partition, such that the nodes inside each member of the partition (a group) are more related to each other than to any node in another group.

Subiza et al. (2016) study the folk rule in simple mcst problems, where only two different costs are possible.

Finally, from a non-cooperative point of view, the folk rule appears as subgame perfect Nash equilibrium cost allocation in several mechanisms applied to mcst problems (Bergantiños and Vidal-Puga, 2010; Hougaard and Tvede, 2013).

8 Conclusion

In this chapter we have reviewed the literature on the minimum cost spanning tree problems. This literature is unique in that most of the allocation methods considered are Shapley values. There are (at least) five different ways to define a game based on a mcst problem, before taking the Shapley value.

The games vary depending on how we set the ground rules: who has access to which nodes, what cost we consider for each edge, etc. The solutions vary depending on whether we care about core stability, if we allow coalitions to use the nodes of their neighbours, and if we take an optimistic or pessimistic view of the game. The corresponding axiomatic analysis reflects the choices made in how we define the game.

Acknowledgments

References

Ando, K. (2012). Computation of the Shapley value of minimum cost spanning tree games: #P-hardness and polynomial cases. *Japan Journal of Industrial and Applied Mathematics*, 29(3):385–400.

Bahel, E. and Trudeau, C. (2017). Minimum incoming cost rules for arborescences. *Social Choice and Welfare*, 49(2):287–314.

Bergantiños, G. and Gómez-Rúa, M. (2010). Minimum cost spanning tree problems with groups. *Economic Theory*, 43(2):227–262.

Bergantiños, G. and Gómez-Rúa, M. (2015). An axiomatic approach in minimum cost spanning tree problems with groups. *Annals of Operations Research*, 225(1):45–63.

Bergantiños, G. and Kar, A. (2010). On obligation rules for minimum cost spanning tree problems. *Games and Economic Behavior*, 69(2):224–237.

Bergantiños, G. and Lorenzo, L. (2004). A non-cooperative approach to the cost spanning tree problem. *Mathematical Methods of Operations Research*, 59(3):393–403.

Bergantiños, G., Lorenzo, L., and Lorenzo-Freire, S. (2010). The family of cost monotonic and cost additive rules in minimum cost spanning tree problems. *Social Choice and Welfare*, 34(4):695–710.

Bergantiños, G., Lorenzo, L., and Lorenzo-Freire, S. (2011). A generalization of obligation rules for minimum cost spanning tree problems. *European Journal of Operational Research*, 211(1):122–129.

Bergantiños, G. and Lorenzo-Freire, S. (2008a). A characterization of optimistic weighted Shapley rules in minimum cost spanning tree problems. *Economic Theory*, 35(3):523–538.

Bergantiños, G. and Lorenzo-Freire, S. (2008b). “Optimistic” weighted Shapley rules in minimum cost spanning tree problems. *European Journal of Operational Research*, 185(1):289–298.

Bergantiños, G. and Vidal-Puga, J. (2007a). A fair rule in minimum cost spanning tree problems. *Journal of Economic Theory*, 137(1):326–352.

Bergantiños, G. and Vidal-Puga, J. (2007b). The optimistic TU game in minimum cost spanning tree problems. *International Journal of Game Theory*, 36(2):223–239.

Bergantiños, G. and Vidal-Puga, J. (2008). On some properties of cost allocation rules in minimum cost spanning tree problems. *AUCO Czech Economic Review*, 2(3):251–267.

Bergantiños, G. and Vidal-Puga, J. (2009). Additivity in minimum cost spanning tree problems. *Journal of Mathematical Economics*, 45(1-2):38–42.

Bergantiños, G. and Vidal-Puga, J. (2010). Realizing fair outcomes in minimum cost spanning tree problems through non-cooperative mechanisms. *European Journal of Operational Research*, 201(3):811–820.

Bergantiños, G. and Vidal-Puga, J. (2011). The folk solution and Boruvka's algorithm in minimum cost spanning tree problems. *Discrete Applied Mathematics*, 159(12):1279–1283.

Bird, C. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks*, 6:335–350.

Bogomolnaia, A. and Moulin, H. (2010). Sharing the cost of a minimal cost spanning tree: beyond the folk solution. *Games and Economic Behavior*, 69(2):238–248.

Borůvka, O. (1926). O jistém problému minimálním (about a certain minimal problem). *Práce moravské přírodovědecké společnosti v Brne*, 3:37–58. In Czech.

Branzei, R., Moretti, S., Norde, H., and Tijs, S. (2004). The P-value for cost sharing in minimum cost spanning tree situations. *Theory and Decision*, 56(1):47–61.

Curiel, I. (1997). *Minimum Cost Spanning Tree Games*, volume 16 of *Theory and Decision Library*, chapter 6, pages 129–148. Academic Publishers, Boston, MA.

Dutta, B. and Kar, A. (2004). Cost monotonicity, consistency and minimum cost spanning tree games. *Games and Economic Behavior*, 48(2):223–248.

Dutta, B. and Mishra, D. (2012). Minimum cost arborescences. *Games and Economic Behavior*, 74(1):120–143.

Feltkamp, V., Tijs, S., and Muto, S. (1994). On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems. Technical Report 106, CentER DP 1994, Tilburg University, The Netherlands.

Gómez-Rúa, M. and Vidal-Puga, J. (2017). A monotonic and merge-proof rule in minimum cost spanning tree situations. *Economic Theory*, 63:813–826.

Hougaard, J. L. and Tvede, M. (2013). Truth-telling and nash equilibria in minimum cost spanning tree models. *European Journal of Operational Research*, 222(3):566–570.

Kalai, E. and Samet, D. (1987). On weighted Shapley values. *International Journal of Game Theory*, 16(3):205–222.

Kar, A. (2002). Axiomatization of the Shapley value on minimum cost spanning tree games. *Games and Economic Behavior*, 38(2):265–277.

Kruskal, J. B. (1956). On the shortest spanning subtree of a graph and the travelling salesman problem. *Proceedings of the American Mathematical Society*, 7:48–50.

Kuipers, J. (1993). On the core of information graph games. *International Journal of Game Theory*, 21(4):339–350.

Lorenzo, L. and Lorenzo-Freire, S. (2009). A characterization of Kruskal sharing rules for minimum cost spanning tree problems. *International Journal of Game Theory*, 38(1):107–126.

Maniquet, F. (2003). A characterization of the Shapley value in queueing problems. *Journal of Economic Theory*, 109:90–103.

Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2(3):225–229.

Myerson, R. B. (1980). Conference structures and fair allocation rules. *International Journal of Game Theory*, 9(3):169–182.

Norde, H., Moretti, S., and Tijs, S. (2004). Minimum cost spanning tree games and population monotonic allocation schemes. *European Journal of Operational Research*, 154(1):84–97.

Prim, R. (1957). Shortest connection networks and some generalizations. *Bell Systems Technology Journal*, 36(6):1389–1401.

Shapley, L. S. (1953a). *Additive and non-additive set functions*. PhD thesis, Princeton University.

Shapley, L. S. (1953b). A value for n-person games. In Kuhn, H. and Tucker, A., editors, *Contributions to the theory of games*, volume II of *Annals of Mathematics Studies*, pages 307–317. Princeton University Press, Princeton NJ.

Subiza, B., Giménez-Gómez, J.-M., and Peris, J. E. (2016). Folk solution for simple minimum cost spanning tree problems. *Operations Research Letters*, 44(5):598–601.

Sziklai, B., Fleiner, T., and Solymosi, T. (2016). On the core and nucleolus of directed acyclic graph games. *Mathematical Programming*, 163(1):243–271.

Tijs, S. (2005). The first steps with Alexia, the Average lexicographic value. CentER DP 2005-12, Tilburg University.

Tijs, S., Branzei, R., Moretti, S., and Norde, H. (2006). Obligation rules for minimum cost spanning tree situations and their monotonicity properties. *European Journal of Operational Research*, 175(1):121–134.

Trudeau, C. (2012). A new stable and more responsible cost sharing solution for mcst problems. *Games and Economic Behavior*, 75(1):402–412.

Trudeau, C. (2013). Characterizations of the Kar and folk solutions for minimum cost spanning tree problems. *International Game Theory Review*, 15(2):1340003–16.

Trudeau, C. (2014a). Characterizations of the cycle-complete and folk solutions for minimum cost spanning tree problems. *Social Choice and Welfare*, 42(4):941–957.

Trudeau, C. (2014b). Linking the Kar and folk solutions through a problem separation property. *International Journal of Game Theory*, 43(4):845–870.

Trudeau, C. (2014c). Minimum cost spanning tree problems with indifferent agents. *Games and Economic Behavior*, 84:137–151.

Trudeau, C. and Vidal-Puga, J. (2017a). Clique games: a family of games with coincidence between the nucleolus and the Shapley value. Department of economics working paper series 17-05, University of Windsor.

Trudeau, C. and Vidal-Puga, J. (2017b). On the set of extreme core allocations for minimal cost spanning tree problems. *Journal of Economic Theory*, 169:425–452.

Vidal-Puga, J. (2004). Bargaining with commitments. *International Journal of Game Theory*, 33(1):129–144.

Young, H. P. (1985). Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2):65–72.