A value for PERT problems*

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Abstract

The *PERT* (Program Evaluation Review Technique) is a operational research tool used to schedule and coordinate activities in a complex project. We present two values for measuring the importance of each activity. Both values are obtained through an axiomatic characterization using three properties. The first value is characterized with separability, monotonicity, and order preservation. The second value is characterized with separability, equal treatment inside a component, and independence of large durations. We also present an application to the problem of how to share the surplus obtained when a project finishes before the expected completion time.

Keywords: *PERT* problem, value, game theory.

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1 Introduction

Complex projects require a series of activities, some of which must be performed sequentially and others that can be performed in parallel with other activities. This collection of series and parallel tasks can be modeled as a network. The $PERT$ (Program Evaluation Review Technique) is an operational research tool used to schedule and coordinate these activities. The $PERT$ has been developed by the U.S. Navy in the late 1950's to manage the Polaris submarine missile program, a project having thousands of contractors[1].

$PERT$ planning involves several steps, which include the estimation of the time required for each activity and to determine the *critical activities*, *i.e.* activities which have the potential to delay the whole project.

An important question addressed in the literature of cooperative game theory is how to measure the power (or importance) of each player. In TU games the answer is given by values as, for instance, the Shapley value (Shapley, 1953).

The $PERT$ time is the minimal time required for completing the project. In this paper we try to measure the importance of each activity in the $PERT$ time. To solve this question, we share the $PERT$ time between the activities involved in the project. We can then consider the time assigned to an activity as a measure of its importance. Assume that the activities are lined-up. In this case, the $PERT$ time is the sum of the duration of the activities, and the responsibility of each activity in the $PERT$ time is exactly its duration. Even though in this example the answer is trivial, in general this is not an easy question.

We follow the axiomatic approach. We propose some desirable properties for the aim of fairness and then, we obtain an axiomatic characterization of a unique value satisfying these properties.

Any value dividing the $PERT$ time between the activities must take

[1]The Critical Path Method (CPM) is a similar technique developed in 1957 for project management in the private sector. CPM has become synonymous with PERT, so that the technique is known by any variation on the names: PERT, CPM, or PERT/CPM.

into account two aspects: the duration of the activities and their position in the network defining the project. Thus, we formulate two properties. Monotonicity refers to the duration of the activities. Separability refers to the network.

Consider two $PERT$ problems with the same associated network. Assume that the duration of each activity in the second problem is not smaller than in the first problem. Monotonicity says that the value of each activity in the second problem is not smaller than the value of this activity in the first problem.

Assume that we can divide the $PERT$ problem in several $PERT$ subproblems in such a way that the $PERT$ problem is the disjoint union of the subproblems. Separability says that for each activity, the value in the whole problem must coincide with the value in the subproblem the activity belongs to.

We first study the class of rules satisfying monotonicity and separability. Since there are many rules we impose an additional property called order preservation. This property says that if all activities but one have a duration of 0, the value of the activity with a duration larger than 0 cannot be smaller than the value of each one of the rest of the activities.

Our main result says that there is a unique value satisfying monotonicity, separability, and order preservation. This rule is defined as follows. We first assign to each subproblem its $PERT$ time. Second, we divide the $PERT$ time of each subproblem equally among their activities.

We also present an application of this result. Suppose that a firm should carry out a project. This firm has several departments and each department is assigned to perform a different activity. If the project is completed before the expected completion time a surplus is generated. An interesting question is how to divide this surplus between the departments of the firm.

We propose a rule for dividing the surplus between the activities. We first compute for each activity the time assigned by the value assuming that all the activities employ exactly their scheduled time. Later, given the actual completion time, we compute, for each activity, the time assigned by the value. The responsibility of each activity of the time saved in the whole project is the difference between the value when each activity employs exactly its expected completion time, and the value assuming that each activity employs the actual completion time. The surplus is divided proportionally to this responsibility.

The value characterized above could not be suitable in some $PERT$ problems. For instance, the value of each activity depends on the total duration of the component this activity belongs to, but not on the duration of this activity. Thus, we present a value that takes into account both, the duration of the component and the duration of the activity. This value is inspired in the serial cost sharing rule introduced by Moulin and Shenker (1992). We characterize this value as the unique value satisfying separability, independence of large durations (the value of an activity does not depend on the duration of the activities with a larger duration), and equal treatment inside each component (if two activities have the same duration and belong to the same component, then both activities must have the same value).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we introduce and characterize the first value. Section 4 is devoted to an application on the first value. In Section 5 we introduce and characterize the second value. Finally, in the Appendix it is possible to find the proof of the results.

2 The model

We denote the set of nonnegative real numbers as $\mathbb{R}_+$. Given a finite set N, we denote the cardinality of N as $|N|$, and the set of all functions from N to $\mathbb{R}$ as $\mathbb{R}^N$. A member x of $\mathbb{R}^N$ is an n-dimensional vector whose coordinates are indexed by members of N; thus, when $a \in N$, we write x_a for $x(a)$.

A $PERT$ *problem* is a triple $(N, \prec, x)$ where N is a finite set of *activities*, $\prec$ is a relation in N satisfying *transitivity* ($a \prec b$ and $b \prec c$ implies $a \prec c$)

and *strict anti-symmetry* ($a \prec b$ implies $b \nprec a$), and $x \in \mathbb{R}_+^N$.

Given $a, a' \in N$, $a \prec a'$ means that activity a' cannot begin until activity a is finished. Given $a \in N$, x_a represents the expected completion time of activity a. We call x_a the duration of activity a.

Given $(N, \prec, x)$ and $M \subset N$, we denote as $(M, \prec, x)$ the $PERT$ problem that arises in the restriction of $(N, \prec, x)$ to M.

Given $a, a' \in N$, $a \sim a'$ means that either $a \prec a'$ or $a' \prec a$. Given $M, M' \subset N$, the expression $M \prec M'$ means that $a \prec a'$ for all $a \in M$, $a' \in M'$. The expression $M \sim M'$ means that $a \sim a'$ for all $a \in M$, $a' \in M'$.

A *pseudo-path* in $(N, \prec)$ is a nonempty set $P = \{a_i\}_{i=1}^p \subset N$ such that $a_{i-1} \prec a_i$ for all $i = 2, ..., p$. We denote the set of all pseudo-paths in $(N, \prec)$ as $\mathcal{P}_{(N,\prec)}$. $\mathcal{P}_{(N,\prec)} \neq \emptyset$ because $\{a\} \in \mathcal{P}_{(N,\prec)}$ for all $a \in N$. Given $a \in N$, we denote the set of all pseudo-paths in $(N, \prec)$ that contain a as $\mathcal{P}^a_{(N,\prec)}$. For all $a \in N$, $\mathcal{P}^a_{(N,\prec)} \neq \emptyset$ because $\{a\} \in \mathcal{P}_{(N,\prec)}$.

Given $x \in \mathbb{R}_+^N$, we define the $PERT$ *time* as the minimum time we need to finish the project, *i.e.*

$$\tau(N, \prec, x) = \max_{P \in \mathcal{P}_{(N,\prec)}} \sum_{a \in P} x_a.$$

A *critical pseudo-path* in $(N, \prec, x)$ is a pseudo-path $P \in \mathcal{P}_{(N,\prec)}$ such that

$$\sum_{a \in P} x_a = \tau(N, \prec, x).$$

A *critical activity* in $(N, \prec, x)$ is an activity that belongs to a critical pseudo-path in $(N, \prec, x)$.

A *component* is a nonempty subset $C \subset N$ such that $\tau(N, \prec, x) = \tau(C, \prec, x) + \tau(N \backslash C, \prec, x)$ for all $x \in \mathbb{R}_+^N$. Let $\mathcal{C}_{(N,\prec)}$ be the set of components in $(N, \prec)$. In particular, $N \in \mathcal{C}_{(N,\prec)}$. A *minimal component* is a component without proper subcomponents, *i.e.* C is a minimal component if there does not exist any component $C' \subsetneq C$. Let $\mathcal{C}^m_{(N,\prec)}$ be the set of minimal components in $(N, \prec)$.

A *value* is a function f that assigns to each $PERT$ problem $(N, \prec, x)$ a vector $\{f_a(N, \prec, x)\}_{a \in N} \in \mathbb{R}_+^N$ such that

$$\sum_{a \in N} f_a(N, \prec, x) = \tau(N, \prec, x).$$

Each $f_a(N, \prec, x)$ represents the share of the $PERT$ time that is assigned to a.

3 Characterization of the first value

In this section we first prove that the set of minimal components are a partition of N. Later, we introduce the properties of monotonicity, separability, and order preservation. In Theorem 1 we characterize a value for $PERT$ problems using these properties.

Proposition 1 *Given a $PERT$ problem $(N, \prec, x)$, the set of minimal components is a partition of N.*

Proof. See the Appendix. ■

We consider several properties for a value f. Since our objective is to divide the $PERT$ time between the activities, we must take into account the duration of the activities and their position in the network defining the project. We first define two properties: monotonicity, which refers to the duration of the activities; and separability, which refers to the network.

Monotonicity (MON) Let $(N, \prec, x)$ and $(N, \prec, y)$ be two $PERT$ problems such that $x_a \leq y_a$ for all $a \in N$. Then, $f_a(N, \prec, x) \leq f_a(N, \prec, y)$ for all $a \in N$.

MON says that the value must be a nondecreasing function of the vector of durations.

Remark 1. It is easy to check that MON can also be defined in the following way. Let $(N, \prec, x)$ and $(N, \prec, y)$ be two $PERT$ problems such that

$x_b < y_b$ for some $b \in N$ and $x_a = y_a$ for all $a \in N \backslash \{b\}$. Then, $f_a(N, \prec, x) \leq f_a(N, \prec, y)$ for all $a \in N$.

Separability (SEP) Let $(N, \prec, x)$ be a $PERT$ problem and $C \subset N$ be a component. Then, $f_a(N \prec, x) = f_a(C \prec, x)$ for all $a \in C$.

SEP says that if the $PERT$ time of a group of activities is independent on what the other activities do (*i.e.* they are a component), the value should not take into account the other activities.

In the next proposition we prove that the values satisfying MON and SEP only depend on $(N, \prec)$ and $\{\tau(C, \prec, x)\}_{C \in \mathcal{C}^m_{(N,\prec)}}$.

Proposition 2 *Let $(N, \prec, x)$ and $(N, \prec, y)$ be two $PERT$ problems satisfying that $\tau(C, \prec, x) = \tau(C, \prec, y)$ for all $C \in \mathcal{C}^m_{(N,\prec)}$. If f is a value that satisfies MON and SEP, then $f(N, \prec, x) = f(N, \prec, y)$.*

Proof. See the Appendix. ■

We now present other desirable property.

Order Preservation (OP) Let $(N, \prec, x)$ be a $PERT$ problem and $b \in N$ such that $x_a = 0$ for all $a \neq b$. Then $f_a(N, \prec, x) \leq f_b(N, \prec, x)$ for all $a \neq b$.

OP says that if all the activities but one finish immediately, then the time assigned to the lengthiest activity should not be less than the time assigned to the others.

Our main result of the section is the following:

Theorem 1 *There exists a unique value satisfying MON, SEP and OP, and it is given by*

$$f^0_a(N, \prec, x) = \frac{\tau(C, \prec, x)}{|C|}$$

for each $a \in C \in \mathcal{C}^m_{(N,\prec,x)}$.

Proof. See the Appendix. ■

f^0 is a value dividing the time of each component equally among the activities of the component. We admit that examples of $PERT$ problems exist where f^0 is not adequate (in Section 5 we give an example). Nevertheless, there exist subclasses of $PERT$ problems, as the ones we present in the next section, where f^0 could be adequate.

If we observe the three properties characterizing f^0 we realize that MON is the property that can be more easily criticized. Also, observing the proofs, we realize that MON is the property that "makes" the value to equally divide the time among the activities of the same component.

MON is a quite standard property in the literature and it is used in many kind of problems. We discuss the impact of the monotonicity property in two problems: bargaining problems and minimum cost spanning tree problems. In bargaining problems an egalitarian rule is characterized with the monotonicity property. In minimum cost spanning tree problems a non-egalitarian rule is characterized with the monotonicity property.

Nash (1950) introduced the bargaining problem. Kalai (1977) characterized the egalitarian rule with three properties: symmetry, weak Pareto optimality and monotonicity (if the bargaining set increases, nobody can be worse off). This result has some common aspects with our result. First, we characterize an egalitarian rule with monotonicity and two weak properties. The characterized rule cannot be adequate in some class of problems. For instance, the egalitarian rule does not satisfy Pareto optimality in some class of bargaining problems.

In minimum cost spanning tree problems Bergantiños and Vidal-Puga (2007) characterize a rule with three properties: *population monotonicity*, *equal share of extra costs*, and a monotonicity property called *solidarity*. This monotonicity property says that if the connection cost between two agents increases, no agent can be better off. The rule characterized is the Shapley value of a game and it does not divide the cost equally among the agents.

We end this section by proving that the properties used in Theorem 1 are independent.

- Let $\mathcal{P}^0_{(N,\prec)}$ be the set of critical pseudo-paths in $(N,\prec)$. Given $P \in \mathcal{P}^0_{(N,\prec)}$ and $x \in \mathbb{R}^N_+$, let $f^P(N,\prec,x) \in \mathbb{R}^N_+$ be defined as $f^P_a(N,\prec,x) = x_a$ if $a \in P$ and $f^P_a(N,\prec,x) = 0$ otherwise. We define f^1 as

$$f^1(N,\prec,x) = \frac{1}{\left|\mathcal{P}^0_{(N,\prec)}\right|} \sum_{P \in \mathcal{P}^0_{(N,\prec)}} f^P(N,\prec,x).$$

  It is trivial to see that f^1 satisfies SEP and OP. However, f^1 does not satisfy MON. For example, assume that $N = \{a,b\}$ and $a \not\prec b$, then $f^1(N,\prec,(1,0)) = (1,0)$ and $f^1(N,\prec,(1,2)) = (0,2)$.

- We define the value f^2 as

$$f^2_a(N,\prec,x) = \frac{\tau(N,\prec,x)}{|N|} \text{ for all } a \in N.$$

  It is trivial to see that f^2 satisfies MON and OP. However, f^2 does not satisfy SEP. For example, assume that $N = \{a,b\}$ and $a \prec b$, both $\{a\}$ and $\{b\}$ are components. Then, $f^2(N,\prec,(1,3)) = (2,2)$.

- Let $\sigma : N \to \{1,...,|N|\}$ be a one-to-one function. Let f^σ be defined as $f^\sigma_a(N,\prec,x) = \tau(C,\prec,x)$ for each $a \in C \in \mathcal{C}^m_{(N,\prec)}$ such that $\sigma(a) = \min_{b \in C} \sigma(b)$ and $f^\sigma_a(x) = 0$ otherwise. It is trivial to see that f^σ satisfies MON and SEP. However, f^σ does not satisfy OP. For example, assume that $N = \{a,b\}$ and $a \not\prec b$, when $\sigma(a) = 1$ and $\sigma(b) = 2$, $f^\sigma(N,\prec,(0,1)) = (1,0)$.

4 An application

Suppose that a firm should carry out a project. This firm has several departments and each department is assigned to perform a different activity. If the project is completed before the $PERT$ time a surplus is generated. An interesting question is how to divide this surplus between the departments of the firm. In this section we show that the results stated in the previous section can be applied to this problem.

A "dual" problem has been previously studied by Bergantiños and Sánchez (2002a), Branzei *et al* (2002), and Castro *et al* (2007, 2008a). In these papers the project is delayed and a cost is generated. The question addressed is how to divide this cost between the activities of the project. Bergantiños and Sánchez (2002a) present two rules for sharing the cost generated by the delay of the project. One of the rules is based on serial cost sharing problems and the other is based on game theory. Branzei *et al* (2002) take also two different approaches. First, they assign to each $PERT$ problem a bankruptcy problem. Later, they apply well-known rules of the associated bankruptcy problem to the $PERT$ problem. Second, they define rules based on delays of paths. Castro *et al* (2007) model the problem as a cooperative game. Castro *et al* (2008a) present a rule which is a weighted version of serial cost sharing problems. There exists another difference between these papers. Branzei *et al* (2002) and Castro *et al* (2007) divide the delay of the project. Bergantiños and Sánchez (2002a) and Castro *et al* (2008a) divide the total cost caused by the delay.

In this paper we follow the approach of Branzei *et al* (2002) and Castro *et al* (2007). We divide the time saved in the project (the difference between the $PERT$ time and the actual completion time) among the activities. Later, we divide the surplus proportionally to the time saved assigned to the activities.

We use the value f^0 obtained in Theorem 1 for computing the time saved assigned to each activity.

Let $(N, \prec, x)$ be a $PERT$ problem where x represents the (estimated) completion times of the activities involved in the project. Assume that the project is completed before the expected time $\tau(N, \prec, x)$. Let $(N, \prec, y)$ be the $PERT$ problem where for all $a \in N$, y_a represent the actual completion time of activity a. We assume that $x_a \geq y_a$ for all $a \in N$.

Notice that the total time saved in the project is $t = \tau(N, \prec, x) - \tau(N, \prec, y)$. We divide this time between the different activities.

Given $a \in N$, we define the time saved by activity a as

$$t_a = f_a^0(N, \prec, x) - f_a^0(N, \prec, y).$$

We must prove that $t_a \geq 0$ for all $a \in N$ and $\sum_{a \in N} t_a = t$.

Since $x_a \geq y_a$ for all $a \in N$ and f^0 satisfies MON, we conclude that $f_a^0(N, \prec, x) \geq f_a^0(N, \prec, y)$ for all $a \in N$. Thus, $t_a \geq 0$ for all $a \in N$.

Moreover,

$$\begin{aligned} \sum_{a \in N} t_a &= \sum_{a \in N} f_a^0(N, \prec, x) - \sum_{a \in N} f_a^0(N, \prec, y) \\ &= \tau(N, \prec, x) - \tau(N, \prec, y) = t. \end{aligned}$$

Let s be the surplus generated when the project is finished t units before his expected completion time. Thus, the surplus assigned to activity a (or equivalently the department of the firm responsible for activity a) is defined as

$$s_a = \frac{t_a}{t} s.$$

Remark 2. Assume that the surplus s depends linearly of the total time saved of the project, *i.e.* $s = \alpha t$ where $\alpha \in \mathbb{R}_+$. Giving a time saving vector $\{t_a\}_{a \in N}$, we believe that to divide the surplus proportionally to this vector is the most reasonable way. We also think that if the surplus does not depend linearly on the total time saved, more ways of dividing the surplus could be reasonable.

We mentioned in the previous section that MON is a very demanding property. Moreover, as in bargaining problems, in some subclass of problems if we insist in MON we do not obtain very interesting rules or values. Nevertheless, there are other subclasses of problems where MON is a very appealing property. For instance, the subclass of problems studied in this section. Let us clarify this point with two arguments. The first one is based on fairness arguments and the second one on incentive arguments.

1. The fairness argument.

   We will use the definition of MON of Remark 1. We will assume that a firms department reduces the completion time of its activity, while for the other activities it does not change. It seems clear that the surplus assigned to this department should not decrease. How should the surplus of other activities be affected?

   Notice that the reduction of an activity's completion time implies that the project's total time is also reduced or remains the same. Thus, the surplus of the firm increases or remains the same. We believe that the surplus of the other departments should not decrease.

   MON guarantees it. Let $(N, \prec, x)$ be a $PERT$ problem where x is the expected completion time of the activities. Let $(N, \prec, y)$ and $(N, \prec, y')$ be two $PERT$ problems in which the departments of the firm reduce the completion time of the activities, *i.e* for all $a \in N$, $y_a \leq x_a$ and $y'_a \leq x_a$. Assume that the department of activity b reduces its completion time more in y' than in y, *i.e.* $y_b > y'_b$. Moreover, all departments but b reduce the same time in y than in y', *i.e.* $y_a = y'_a$ for all $a \in N \backslash \{b\}$. If a value f satisfies MON, then $f_a(N, \prec, y) \geq f_a(N, \prec, y')$ for all $a \in N$. If we compare the time assigned to the activities in y and y' we obtain that for all $a \in N$,

   $$t_a = f_a^0(N, \prec, x) - f_a^0(N, \prec, y) \leq f_a^0(N, \prec, x) - f_a^0(N, \prec, y') = t'_a.$$

   Now, it is easy to conclude that $s_a \leq s'_a$ for all $a \in N$. Then, the surplus of other departments does not decrease.

2. The incentive argument.

   Assume that all the departments in a firm are trying to reduce the completion time of the activities. Even though each activity is assigned to a specific department, departments should often communicate between them, for instance by sharing information on the activity's progress.

> Suppose that department a needs information from department b in order to decrease the completion time of activity a. Department b knows that by giving this information to department a, department a will reduce his completion time. What are the incentives, in terms of the surplus that department b will receive, of department b?
>
> If the value satisfies MON, department b will receive more surplus because department a will reduce its completion time. Thus, department b has incentives to give this information to department a.
>
> If the value does not satisfy MON, it could be the case that the surplus received by department b is smaller when department a reduces his completion time. Thus, department b has incentives to not pass this information onto department a. Of course, this is an undesirable situation for the firm as the total surplus for reducing the project could be smaller.
>
> From the firms point of view, MON guarantees that the different departments have incentives to collaborate between them.

We end this section with a numerical example.

Let $(N, \prec, x)$ be a $PERT$ problem where $N = \{a, b, c\}$, $a \prec b$, and $a \prec c$. Assume that $x = (5, 6, 7)$ and $y = (4, 3, 6)$. Moreover, we assume that the surplus is ten times the total saved time, *i.e.* $s = 10t$.

Making some computations we obtain the following. The total time saved is $t = \tau(N, \prec, x) - \tau(N, \prec, y) = 12 - 10 = 2$.

There is two minimal components $C_1 = \{a\}$ and $C_2 = \{b, c\}$. Thus, $f^0(N, \prec, x) = (5, 3.5, 3.5)$, $f^0(N, \prec, y) = (4, 3, 3)$, $t_a = 1$, and $t_b = t_c = 0.5$.

The surplus assigned to each activity is $(10, 5, 5)$.

5 Characterization of the second value

Bergantiños and Sánchez (2002b, 2002c) introduce a value for dividing the slack of the project among the activities. This value, as f^0, does not depend directly on the duration of the activities. Later, Castro *et al* (2008b) define a value, for dividing the slack of the project among the activities, which depends directly on the duration of the activities. In this section we present, in $PERT$ problems, a new value which depends directly on the duration of the activities. The new value is defined by applying the ideas of the serial cost sharing rule introduced in Moulin and Shenker (1992).

Let $(N, \prec, x)$ be a $PERT$ problem. We assume without loss of generality that $x_a \leq x_b$ when $a < b$. For each $(N, \prec, x)$ and each $a \in N$, we consider the vector of durations x^a where $x_b^a = \min\{x_b, x_a\}$ for all $b \in N$.

We define the value f^d as follows. For each $a \in C \in \mathcal{C}^m_{(N,\prec,x)}$, we define

$$f_a^d(N, \prec, x) = \sum_{b=1}^{a} \frac{\tau\left(C, \prec, x^b\right) - \tau\left(C, \prec, x^{b-1}\right)}{|\{b' \in C : x_{b'} \geq x_b\}|}$$

where, by convention, $\tau(C, \prec, x^0) = 0$.

Let us clarify this definition with an example. Let $(N, \prec, x)$ be a $PERT$ problem where $N = \{a, b, c\}$, $a \prec b$, and $x = (1, 1, 9)$. Thus, $C = N$, $x^a = x^b = (1, 1, 1)$, $x^c = x$, $\{b' \in C : x_{b'} \geq x_a\} = \{b' \in C : x_{b'} \geq x_b\} = N$, and $\{b' \in C : x_{b'} \geq x_c\} = \{3\}$. Now

$$\begin{aligned} f_a^d(N, \prec, x) &= f_a^d(N, \prec, x) = \frac{\tau(C, \prec, x^a)}{3} = \frac{2}{3}, \\ f_c^d(N, \prec, x) &= \frac{\tau(C, \prec, x^a)}{|N|} + \frac{\tau\left(C, \prec, x^b\right) - \tau(C, \prec, x^a)}{|N|} + \frac{\tau(C, \prec, x^c) - \tau\left(C, \prec, x^b\right)}{|\{3\}|} \\ &= \frac{2}{3} + \frac{0}{3} + 7 = \frac{23}{3}. \end{aligned}$$

If we compute the first value we obtain that $f_a^0(N, \prec, x) = (3, 3, 3)$. In this example, f^d seems more reasonable. The main difference between both values is that with f^0 two activities in the same component have the same value always, independently of their durations. Nevertheless, with f^d two activities in the same component with different durations may have different values, as shown in this example.

We now introduce two properties, which will be used in the characterization of f^d.

Independence of large durations (ILD) For each $(N, \prec, x)$ and each $a \in N$, $f_a(N, \prec, x) = f_a(N, \prec, x^a)$.

ILD says that the value of an activity does not depend on the durations of activities with a larger duration.

Equal treatment inside a component (ETC) For each $(N, \prec, x)$, each component C, and each $a, b \in C$ such that $x_a = x_b$, then $f_a(N, \prec, x) = f_b(N, \prec, x)$.

ETC says that the if two activities have the same duration and belong to the same component, then both activities must have the same value.

In the next theorem we provide a characterization of f^d.

Theorem 2 *f^d is the unique value satisfying SEP, ILD, and ETC.*

Proof. See the Appendix. ■

We end this section by proving that the properties used in 2 are independent.

- Let $(N, \prec, x)$ be a $PERT$ problem. We assume without loss of generality that $x_a \leq x_b$ when $a < b$. We define the value f^α by applying the ideas of the serial cost sharing rule as follows. For each a, we define

$$f_a^d(N, \prec, x) = \sum_{b=1}^{a} \frac{\tau(N, \prec, x^b) - \tau(N, \prec, x^{b-1})}{|\{b' \in N : x_{b'} \geq x_b\}|}.$$

  f^α satisfies ETC and ILD but fails SEP.

- f^0 satisfies SEP and ETC but fails ILD.

- Let $\sigma : N \to \{1, ..., |N|\}$ be a one-to-one function. For each $a \in C \in \mathcal{C}^m_{(N,\prec)}$, let $f^{*\sigma}$ be defined as

$$f^{*\sigma}_a (N, \prec, x) = \tau \left(C, \prec, x^{\sigma(a)}\right) - \tau \left(C, \prec, x^{pre(a,C,\sigma)}\right)$$

where $pre\,(a, C, \sigma)$ denotes the agent in C which comes just before a in the order given by σ, namely $\{b \in C : pre\,(a, C, \sigma) < \sigma (b) < \sigma (a)\} = \varnothing$. If a is the first agent in C in the order given by σ we take, by convention, $\tau \left(C, \prec, x^{pre(a,C,\sigma)}\right) = 0$.

$f^{*\sigma}$ satisfies SEP and ILD but fails ETC.

6 Appendix

In this section we prove the results stated in the paper.

6.1 Proof of Proposition 1

We proceed by a series of lemmas.

Lemma 1 *$P \subset N$ is a pseudo-path if and only if $a \sim a'$ for all $a, a' \in P$ with $a \neq a'$.*

Proof. ($\Rightarrow$) Let $P = \{a_j\}_{j=1}^p$. Let a_j and a_k be such that $j < k$. If $k = j+1$, then $a_j \prec a_{j+1}$. If $k > j + 1$, then $a_j \prec a_k$ because $\prec$ is transitive. Hence, $a_j \sim a_k$.

($\Leftarrow$) Since P is finite and $\prec$ is anti-symmetric, there exists $a_1 \in P$ such that $a \not\prec a_1$ for all $a \in P$. Since $a_1 \sim a$ for all $a \in P \backslash \{a_1\}$, $a_1 \prec P \backslash \{a_1\}$. Since $P \backslash \{a_1\}$ is finite and $\prec$ is anti-symmetric, there exists $a_2 \in P \backslash \{a_1\}$ such that $a \not\prec a_2$ for all $a \in P \backslash \{a_1\}$. Since $a_2 \sim a$ for all $a \in P \backslash \{a_1, a_2\}$, $a_2 \prec P \backslash \{a_1, a_2\}$. Following the same reasoning, we deduce that $P = \{a_j\}_{j=1}^p$ such that $a_k \prec P \backslash \{a_j\}_{j=1}^k$ for all $k = 1, 2, ..., p$. Hence, P is a pseudo-path. ■

We define

$$\mathcal{E}_{(N,\prec)} = \{E \subset N : E \sim (N \backslash E)\}.$$

We also define $\mathcal{E}^m_{(N,\prec)}$ as the set of minimal nonempty elements of $\mathcal{E}_{(N,\prec)}$, namely

$$\mathcal{E}^m_{(N,\prec)} = \left\{E \in \mathcal{E}_{(N,\prec)} \backslash \{\emptyset\} : E' \subsetneq E \Rightarrow E' \notin \mathcal{E}_{(N,\prec)} \backslash \{\emptyset\}\right\}.$$

Lemma 2 *Let $E, E' \in \mathcal{E}_{(N,\prec)}$ be such that $E' \subsetneq E$. Then, $E \backslash E' \in \mathcal{E}_{(N,\prec)}$.*

Proof. Let $a \notin E \backslash E'$. We need to prove that $a \sim E \backslash E'$.

Assume first $a \notin E$. Then $a \sim E$ because $E \in \mathcal{E}_{(N,\prec)}$. Thus, $a \sim E \backslash E'$.

Assume now $a \in E'$. Let $b \in E \backslash E'$. Since E' is a component, we have $b \sim a$. Thus, $a \sim E \backslash E'$. ■

Lemma 3 *$\mathcal{E}^m_{(N,\prec)}$ is a partition of N.*

Proof. Since N is finite and $N \in \mathcal{E}_{(N,\prec)}$, by Lemma 2, each activity belongs to one element of $\mathcal{E}^m_{(N,\prec)}$.

We should only prove that the elements of $\mathcal{E}^m_{(N,\prec)}$ do not overlap. Namely, given $E, E' \in \mathcal{E}^m_{(N,\prec)}$, $E \cap E' \neq \emptyset$ implies $E = E'$. Since they are minimal, it is enough to prove that $E \cap E' \in \mathcal{E}_{(N,\prec)}$. Let $a \notin E \cap E'$. We need to prove that $a \sim E \cap E'$. If $a \notin E$, then $a \sim E$ because $E \in \mathcal{E}_{(N,\prec)}$. If $a \notin E'$, then $a \sim E'$ because $E' \in \mathcal{E}_{(N,\prec)}$. In any case, $a \sim E \cap E'$. ■

Lemma 4 *Let $E \in \mathcal{E}_{(N,\prec)}$, $P \in \mathcal{P}_{(E,\prec)}$, and $P' \in \mathcal{P}_{(N \backslash E,\prec)}$. Then, $P \cup P' \in \mathcal{P}_{(N,\prec)}$*

Proof. Let $a, a' \in P \cup P'$ with $a \neq a'$. If $a, a' \in P$ or $a, a' \in P'$, then $a \sim a'$ because P, P' are pseudo-paths. If $a \in P$ and $a' \in P'$, we have $a \in E$ and $a' \in N \backslash E$. Since $E \in \mathcal{E}_{(N,\prec)}$, we have $a \sim a'$. Thus, by Lemma 1, $P \cup P'$ is a pseudo-path in $(N, \prec)$. ■

Lemma 5 *Let $E \in \mathcal{E}_{(N,\prec)}$, $P \in \mathcal{P}_{(N,\prec)}$, and $x \in \mathbb{R}^N$. Then, P is critical in $(N, \prec, x)$ if and only if $P \cap E$ and $P \cap (N \backslash E)$ are critical in $(E, \prec, x)$ and $(N \backslash E, \prec, x)$, respectively.*

Proof. ($\Rightarrow$) Assume $P \cap E$ is not critical in $(E, \prec, x)$, *i.e.* there exists $Q \in \mathcal{P}_{(E,\prec)}$ such that $\sum_{a \in P \cap E} x_a < \sum_{a \in Q} x_a$. Consider $R = Q \cup (P \cap (N \backslash E))$. By Lemma 4, $R \in \mathcal{P}_{(N,\prec)}$. Moreover,

$$\sum_{a \in P} x_a = \sum_{a \in P \cap E} x_a + \sum_{a \in P \cap (N \backslash E)} x_a < \sum_{a \in Q} x_a + \sum_{a \in P \cap (N \backslash E)} x_a = \sum_{a \in R} x_a$$

and thus P is not critical in $(N, \prec x)$.

Since $E \in \mathcal{E}_{(N,\prec)}$ if and only if $N \backslash E \in \mathcal{E}_{(N,\prec)}$, we can obtain the same contradiction if $P \cap (N \backslash E)$ is not critical in $(N \backslash E, \prec, x)$.

($\Leftarrow$) Assume P is not critical in $(N, \prec, x)$, *i.e.* there exists $R \in \mathcal{P}_{(N,\prec)}$ such that $\sum_{a \in P} x_a < \sum_{a \in R} x_a$. Then,

$$\sum_{a \in P \cap E} x_a + \sum_{a \in P \cap (N \backslash E)} x_a < \sum_{a \in R \cap E} x_a + \sum_{a \in R \cap (N \backslash E)} x_a$$

which means that either $\sum_{a \in P \cap E} x_a < \sum_{a \in R \cap E} x_a$ or $\sum_{a \in P \cap (N \backslash E)} x_a < \sum_{a \in R \cap (N \backslash E)} x_a$. It is obvious that $R \cap E \in \mathcal{P}_{(E,\prec)}$ and $R \cap (N \backslash E) \in \mathcal{P}_{(N \backslash E,\prec)}$. In the first case, $P \cap E$ is not critical in $(E, \prec, x)$. In the second case, $P \cap (N \backslash E)$ is not critical in $(N \backslash E, \prec, x)$. ■

Lemma 6 *For any nonempty set $C \subset N$, $C \in \mathcal{C}_{(N,\prec)}$ if and only if $C \in \mathcal{E}_{(N,\prec)}$.*

Proof. ($\Rightarrow$) Assume that $C \notin \mathcal{E}_{(N,\prec)}$. Then, there exists $a_1 \in C$ and $a_2 \notin C$ such that $a_1 \nsim a_2$. By Lemma 1, this means that *no pseudo-path exists containing* both a_1 and a_2. Let $x \in \mathbb{R}^N$ be defined by $x_{a_1} = x_{a_2} = 1$ and $x_a = 0$ otherwise. For each $j = 1, 2$ let $P_j \in \mathcal{P}^{a_j}_{(N,\prec)}$ be such that

$$\sum_{a \in P_j} x_a = \max_{P \in \mathcal{P}^{a_j}_{(N,\prec)}} \sum_{a \in P} x_a.$$

Hence,

$$\begin{aligned}
\tau(N,\prec,x) &= \max_{P\in P_{(N,\prec)}}\sum_{a\in P}x_a\\
&= \max\left\{\max_{P\in\mathcal{P}^{a_1}_{(N,\prec)}}\sum_{a\in P}x_a,\max_{P\in\mathcal{P}^{a_2}_{(N,\prec)}}\sum_{a\in P}x_a,\max_{P\in\mathcal{P}_{(N,\prec)}:a_1,a_2\notin P}\sum_{a\in P}x_a\right\}\\
&= \max\{1,1,0\}=1\\
\tau(C,\prec,x) &= \max_{P\in\mathcal{P}_{(C,\prec)}}\sum_{a\in P}x_a\\
&= \max\left\{\max_{P\in\mathcal{P}^{a_1}_{(C,\prec)}}\sum_{a\in P}x_a,\max_{P\notin\mathcal{P}^{a_1}_{(C,\prec)}}\sum_{a\in P}x_a\right\}\\
&= \max\{1,0\}=1.
\end{aligned}$$

Analogously, $\tau(N\backslash C,\prec,x)=1$. Thus, $\tau(N,\prec,x)=1\neq 2=\tau(C,\prec,x)+\tau(N\backslash C,\prec,x)$, which contradicts that C is a component.

($\Leftarrow$) Let P be a critical pseudo-path in $(N,\prec,x)$. Then, $\tau(N,\prec,x)=\sum_{a\in P}x_a$. By Lemma 5, $P\cap C$ is a critical path in $(C,\prec,x)$ and $P\cap(N\backslash C)$ is a critical path in $(N\backslash C,\prec,x)$. Then, $\sum_{a\in P\cap C}x_a=\tau(C,\prec,x)$ and $\sum_{a\in P\cap(N\backslash C)}x_a=\tau(N\backslash C,\prec,x)$. Thus,

$$\tau(N,\prec,x)=\sum_{a\in P}x_a=\sum_{a\in P\cap C}x_a+\sum_{a\in P\cap(N\backslash C)}x_a=\tau(C,\prec,x)+\tau(N\backslash C,\prec,x).$$

■

Lemma 7 *$\mathcal{C}^m_{(N,\prec)}$ is a partition of N.*

Proof. By Lemma 6, $\mathcal{C}_{(N,\prec)}=\mathcal{E}_{(N,\prec)}\backslash\{\emptyset\}$ and thus $\mathcal{C}^m_{(N,\prec)}=\mathcal{E}^m_{(N,\prec)}$. The result follows from Lemma 3. ■

6.2 Proof of Proposition 2

Let $(N,\prec,x)$ be a $PERT$ problem. An *initial activity* is an activity $a\in N$ such that $a'\not\prec a$ for all $a'\in N$. Since N is finite and $\prec$ is anti-symmetric, we conclude that each $(N, \prec)$ has at least one initial activity. We denote the set of initial activities as $S^{N,\prec}$. Moreover, we define $T^{N,\prec} = N \backslash S^{N,\prec}$. For simplicity, we write S and T instead of $S^{N,\prec}$ and $T^{N,\prec}$, respectively.

Let Π be the set of all orders in T, i.e.

$$\Pi = \{\sigma : \{1, ..., |T|\} \to T : \sigma \text{ is a one-to-one function}\}.$$

Given an order $\sigma \in \Pi$, we denote $\sigma(j)$ as σ_j. Given $\sigma \in \Pi$ and $j \leq |T|$, we define $Pre(\sigma, j)$ as the set of predecessors of σ_j in the order, *i.e.*

$$Pre(\sigma, j) = \{\sigma_k \in T : k < j\}.$$

Let f be a value that satisfies MON and SEP. By SEP, we can assume without loss of generality that $\mathcal{C}^m_{(N,\prec)} = \{N\}$.

We say that $z \in \mathbb{R}^N$ is a *reduced vector* for $(N, \prec, x)$ if it satisfies

Definition 1 *1. $z_a \geq 0$ for all $a \in N$;*

2. $\tau(N, \prec, z) = \tau(N, \prec, x)$;

3. every $a \in S$ is a critical activity in $(N, \prec, z)$;

4. $z_a = 0$ for all $a \in T$.

Condition 1 says that $(N, \prec, z)$ is a well-defined $PERT$ problem. Condition 2 says that the $PERT$ time is the same for $(N, \prec, x)$ as for $(N, \prec, z)$. Conditions 3 and 4 say that all $PERT$ time is due to the initial activities.

Lemma 8 *For every $PERT$ problem $(N, \prec, x)$ there exists a unique reduced vector z.*

Proof. *Existence.* We consider $z \in \mathbb{R}^N$ given by

- $z_a = \tau(N, \prec, x)$ for each $a \in S$;
- $z_a = 0$ for each $a \in T$.

We check that z is a reduced vector for $(N, \prec, x)$. By definition, $z_a = 0$ for all $a \in T$ (condition 4 in the definition). Moreover, it is clear that $z_a \geq 0$ for all $a \in N$ (condition 1).

Let $a \in S$ and $P^a \in \mathcal{P}^a_{(N,\prec)}$. Since there cannot be more than one initial activity in the same pseudo-path, we have $z_b = 0$ for all $b \in P^a \backslash \{a\}$. Hence

$$\sum_{b \in P^a} z_b = z_a = \tau(N, \prec, x). \tag{1}$$

Let P be a pseudo-path in $(N, \prec)$. If $P \cap S = \emptyset$, then

$$\sum_{b \in P} z_b = 0 \leq \tau(N, \prec, x).$$

If $P \cap S \neq \emptyset$, then P has a unique initial activity $a \in S$ (because there cannot be more than one initial activity in the same pseudo-path) and

$$\sum_{b \in P} z_b = z_a = \tau(N, \prec, x).$$

Hence

$$\tau(N, \prec, z) = \max_{P \in \mathcal{P}_{(N,\prec)}} \sum_{b \in P} z_b = \tau(N, \prec, x)$$

which is condition 2 in the definition. Moreover, by (1) every initial activity is a critical activity in (N, z) (condition 3).

Uniqueness. Let z and z' be two reduced vectors for $(N, \prec, x)$. By definition, $z_a = z'_a = 0$ for all $a \in T$. Given $a \in S$, let P be a critical pseudo-path in $(N, \prec, z)$ with $a \in P$. This pseudo-path exists by condition 3. Since $a \in S$, we have $P \backslash \{a\} \subset T$. Then,

$$z'_a = \sum_{b \in P} z'_b \leq \tau(N, \prec, z') = \tau(N, \prec, z) = \sum_{b \in P} z_b = z_a.$$

By an analogous reasoning, $z_a \leq z'_a$ and thus $z_a = z'_a$. ■

We now prove Proposition 2

Let $x \in \mathbb{R}^N_+$. As a consequence of the proof of Lemma 8, a reduced vector for $(N, \prec, x)$ only depends on $\tau(N, \prec, x)$ (we are assuming $\mathcal{C}^m_{(N,\prec)} = \{N\}$).

Hence, it is enough to prove that $f(N,\prec,x) = f(N,\prec,z)$ when z is the reduced vector for $(N,\prec,x)$.

When $T \neq \emptyset$, we define an order $\sigma \in \Pi$ as follows: The set S is not a component *as* $\mathcal{C}^m_{(N,\prec)} = \{N\}$. Then, there exists $a \in S$, $\sigma_1 \in T$ such that $a \nsim \sigma_1$.

Assume we have defined $\sigma_1, ..., \sigma_{j-1}$ and $S \cup \{\sigma_k\}_{k=1}^{j-1} \neq N$. We define σ_j. Since $S \cup \{\sigma_k\}_{k=1}^{j-1}$ is not a component, there exists $a \in S \cup \{\sigma_k\}_{k=1}^{j-1}$, $\sigma_j \notin S \cup \{\sigma_k\}_{k=1}^{j-1}$ such that $a \nsim \sigma_j$. Thus, we have an order $\sigma \in T$ such that $T = \{\sigma_j\}_{j=1}^{|T|}$.

Let $D = \{a \in T : x_a > 0\}$. When $D \neq \emptyset$, let $j_0 \in \{1, ..., |T|\}$ be such that $\sigma_j \notin D$ for all $j < j_0$ and $\sigma_{j_0} \in D$.

We proceed by double induction on $|D|$ and j_0.

Assume first $|D| = 0$ (this includes the case $T = \emptyset$). Then, $x_a = 0$ for all $a \in T$. We consider the reduced vector $z \in \mathbb{R}^N$ given by

- $z_a = \tau(N,\prec,x)$ for all $a \in S$
- $z_a = 0$ for all $a \in T$.

It is straightforward to check that $x_a \leq z_a$ for all $a \in N$. Hence, by MON we have $f_a(N,\prec,x) \leq f_a(N,\prec,z)$ for all $a \in N$. Moreover, $\tau(N,\prec,x) = \tau(N,\prec,z)$ and thus $\sum\limits_{a\in N} f_a(N,\prec,x) = \sum\limits_{a\in N} f_a(N,\prec,z)$. Hence, $f(N,\prec,x) = f(N,\prec,z)$.

Assume now the result is true for $|D| - 1$. We prove that the result holds for $|D|$ by induction on j_0.

Assume $j_0 = 1$ (*i.e.* $\sigma_1 \in D$). Thus, $x_{\sigma_1} > 0$. By the definition of σ, there exists $a \in S$ such that $a \nsim \sigma_1$. If needed, we can increase x_a until a is critical in $(N,\prec,x)$. Namely, we define $x^1 \in \mathbb{R}^N_+$ as follows:

$$x^1_a = x_a + \tau(N,\prec,x) - \max_{P \in \mathcal{P}^a_{(N,\prec)}} \sum_{b\in P} x_b$$

and $x^1_b = x_b$ for $b \in N \backslash \{a\}$. It is not difficult to check that a is critical in $(N,\prec,x^1)$, $x \leq x^1$ and $\tau(N,\prec,x) = \tau(N,\prec,x^1)$. By MON, $f_b(N,\prec,x^1) \geq$

$f_b(N, \prec, x)$ for all $b \in N$. Since $\tau(N, \prec, x^1) = \tau(N, \prec, x)$, we conclude that $f(N, \prec, x^1) = f(N, \prec, x)$.

We now decrease $x^1_{\sigma_1}$ to 0. Namely, we define $y^1 \in \mathbb{R}^N_+$ as $y^1_{\sigma_1} = 0$ and $y^1_b = x^1_b$ for all $b \in N \setminus \{\sigma_1\}$. Since $a \not\sim \sigma_1$, no pseudo-path is shared by a and σ_1 and thus the $PERT$ time remains unchanged, *i.e.* $\tau(N, \prec, x^1) = \tau(N, \prec, y^1)$. By MON, $f_b(N, \prec, y^1) \leq f_b(N, \prec, x^1)$ for all $b \in N$. Since $\tau(N, \prec, y^1) = \tau(N, \prec, x)$, we conclude that $f(N, \prec, y^1) = f(N, \prec, x^1)$.

Since $y^1_{\sigma_1} = 0$, we can apply the induction hypothesis on $|D|$ to deduce $f(N, \prec, y^1) = f(N, \prec, z')$ where z' is the reduced vector for y^1. Since $\tau(N, \prec, y) = \tau(N, \prec, x)$ we have that $z' = z$. Hence, $f(N, \prec, x) = f(N, \prec, z)$.

Assume now the result is true for $j_0 - 1$. We prove it for j_0. We know that $x_{j_0} > 0$. By definition of σ, there exists $a \in S \cup \{\sigma_j\}_{j=1}^{j_0-1}$ such that $a \not\sim \sigma_{j_0}$. We have two cases:

- If $a \in S$, we proceed as before and deduce the result by induction hypothesis on $|D|$.
- If $a = \sigma_j$ for some $j < j_0$, we proceed as follows:
  - If σ_j is not critical, we proceed as before to increase x_j until σ_j is critical (as in the construction of x^1). Again, f is not affected. We now decrease $x_{\sigma_{j_0}}$ to 0 (as in the construction of y^1). Since $\sigma_j \not\sim \sigma_{j_0}$, no pseudo-path is shared by σ_j and σ_{j_0} and thus the $PERT$ time remains unchanged. By MON, f is not affected. But now $x_{\sigma_{j_0}} = 0$ and thus we can apply the induction hypothesis on $|D|$ to deduce the result.
  - If σ_j is critical, we proceed as before to decrease $x_{\sigma_{j_0}}$ to 0 (as in the construction of y^1). Since $\sigma_j \not\sim \sigma_{j_0}$, no pseudo-path is shared by σ_j and σ_{j_0} and thus the total time remains unchanged. By MON, f is not affected. But now $|D|$ has decreased and thus we can apply the induction hypothesis on $|D|$ to conclude the result.

■

6.3 Proof of Theorem 1

It is straightforward to check that f^0 satisfies these properties.

Let f be a value satisfying MON, SEP and OP. We will prove that $f = f^0$. Since f satisfies SEP, we can assume that there exists a unique component, *i.e.* $\mathcal{C}^m_{(N,\prec)} = \{N\}$.

By the proof of Proposition 2, we know that $f(N, \prec, x) = f(N, \prec, z)$ and $f^0(N, \prec, x) = f^0(N, \prec, z)$ where z is the reduced vector for $(N, \prec, x)$. Then, it is enough to prove that $f(N, \prec, z) = f^0(N, \prec, z)$.

Given $a \in N$, we define $z^a \in \mathbb{R}^N_+$ as follows:

$$z^a_b = \begin{cases} \tau(N, \prec, z) & \text{if } b = a \\ 0 & \text{if } b \neq a. \end{cases}$$

Since f satisfies OP, $f_b(N, \prec, z^a) \leq f_a(N, \prec, z^a)$ for all $b \in N \backslash \{a\}$. Moreover, z is also the reduced vector for $(N, \prec, z^a)$ for all $a \in N$. By the proof of Proposition 2, $f(N, \prec, z) = f(N, \prec, z^a)$ for all $a \in N$. Thus, given $a, b \in N$,

$$f_b(N, \prec, z) = f_b(N, \prec, z^a) \leq f_a(N, \prec, z^a) = f_a(N, \prec, z).$$

Hence, $f_a(N, \prec, z) = f_b(N, \prec, z)$ for all $a, b \in N$, *i.e.* $f = f^0$. ■

6.4 Proof of Theorem 2

It is straightforward to check that f^d satisfies these properties.

Let f be a value satisfying SEP, ILD, and ETC. We will prove that $f = f^d$. Since f satisfies SEP, we can assume that there exists a unique component, *i.e.* $\mathcal{C}^m_{(N,\prec)} = \{N\}$.

Let $a_1 \in N$ be such that $x_{a_1} \leq x_b$ for all $b \in N$. By ILD, $f_{a_1}(N, \prec, x) = f_{a_1}(N, \prec, x^{a_1})$. Since $x^{a_1}_b = x_{a_1}$ for all $b \in N$, by ETC, $f_{a_1}(N, \prec, x^{a_1}) = \frac{\tau(N,\prec,x^{a_1})}{|N|}$. Thus,

$$f_{a_1}(N, \prec, x) = \frac{\tau(N, \prec, x^{a_1})}{|N|} = f^d_{a_1}(N, \prec, x).$$

Let $a_2 \in N$ be such that $x_{a_2} \leq x_b$ for all $b \in N \backslash \{a_1\}$. By ILD, $f_{a_2}(N, \prec, x) = f_{a_2}(N, \prec, x^{a_2})$. By ILD, $f_{a_1}(N, \prec, x^{a_2}) = f_{a_1}(N, \prec, x^{a_1}) = f_{a_1}(N, \prec, x)$. Since $x_b^{a_2} = x_{a_2}$ for all $b \in N \backslash \{a_1\}$, by ETC,

$$\begin{aligned} f_{a_2}(N, \prec, x^{a_2}) &= \frac{\tau(N, \prec, x^{a_2}) - f_{a_1}(N, \prec, x^{a_2})}{|N| - 1} \\ &= \frac{\tau(N, \prec, x^{a_1})}{|N|} + \frac{\tau(N, \prec, x^{a_2}) - \tau(N, \prec, x^{a_1})}{|N| - 1}. \end{aligned}$$

Thus, $f_{a_2}(N, \prec, x) = f_{a_2}^d(N, \prec, x)$.

Repeating the same argument we can prove that $f_a(N, \prec, x) = f_a^d(N, \prec, x)$ for each $a \in N$. ■

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