1 Supplementary material

Proof of Lemma 6. It is straightforward to prove that $\alpha+\omega_i$ is feasible for i, or $\gamma_0^i \le \alpha+\omega_i \le \delta^i$:

$$\begin{aligned}
\alpha &\le \delta^{i+1} \Longrightarrow \alpha+\omega_i \le \delta^{i+1}+\omega_i = \delta^i. \\
\alpha &\ge \gamma_0^{i+1} \ge q-\omega_{P_{i+1}} \Longrightarrow \alpha+\omega_i \ge q-\omega_{P_{i+1}}+\omega_i = q-\omega_{P_i}.
\end{aligned}$$

Since $\alpha+\omega_i \ge 0$, it follows that $\alpha+\omega_i \ge \max(q-\omega_{P_i},0)=\gamma_0^i$.

a) If $\alpha<\gamma_0^i$, every $T\subset P_{i+1}$ with $\omega_T \ge q-\alpha$ satisfies $i\in T$. Then:

$$\begin{aligned}
b\left(i+1,\alpha\right) &= \max_{T\subset P_{i+1}:\omega_T\ge q-\alpha}(1-d_T) = \max_{T\subset P_{i+1}:i\in T,\omega_T\ge q-\alpha}(1-d_T) \\
&= \max_{T\subset P_i:\omega_T\ge q-\alpha-\omega_i}(1-d_T)-d_i = b\left(i,\alpha+\omega_i\right)-d_i.
\end{aligned}$$

b) If $\alpha\ge\gamma_0^i$, $b\left(i,\alpha\right)$ is well defined and

$$\begin{aligned}
b\left(i+1,\alpha\right) &= \max_{T\subset P_{i+1}:\omega_T\ge q-\alpha}(1-d_T) \\
&= \max\left\{\max_{T\subset P_{i+1}:i\notin T,\omega_T\ge q-\alpha}(1-d_T), \max_{T\subset P_{i+1}:i\in T,\omega_T\ge q-\alpha}(1-d_T)\right\} \\
&= \max\left\{\max_{T\subset P_i:\omega_T\ge q-\alpha}(1-d_T), \max_{T\subset P_i:\omega_T\ge q-\alpha-\omega_i}(1-d_T)-d_i\right\} \\
&= \max\left\{b\left(i,\alpha\right), b\left(i,\alpha+\omega_i\right)-d_i\right\}.
\end{aligned}$$

■

Proof of Lemma 7. Let $\sigma^i\in\Sigma^i$ such that $\sigma^i>\omega_i$. The reader can check that $\gamma^{i+1}\le\sigma^i-\omega_i\le\delta^{i+1}$, hence $\sigma^i-\omega_i$ is a feasible value of α for player $i+1$. Note that for $b\left(i,\sigma^i\right)>0$, homogeneity implies $\sigma^i-\omega_i\ge\omega_{i+1}$.

We now prove (2). Under Lemma 6a), (2) is true when $\sigma^i-\omega_i<\gamma_0^i$. Assume then $\sigma^i-\omega_i\ge\gamma_0^i$.

Since $\gamma^i\le\sigma^i-\omega_i\le\delta^{i+1}$, it is the case that for any $\tau^i\in T^i$

$$\frac{b\left(i,\tau^i\right)}{\tau^i} \ge \frac{b\left(i,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

Re-arranging terms,

$$\begin{aligned} b\left(i, \sigma^i - \omega_i\right) &\leq \frac{\sigma^i - \omega_i}{\tau^i} b\left(i, \tau^i\right) \\ &= b\left(i, \sigma^i\right) - \left[b\left(i, \sigma^i\right) - \frac{\sigma^i - \omega_i}{\tau^i} b\left(i, \tau^i\right)\right] \\ &\leq b\left(i, \sigma^i\right) - d_i. \end{aligned}$$

Hence, (2) follows under lemma 6b). To show (3), we replace $\frac{b\left(i+1, \sigma^i - \omega_i\right)}{\sigma^i - \omega_i}$ by $\frac{b\left(i, \sigma^i\right) - d_i}{\sigma^i - \omega_i}$ and use $d_i \leq d_i^*$. We distinguish two subcases:

1. If $b\left(i, \tau^i\right) \geq 0$, then $d_i^* = b\left(i, \sigma^i\right) - \frac{\sigma^i - \omega_i}{\tau^i} b\left(i, \tau^i\right)$.

$$\begin{aligned} \frac{b\left(i+1, \sigma^i - \omega_i\right)}{\sigma^i - \omega_i} &= \frac{b\left(i, \sigma^i\right) - d_i}{\sigma^i - \omega_i} \\ &\geq \frac{b\left(i, \sigma^i\right) - b\left(i, \sigma^i\right) + \frac{\sigma^i - \omega_i}{\tau^i} b\left(i, \tau^i\right)}{\sigma^i - \omega_i} \\ &= \frac{b\left(i, \tau^i\right)}{\tau^i} \end{aligned}$$

   with strict inequality iff $d_i < d_i^*$.

2. If $b\left(i, \tau^i\right) < 0$, then $d_i^* = b\left(i, \sigma^i\right)$ and thus

$$\frac{b\left(i+1, \sigma^i - \omega_i\right)}{\sigma^i - \omega_i} = \frac{b\left(i, \sigma^i\right) - d_i}{\sigma^i - \omega_i} \geq 0$$

   with strict inequality iff $d_i < d_i^*$.

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Proof of Lemma 8. Let α such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 6, either $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$ or $b(i+1, \alpha) = b(i, \alpha)$. We have to prove that

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b\left(i+1, \sigma^i - \omega_i\right)}{\sigma^i - \omega_i}.$$

If $b(i+1, \alpha) = b(i, \alpha)$, the result follows from (3).

If $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$, we have three cases:

1. If $\sigma^i \leq \delta^{i+1}$, then

$$\frac{b(i+1,\alpha)}{\alpha} = \frac{b(i,\alpha+\omega_i) - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\alpha} \leq \frac{\frac{b\left(i,\sigma^i\right)}{\sigma^i}(\alpha+\omega_i) - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\alpha}$$
$$= \frac{b\left(i,\sigma^i\right)}{\sigma^i} \overset{\text{(Lemma 7)}}{\leq} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

2. If $\sigma^i > \delta^{i+1}$ and $b\left(i,\tau^i\right) \geq 0$ for some/all $\tau^i \in T^i$, then either $\frac{b(i,\alpha+\omega_i)}{\alpha+\omega_i} \leq \frac{b\left(i,\tau^i\right)}{\tau^i}$ (if $\alpha+\omega_i \leq \delta^{i+1}$) or $b(i,\alpha+\omega_i) = b\left(i,\sigma^i\right)$ (if $\alpha+\omega_i > \delta^{i+1}$, by Corollary 3).

   If $\alpha+\omega_i \leq \delta^{i+1}$,

$$\frac{b(i+1,\alpha)}{\alpha} = \frac{b(i,\alpha+\omega_i) - b\left(i,\sigma^i\right) + \frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right)}{\alpha}$$
$$\leq \frac{\frac{\alpha+\omega_i}{\tau^i} b\left(i,\tau^i\right) - \frac{\sigma^i}{\tau^i} b\left(i,\tau^i\right) + \frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right)}{\alpha}$$
$$= \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{\text{(Lemma 7)}}{\leq} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

   If $\alpha+\omega_i > \delta^{i+1}$,

$$\frac{b(i+1,\alpha)}{\alpha} = \frac{b(i,\alpha+\omega_i) - b\left(i,\sigma^i\right) + \frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right)}{\alpha} = \frac{\sigma^i-\omega_i}{\alpha} \frac{b\left(i,\tau^i\right)}{\tau^i}.$$

   If $b\left(i,\sigma^i\right) > 0$, corollary 3 implies $\sigma^i = \delta^{i+1}+1$. Then $\alpha+\omega_i > \delta^{i+1}$ implies $\alpha+\omega_i \geq \delta^{i+1}+1 = \sigma^i$, or $\left(\sigma^i-\omega_i\right)/\alpha \leq 1$. If $b(i,\sigma^i) = 0$, $b(i,\tau^i) = 0$, implying $\frac{\sigma^i-\omega_i}{\alpha}\frac{b(i,\tau^i)}{\tau^i} = \frac{b(i,\tau^i)}{\tau^i}$. In either case,

$$\frac{b(i+1,\alpha)}{\alpha} \leq \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{\text{(Lemma 7)}}{\leq} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

3. If $\sigma^i > \delta^{i+1}$ and $b\left(i,\tau^i\right) < 0$ for some/all $\tau^i \in T^i$, then either $b(i,\alpha+\omega_i) < 0$ (if $\alpha+\omega_i \leq \delta^{i+1}$) or $b(i,\alpha+\omega_i) = b\left(i,\sigma^i\right)$ (if $\alpha+\omega_i > \delta^{i+1}$, by Corollary 3).

If $b\left(i,\alpha+\omega_i\right)<0$,

$$\frac{b\left(i+1,\alpha\right)}{\alpha}=\frac{b\left(i,\alpha+\omega_i\right)-d_i}{\alpha}<-\frac{d_i}{\alpha}\leq 0\overset{\text{(Lemma 7)}}{\leq}\frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

If $b\left(i,\alpha+\omega_i\right)=b\left(i,\sigma^i\right)$,

$$\frac{b\left(i+1,\alpha\right)}{\alpha}=\frac{b\left(i,\alpha+\omega_i\right)-d_i}{\alpha}=0\overset{\text{(Lemma 7)}}{\leq}\frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

■

Proof of Lemma 9. a) Let $S\in\underset{T\subset P_i:\omega_T\geq q-\sigma^i}{\arg\max}(1-d_T)$. We have to prove $b\left(i+1,\sigma^i-\omega_i\right)=1-d_{S\cup\{i\}}$. Using (2),

$$b\left(i+1,\sigma^i-\omega_i\right)=b\left(i,\sigma^i\right)-d_i=1-d_S-d_i=1-d_{S\cup\{i\}}.$$

b) Since $i\in S$, $b(i+1,\sigma^{i+1})=b(i,\sigma^{i+1}+\omega_i)-d_i$, or

$$b(i,\sigma^{i+1}+\omega_i)=b(i+1,\sigma^{i+1})+d_i. \tag{4}$$

Let $\sigma^i>\omega_i$. We have shown that $\sigma^i-\omega_i\in\Sigma^{i+1}$, thus

$$\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}}=\frac{b(i+1,\sigma^i-\omega_i)}{\sigma^i-\omega_i}. \tag{5}$$

1. If $\sigma^i\leq\delta^{i+1}$ for some $\sigma^i\in\Sigma^i$, it follows from (5) and (3) that

$$\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}}=\frac{b(i,\sigma^i)}{\sigma^i}. \tag{6}$$

Then

$$\begin{aligned}\frac{b(i,\sigma^{i+1}+\omega_i)}{\sigma^{i+1}+\omega_i}\overset{(4)}{=}\frac{b(i+1,\sigma^{i+1})+d_i}{\sigma^{i+1}+\omega_i}&=\frac{b(i+1,\sigma^{i+1})+\frac{\omega_i}{\sigma^i}b\left(i,\sigma^i\right)}{\sigma^{i+1}+\omega_i}\\ \overset{(6)}{=}\frac{\frac{\sigma^{i+1}}{\sigma^i}b\left(i,\sigma^i\right)+\frac{\omega_i}{\sigma^i}b\left(i,\sigma^i\right)}{\sigma^{i+1}+\omega_i}&=\frac{b(i,\sigma^i)}{\sigma^i}.\end{aligned}$$

Hence $\sigma^{i+1}+\omega_i\in\Sigma^i$ and $b\left(i,\sigma^{i+1}+\omega_i\right)=b\left(i+1,\sigma^{i+1}\right)+d_i=1-d_{S\cap P_i}$.

2. If $\sigma^i > \delta^{i+1}$ for all $\sigma^i \in \Sigma^i$, $\delta^{i+1}+1$ always belongs to Σ^i.

   Suppose $S \cap P_i \notin \underset{T\subset P_i:\omega_T \geq q-\sigma^i}{\arg\max} (1-d_T)$ for all $\sigma^i \in \Sigma^i$. Then it must be the case that for any σ^i either $\omega_{S\cap P_i} < q-\sigma^i$, or $\omega_{S\cap P_i} \geq q-\sigma^i$ but $1-d_{S\cap P_i}$ is not maximal.

   Suppose $\omega_{S\cap P_i} < q-\sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $\delta^{i+1}+1 \in \Sigma^i$, it follows from Lemma 2 that $\omega_{S\cap P_i} < q-\delta^i$. But then $\omega_{S\cap P_i}+\omega_{N\backslash P_i} = \omega_{S\cap P_{i+1}}+\omega_{N\backslash P_{i+1}} < q$, contradicting the assumption that $\omega_{S\cap P_{i+1}} \geq q-\sigma^{i+1}$.

   Suppose $\omega_{S\cap P_i} \geq q-\sigma^i$ but $1-d_{S\cap P_i} < 1-d_T$ for some $\sigma^i \in \Sigma^i$ and $T \subset P_i$ with $\omega_T \geq q-\sigma^i$.

   If $\sigma^{i+1}+\omega_i > \delta^{i+1}$,

$$b(i,\sigma^{i+1}+\omega_i) \overset{\text{(Lemma 2)}}{=} b(i,\sigma^i) > 1-d_{S\cap P_i} = b(i+1,\sigma^{i+1})+d_i$$

   contradicting (4).

   If $\sigma^{i+1}+\omega_i \leq \delta^{i+1}$, $\frac{b(i,\sigma^{i+1}+\omega_i)}{\sigma^{i+1}+\omega_i} \leq \frac{b(i,\tau^i)}{\tau^i}$. There are two possibilities:

   - If $b(i,\tau^i) \geq 0$, it follows from (5) and (3) that

$$\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i,\tau^i)}{\tau^i}. \tag{7}$$

     Then

$$\begin{aligned}\frac{b(i,\tau^i)}{\tau^i}(\sigma^{i+1}+\omega_i) &\geq b(i,\sigma^{i+1}+\omega_i) \overset{(4)}{=} b(i+1,\sigma^{i+1})+d_i = \\ &= \frac{b(i,\tau^i)}{\tau^i}\sigma^{i+1}+b\left(i,\sigma^i\right)-(\sigma^i-\omega_i)\frac{b\left(i,\tau^i\right)}{\tau^i}\end{aligned}$$

     implying $\frac{b(i,\tau^i)}{\tau^i} \geq \frac{b(i,\sigma^i)}{\sigma^i}$, which is not possible because $\sigma^i > \delta^{i+1}$ for all $\sigma^i \in \Sigma^i$ implies $\frac{b(i,\tau^i)}{\tau^i} < \frac{b(i,\sigma^i)}{\sigma^i}$. Hence, this subcase is void.
   - If $b(i,\tau^i) < 0$, it follows from (5) and (3) that $\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}} = 0$. Then

$$b(i,\sigma^{i+1}+\omega_i) = b(i+1,\sigma^{i+1})+d_i = b(i,\sigma^i).$$

     Hence $b(i,\sigma^i) = 1-d_{S\cap P_i}$ and the result follows.

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Proof of Lemma 10. $a)$ Let $\sigma^{i+1} \in \Sigma^{i+1}$ and $\tau^i \in T^i$. We need to prove that $b\left(i, \sigma^{i+1}\right)$ exists and $b\left(i, \sigma^{i+1}\right) > b\left(i, \sigma^{i+1} + \omega_i\right) - d_i$. This will be due to player $i+1$ having the option of setting $\alpha = \sigma^i$ (if $\sigma^i \leq \delta^{i+1}$) or $\alpha = \tau^i$ (if $\sigma^i > \delta^{i+1}$) without buying player i's votes. We examine each case in turn:

1. If $\sigma^i \leq \delta^{i+1}$, then $d_i > \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}$.

   Since $\sigma^i \leq \delta^{i+1}$, $b(i+1, \sigma^i)$ exists. Moreover, lemma 6$b)$ implies

   $$b(i+1, \sigma^i) \geq b(i, \sigma^i). \tag{8}$$

   In principle, there are three possibilities for σ^{i+1}: either $\sigma^{i+1} < \gamma_0^i$, or $\sigma^{i+1} \geq \gamma_0^i$ and $b\left(i, \sigma^{i+1}\right) \leq b\left(i, \sigma^{i+1} + \omega_i\right) - d_i$, or $\sigma^{i+1} \geq \gamma_0^i$ and $b\left(i, \sigma^{i+1}\right) > b\left(i, \sigma^{i+1} + \omega_i\right) - d_i$. We will show that the first two possibilities lead to a contradiction. In both cases, Lemma 6 implies

   $$b\left(i+1, \sigma^{i+1}\right) = b\left(i, \sigma^{i+1} + \omega_i\right) - d_i. \tag{9}$$

   From (9) we can deduce:

   $$\begin{aligned} \frac{b\left(i+1, \sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i, \sigma^{i+1} + \omega_i\right) - d_i}{\sigma^{i+1}} \\ &< \frac{b\left(i, \sigma^{i+1} + \omega_i\right) - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\sigma^{i+1}} \\ &\leq \frac{\frac{(\sigma^{i+1}+\omega_i) b\left(i,\sigma^i\right)}{\sigma^i} - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\sigma^{i+1}} \\ &= \frac{b\left(i, \sigma^i\right)}{\sigma^i} \overset{(8)}{\leq} \frac{b\left(i+1, \sigma^i\right)}{\sigma^i}. \end{aligned}$$

   which contradicts that $\sigma^{i+1} \in \Sigma^{i+1}$. Thus, $\sigma^{i+1} \geq \gamma_0^i$ (i.e. $b\left(i, \sigma^{i+1}\right)$ does exist) and $b\left(i, \sigma^{i+1}\right) > b\left(i, \sigma^{i+1} + \omega_i\right) - d_i$. We conclude then that $i \notin S$ for all $S \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} \left(1 - d_T\right)$.

2. If $\sigma^i > \delta^{i+1}$, then $d_i > b\left(i, \sigma^i\right) - \frac{\sigma^i - \omega_i}{\tau^i} b\left(i, \tau^i\right)$.

Under Lemma 6*b)*:

$$b\left(i+1,\tau^{i}\right)=\max\left\{b\left(i,\tau^{i}\right),b\left(i,\tau^{i}+\omega_{i}\right)-d_{i}\right\}\geq b\left(i,\tau^{i}\right). \qquad (10)$$

Suppose $b\left(i,\sigma^{i+1}\right)$ does not exist (i.e. $\sigma^{i+1}<\gamma_{0}^{i}$), or $b\left(i,\sigma^{i+1}\right)$ exists and $b\left(i,\sigma^{i+1}\right)\leq b\left(i,\sigma^{i+1}+\omega_{i}\right)-d_{i}$. In both cases, under Lemma 6,

$$b\left(i+1,\sigma^{i+1}\right)=b\left(i,\sigma^{i+1}+\omega_{i}\right)-d_{i}. \qquad (11)$$

We will prove that (11) leads to a contradiction, so that $b\left(i,\sigma^{i+1}\right)$ exits and $b\left(i,\sigma^{i+1}\right)>b\left(i,\sigma^{i+1}+\omega_{i}\right)-d_{i}$, which implies $i\notin S$ for all $S\in\underset{T\subset P_{i+1}:\omega_{T}\geq q-\sigma^{i+1}}{\arg\max}\left(1-d_{T}\right)$ as desired.

We have two cases:

- If $\sigma^{i+1}+\omega_{i}\leq\delta^{i+1}$. Then $\frac{b(i,\sigma^{i+1}+\omega_{i})}{\sigma^{i+1}+\omega_{i}}\leq\frac{b(i,\tau^{i})}{\tau^{i}}$ and

$$\begin{aligned}\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i,\sigma^{i+1}+\omega_{i}\right)-d_{i}}{\sigma^{i+1}}\\ &< \frac{b\left(i,\sigma^{i+1}+\omega_{i}\right)-b\left(i,\sigma^{i}\right)+\frac{\sigma^{i}-\omega_{i}}{\tau^{i}}b\left(i,\tau^{i}\right)}{\sigma^{i+1}}\\ &\leq \frac{\frac{\sigma^{i+1}+\omega_{i}}{\tau^{i}}b\left(i,\tau^{i}\right)-\frac{\sigma^{i}}{\tau^{i}}b\left(i,\tau^{i}\right)+\frac{\sigma^{i}-\omega_{i}}{\tau^{i}}b\left(i,\tau^{i}\right)}{\sigma^{i+1}}\\ &= \frac{b\left(i,\tau^{i}\right)}{\tau^{i}}\overset{(10)}{\leq}\frac{b\left(i+1,\tau^{i}\right)}{\tau^{i}}\end{aligned}$$

  which is a contradiction.

- If $\sigma^{i+1}+\omega_{i}>\delta^{i+1}$, then under Corollary 3, $b\left(i,\sigma^{i+1}+\omega_{i}\right)=b\left(i,\sigma^{i}\right)$. If $b\left(i,\sigma^{i}\right)>0$, $\sigma^{i}=\delta^{i+1}+1$ and $\sigma^{i+1}+\omega_{i}\geq\sigma^{i}$, which implies $\left(\sigma^{i}-\omega_{i}\right)/\sigma^{i+1}\leq1$. If $b\left(i,\sigma^{i}\right)=0$, $b\left(i,\tau^{i}\right)=0$. Hence:

$$\begin{aligned}\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i,\sigma^{i+1}+\omega_{i}\right)-d_{i}}{\sigma^{i+1}}\\ &< \frac{b\left(i,\sigma^{i+1}+\omega_{i}\right)-b\left(i,\sigma^{i}\right)+\frac{\sigma^{i}-\omega_{i}}{\tau^{i}}b\left(i,\tau^{i}\right)}{\sigma^{i+1}}\\ &= \frac{\sigma^{i}-\omega_{i}}{\sigma^{i+1}}\frac{b\left(i,\tau^{i}\right)}{\tau^{i}}\leq\frac{b\left(i,\tau^{i}\right)}{\tau^{i}}\overset{(10)}{\leq}\frac{b\left(i+1,\tau^{i}\right)}{\tau^{i}}\end{aligned}$$

  which is a contradiction.

b) Recall that we assumed $b(i, \sigma^i) \geq 0$ for all $\sigma^i \in \Sigma^i$. Thus, $b\left(i, \tau^i\right) < 0$ for some $\tau^i \in T^i$ implies $\sigma^i > \delta^{i+1}$. Under Corollary 3, this means $b\left(i, \sigma^i\right) = b\left(i, \delta^i\right)$. Let α be such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 6, we have two cases:

1. $b\left(i+1, \alpha\right) = b\left(i, \alpha + \omega_i\right) - d_i$. Then

   $$b\left(i+1, \alpha\right) < b\left(i, \alpha + \omega_i\right) - b\left(i, \delta^i\right).$$

   Since $\alpha + \omega_i \leq \delta^i$, $b\left(i, \alpha + \omega_i\right) \leq b\left(i, \delta^i\right)$ and thus $b\left(i+1, \alpha\right) < 0$.

2. $b\left(i+1, \alpha\right) = b\left(i, \alpha\right)$. Then $\gamma_0^i \leq \alpha \leq \delta^{i+1}$ and

   $$\frac{b\left(i+1, \alpha\right)}{\alpha} \leq \frac{b\left(i, \tau^i\right)}{\tau^i} < 0$$

   and thus $b\left(i+1, \alpha\right) < 0$.

Since $b\left(i+1, \alpha\right) < 0$ for all α, we conclude $b\left(i+1, \sigma^{i+1}\right) < 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and thus by Lemma 5 all the players get zero. ∎

Proof of Proposition 2. We proceed by backwards induction on i. For $i = n$, its strategy is clearly optimal.

Assume now the result is true for $\mathbb{B}\left(d, i+1\right)$ and moreover assume the following two conditions hold:

Condition 1 The formed coalition maximizes the benefit per vote, that is

$$S \cap P_{i+1} \in \underset{T \subset P_{i+1} : \omega_T \geq q - \sigma^{i+1}}{\arg\max} \left(1 - d_T\right)$$

for some $\sigma^{i+1} \in \Sigma^{i+1}$. (This condition holds trivially for $i+1 = n$ because $\Sigma^n = \{\omega_n\}$).

Condition 2 The above S and σ^{i+1} are such that $S \cap P_{i+1}$ is one of the most favorable sets for player i (*i.e.* $i \notin S$ implies $i \notin T$ for all $T \in \underset{T \subset P_{i+1} : \omega_T \geq q - \sigma^{i+1}}{\arg\max} \left(1 - d_T\right)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$). Among them, it is one of the most favorable to player $i-1$, etc. (This condition holds for $i+1 = n$ because $\Sigma^n = \{\omega_n\}$ and n applies the tie-breaking rule).

We check that this remains true for $\mathbb{B}(d,i)$. Let $\tau^i \in T^i$. We have two cases:

1. If $\sigma^i > \omega_i$ for all $\sigma^i \in \Sigma^i$, then it is straightforward to check that player i obtains strictly less than d_i^* by forming coalition. If i demands d_i^*, there is a coalition $S \subset P_i$ such that $S \cup \{i\} \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ for $\sigma^{i+1} = \sigma^i - \omega_i \in \Sigma^{i+1}$ (Lemma 9a). The induction hypothesis (Conditions 1 and 2) implies that d_i^* will be accepted. Assume player i deviates by demanding $d_i > d_i^*$. If $b\left(i, \tau^i\right) \geq 0$, under Lemma 10$a$) player i does not belong to any coalition in $\underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ for any $\sigma^{i+1} \in \Sigma^{i+1}$ and its final payoff is zero under the induction hypothesis (Condition 1). If $b\left(i, \tau^i\right) < 0$, under Lemma 10$b$), its final payoff is zero.

   Moreover, Conditions 1 and 2 hold for i. Condition 1 follows from Lemma 9b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the player $j > i$ that eventually forms coalition.

2. If $\omega_i \in \Sigma^i$, then $1 - d_S = b(i, \omega_i) = d_i^*$ for all $S \in \underset{T \subset P_i: \omega_T \geq q - \omega_i}{\arg\max} (1 - d_T)$. This means that if player i forms a winning coalition it obtains a final payoff of $b(i, \omega_i)$. Suppose player i deviates and demands $d_i > b(i, \omega_i)$. It is enough to check that $i \notin S$ for all $S \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$. Under the induction hypothesis applied to Condition 1, this means that player i will not be included in any eventual winning coalition, and its final payoff will be zero, while the original strategy yields a nonnegative payoff.

   For constant-sum homogeneous games it is always the case that $\omega_i \leq \delta^{i+1}$, thus $b(i+1, \omega_i)$ is well defined. Under Lemma 6b),

$$b(i+1, \omega_i) = \max\{b(i, \omega_i), b(i, 2\omega_i) - d_i\} \geq b(i, \omega_i) \qquad (12)$$

Suppose that $i \in S$ for some $S \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\min} d_T$ and some $\sigma^{i+1} \in \Sigma^{i+1}$. This means

$$b\left(i+1, \sigma^{i+1}\right) = b\left(i, \sigma^{i+1} + \omega_i\right) - d_i$$

and hence

$$\begin{aligned}
\frac{b\left(i+1, \sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i, \sigma^{i+1} + \omega_i\right) - d_i}{\sigma^{i+1}} \\
&< \frac{b\left(i, \sigma^{i+1} + \omega_i\right) - b\left(i, \omega_i\right)}{\sigma^{i+1}} \\
&\leq \frac{\frac{\sigma^{i+1} + \omega_i}{\omega_i} b\left(i, \omega_i\right) - b\left(i, \omega_i\right)}{\sigma^{i+1}} \\
&= \frac{\left(\sigma^{i+1} + \omega_i\right) b\left(i, \omega_i\right) - \omega_i b\left(i, \omega_i\right)}{\omega_i \sigma^{i+1}} \\
&= \frac{b\left(i, \omega_i\right)}{\omega_i} \overset{(12)}{\leq} \frac{b\left(i+1, \omega_i\right)}{\omega_i}
\end{aligned}$$

which is a contradiction. This contradiction proves that $i \notin S$ for all $S \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\min} d_T$, as desired.

We now check that Conditions 1 and 2 hold for i. If player i forms coalition, Condition 1 holds with $\sigma^i = \omega_i$, and Condition 2 holds because of the tie-breaking rule. If player i demands d_i^* so that S^* is induced, it must be the case that $\{\omega_i\} \subsetneq \Sigma^i$. Hence, there exists $\sigma^i \in \Sigma^i$ with $\sigma^i > \omega_i$. Then, Condition 1 follows from Lemma 9b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the player that eventually forms coalition.

■

Proof of proposition 3. We proceed by backwards induction on i. We prove the following three hypotheses:

1. If $b(i, \sigma^i) < 0$, all players get zero in every SPE of $\mathbb{B}\left(d, i\right)$.

2. If $b(i,\sigma^i) > 0$, player i receives $d_i^* > 0$ in every SPE of $\mathbb{B}(d,i)$ and the coalition that forms satisfies $S \cap P_i \in \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

3. If $b(i,\sigma^i) = 0$,

   a) player i gets $d_i^* = 0$ in every SPE of $\mathbb{B}(d,i)$;

   b) there is a SPE of $\mathbb{B}(d,i)$ in which a winning coalition forms;

   c) if a winning coalition S forms, then $S \cap P_i \in \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

The induction hypothesis holds for player n. Now suppose it holds for player $i+1$. Does it hold for player i?

1. If $b(i,\sigma^i) < 0$, all players get zero (Lemma 5).

2. If $b(i,\sigma^i) > 0$, player i cannot get more than d_i^* by forming coalition. If player i demands more than d_i^* and $b(i,\tau^i) \geq 0$, we know from Lemma 10a) that $i \notin \underset{T \subset P_{i+1} : \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ for all $\sigma^{i+1} \in \Sigma^{i+1}$. The induction hypothesis implies that player i gets zero. If player i demands more than d_i^* and $b(i,\tau^i) < 0$, we know from Lemma 10b) that player i gets zero.

   Now we show that player i can get at least d_i^*. This is immediate if $\omega_i \in \Sigma^i$. Suppose $\omega_i \notin \Sigma^i$. Since $b(i,\sigma^i) > 0$, we know $d_i^* > 0$. The value of d_{i+1}^* induced by d_i^* may be strictly positive or 0. Suppose player i demands $d_i < d_i^*$. Then the corresponding value of d_{i+1}^* is strictly positive. Under Lemma 11, player i belongs to all coalitions associated with some element of Σ^{i+1}, and the induction hypothesis for $d_{i+1}^* > 0$ implies that player i gets d_i. Thus, the perfectness of the equilibrium implies that d_i^* is accepted (otherwise, player i would not have a best response).

Moreover, Lemma 9b), the induction hypothesis and the fact that d_i^* is accepted imply that the coalition that forms satisfies $S \cap P_i \in \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

3. If $b(i, \sigma^i) = 0$, then $d_i^* = 0$ and, moreover, $\alpha \in \Sigma^i$ if and only if $b(i, \alpha) = 0$.

   a) It is trivial that player i gets $d_i^* = 0$. If $d_i > d_i^*$, the induction hypothesis implies that no coalition to which player i belongs will form.

   b) There is an equilibrium of the subgame in which a coalition associated with $\sigma^i \in \Sigma^i$ forms. This is clearly the case for $\omega_i \in \Sigma^i$. Otherwise, it is optimal for player i to demand $d_i^* = 0$. Then $b\left(i+1, \sigma^{i+1}\right) = 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and the induction hypothesis implies that there is a SPE of $\mathbb{B}(i, d)$ in which a winning coalition is formed.

   c) Assume a winning coalition S is formed with $S \cap P_i \notin \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for all $\sigma^i \in \Sigma^i$. This means that, for a given $\sigma^i \in \Sigma^i$, either $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal, or $\omega_{S \cap P_i} < q - \sigma^i$.

   Assume first there exists $\sigma^i \in \Sigma^i$ such that $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal. Since $b\left(i, \sigma^i\right) = 0$, this means $d_{S \cap P_i} > 1$ and it cannot be optimal at any subgame to form S.

   Assume now $\omega_{S \cap P_i} < q - \sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $b\left(i, \sigma^i\right) = 0$ and $b(i, \alpha)$ is nondecreasing in α, $\delta^i \in \Sigma^i$; thus $\omega_{S \cap P_i} < q - \delta^i$. This means $\omega_{S \cap P_i} + \omega_{S \cap (N \setminus P_i)} < q$. Thus, S is not a winning coalition.

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