Implementation of the Levels Structure Value*

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Published in May 2005 in *Annals of Operations Research*‡

Abstract

We implement the levels structure value (Winter, 1989) for cooperative transfer utility games with a levels structure. The mechanism is a generalization of the bidding mechanism by Pérez-Castrillo and Wettstein (2001).

Keywords: Levels structure value, implementation, TU games.

1 Introduction

A cooperative game describes a conflict situation among a finite number of agents or players. Even though players are assumed to have independent interests, they can benefit from cooperation. When cooperation takes place, the question is how the benefits will be distributed among the players, a problem which has been studied from different perspectives. Our aim is to define a *solution concept* which results in a "fair" (or at least "reasonable") allocation for each problem. This allocation must take into account the contribution of each player to the game. Cooperative transfer utility (TU) games have been widely studied. In these games, utility is freely transferable between members of a coalition. A widely studied solution concept for TU games is the Shapley value (described by Shapley in 1953).

*I would like to thank Gustavo Bergantiños for numerous and useful comments on an earlier version of this paper. Naturally, I alone am responsible for any shortcoming. Also gratefully acknowledged is financial support from the Spanish Ministerio de Ciencia y Tecnologia and FEDER through grant BEC2002-04102-C02-01 and from the Xunta de Galicia through grant PGIDIT03PXIC30002PN.

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‡DOI: 10.1007/s10479-005-2255-6.

Once a solution concept has been established, its implementation aims to state a mechanism (or non-cooperative game) such that players, by behaving strategically obtain as the final outcome, the allocation proposed by the solution concept.

In this context, we can say that a mechanism implements the Shapley value (or any other concept) if two properties are satisfied. First, there must be some equilibrium such that the final payoff is the Shapley value. Second, every equilibrium must have the Shapley value as final payoff. The first property is necessary since, even if it is proved that the Shapley value arises in each equilibrium, it may occur that the non-cooperative game has no equilibrium.

Implementation for the Shapley value in TU games has been studied by several authors. For example, Gul (1989), Hart and Moore (1990), Winter (1994), Dasgupta and Chiu (1998), Hart and Mas-Colell (1996) and Evans (1996). Recently, Pérez-Castrillo and Wettstein (2001) presented their *bidding mechanism*, which has novel features.

In the bidding mechanism, one of the players (the *proposer*) should propose an allocation. If all the other players agree, this is the final payoff. If at least one of the other players does not accept the proposed allocation, the proposer leaves the game and the mechanism is repeated with the remaining players.

A key feature in the bidding mechanism is the way the proposer is chosen. Since the final payoff depends on the identity of the proposer, in an initial stage the players should bid for the right to be the proposer. The player who presents the highest net bid is chosen as proposer.

Pérez-Castrillo and Wettstein showed that in equilibrium, all players have the same probability of being chosen as proposer. Furthermore, if the game is zero-monotonic, the equilibrium payoff is the Shapley value.

In equilibrium, the bidding mechanism may finish in one round (when no player drops out) or in more than one. However, the latter only happens when the game is not strictly zero-monotonic.

Frequently, we have more information available than that given by the characteristic function of the game. For example, let us consider the members of the European Parliament. Even though all have the same rights, they do not act independently, since they belong to different political parties. Furthermore, political parties are not completely independent of each other. On a higher level, parties with similar ideologies may be formally associated in larger groups, such as the EPP-ED[1] or the Socialist Group, and so on.

[1]European People's Party (Christian Democrats) and European Democrats.

We call this cooperation description of the players a *levels structure*. Solution concepts which take into account levels structures are the Owen value (described by Owen in 1977) for a single level, and the levels structure value (suggested by Owen in 1977 and studied by Winter in 1989). The levels structure value is a generalization of the Owen value for more than one level. Furthermore, the Owen value is a generalization of the Shapley value.

In Vidal-Puga and Bergantiños (2003), the Pérez-Castrillo and Wettstein bidding mechanism is generalized to take a single-level structure into account. The resulting non-cooperative game implements the Owen value.

In this article, we move this a step further by modifying the bidding mechanism so that a general levels structure is considered. To do so, we generalize the bidding mechanism to a new mechanism, called the *levels bidding mechanism*.

Given a levels structure with h levels, the levels bidding mechanism has h rounds. In Round 1, the members of the same coalition at this level play the bidding mechanism, trying to obtain the resources of the whole coalition. Eventually, we obtain a player (called the *representative*) out of each coalition, who obtains the resources of his own coalition, or of a subcoalition if one or more players are removed. In the second round, the representatives who are in the same coalition at the second level repeat the process, this time with the additional resources obtained in the previous round. The process goes on until level h is reached.

In Section 2 we describe the notation and definitions. In Section 3 we formally define the coalitional bidding mechanism and prove that it implements the levels structure value. A tie-breaking rule is used in the proof. In Section 4 we show that this tie-breaking rule is needed in the model.

2 The model

We consider a cooperative game in characteristic form (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. We denote the set of cooperative games as $TU(N)$.

A *coalition* of (N, v) is a nonempty subset $S \subset N$. We say that (N, v) is *zero-monotonic* if $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ for every $S \subset N \setminus \{i\}$. We say that v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for every $S, T \in N$ such that $S \cap T = \emptyset$.

Notice that superadditivity implies zero-monotonicity.

Unless otherwise specified, we assume that a cooperative game (N, v) is superadditive.

A *coalition structure* on N is a partition $\mathcal{C} = \{C_1, \ldots, C_m\}$ of N, i.e. $C_q \cap C_r = \varnothing$ when $C_q \neq C_r$ and $\bigcup_{C_q \in \mathcal{C}} C_q = N$.

Given $i \in C_q \in \mathcal{C}$, we denote by $\mathcal{C}_{-i}$ the coalition structure on $N \setminus \{i\}$ which equals $\mathcal{C}$ after removing player i, i.e. $\mathcal{C}_{-i} = \{C_1, \ldots, C_{q-1}, C_q \setminus \{i\}, C_{q+1}, \ldots, C_m\}$.

Notice that this means that $\mathcal{C}_{-i}$ may have one less coalition than $\mathcal{C}$.

Given v characteristic function on N, and $S \subset N$, we define $(S, v_S) \in TU(S)$ as the game v restricted to the player set S, i.e. $v_S(T) = v(T)$ for all $T \subset S$.

In particular, we denote $v_{-i} = v_{N \setminus \{i\}}$ and $v_{-S} = v_{N \setminus S}$.

A *levels structure* on N is a sequence $\mathfrak{C} = (\mathcal{C}^0, \mathcal{C}^1, \ldots, \mathcal{C}^h)$, $h \geq 1$ with $\mathcal{C}^l$ $(0 \leq l \leq h)$ coalition structure on N such that:

1. $\mathcal{C}^0 = \{\{1\}, \{2\}, \ldots, \{n\}\}$.
2. $\mathcal{C}^h = \{N\}$.
3. If $C_q \in \mathcal{C}^l$ with $0 < l \leq h$, then $C_q = \bigcup_{S \in Q} S$ for some $Q \subset \mathcal{C}^{l-1}$.

We call $\mathcal{C}^l$ the *l-th level* of $\mathfrak{C}$. We say that $\mathfrak{C}$ is a levels structure of *degree* h. Thus, the levels structure $\mathfrak{C}$ has $h+1$ levels.

If $h = 1$, we say that $\mathfrak{C}$ is a *trivial* levels structure.

Given $i \in C_q \in \mathcal{C}^1$ with $n > 1$, we denote by $\mathfrak{C}_{-i}$ the levels structure on $N \setminus \{i\}$ which equals $\mathfrak{C}$ after removing player i, namely $\mathfrak{C}_{-i} = (\mathcal{C}^0_{-i}, \mathcal{C}^1_{-i}, \ldots, \mathcal{C}^h_{-i})$.

Given $S \in \mathcal{C}^l$, we denote by $\mathfrak{C}_{-S}$ the levels structure on $N \setminus S$ induced by $\mathfrak{C}$.

Assume $h \geq 2$. We define by $\mathfrak{C}/\mathcal{C}^1$ the levels structure induced by $\mathfrak{C}$ by dropping the level $\mathcal{C}^0$ and considering the coalitions $C_q \in \mathcal{C}^1$ as players. Whenever $C_q \in \mathcal{C}^1$ is considered as a player in $\mathfrak{C}/\mathcal{C}^1$, it is denoted by $[C_q]$. We also denote by $[\mathcal{C}^l]$ $(1 \leq l \leq h)$ the coalition structure which comes out from $\mathcal{C}^l$ by considering the coalitions of $\mathcal{C}^1$ as players.

Thus, we have $\mathfrak{C}/\mathcal{C}^1 = ([\mathcal{C}^1], [\mathcal{C}^2], \ldots, [\mathcal{C}^h])$.

In particular for $l = 1$, if $\mathcal{C}^1 = \{C_1, \ldots, C_m\}$, then $[\mathcal{C}^1] = \{\{[C_1]\}, \ldots, \{[C_m]\}\}$.

This new levels structure satisfies conditions 1, 2 and 3. Furthermore, $\mathfrak{C}/\mathcal{C}^1$ has degree $h-1$.

Let $LTU(N)$ be the set of all $(N, v, \mathfrak{C})$ with $(N, v) \in TU(N)$ cooperative game and $\mathfrak{C}$ levels structure on N.

The *quotient game* $(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1)$ is the game $LTU(\mathcal{C}^1)$ defined on the coalition structure $\mathcal{C}^1$ with characteristic function

$$\left(v/\mathcal{C}^1\right)(Q) = v\left(\bigcup_{[C_q] \in Q} C_q\right)$$

for all $Q \subset \mathcal{C}^1$.

A *solution concept* on $LTU(N)$ is a function $f : LTU(N) \to \mathbb{R}^N$ which assigns to each game $(N, v, \mathfrak{C}) \in LTU(N)$ a vector on $\mathbb{R}^N$, so that $f_i(N, v, \mathfrak{C})$ represents the payoff received by player $i \in N$.

In this article, we use two solution concepts for LTU: the *Shapley value* (Shapley, 1953), and the *levels structure value* suggested by Owen (1977) and characterized by Winter (1989).

The Shapley value is given by the following expression: Given $(N, v) \in TU(N)$ with $i \in N$,

$$\varphi_i(N, v) = \sum_{T \subset N \smallsetminus \{i\}} \frac{|T|!(n - |T| - 1)!}{n!} \left[v\left(T \cup \{i\}\right) - v\left(T\right)\right].$$

The levels structure value is a generalization of the Shapley value to games with levels structure, i.e. when the levels structure is trivial, both solution concepts give the same payoff vector. In order to define this, we need some additional notation.

We denote the set of all permutations on N as Π. Given a levels structure $\mathfrak{C}$, we define by induction $\Pi_1(\mathfrak{C}) \subset \Pi_2(\mathfrak{C}) \subset \cdots \subset \Pi_h(\mathfrak{C})$ as follows:

$$\Pi_h(\mathfrak{C}) = \Pi.$$

Given the sets $\Pi_{l+1}(\mathfrak{C}) \subset \Pi_{l+2}(\mathfrak{C}) \subset \cdots \subset \Pi_h(\mathfrak{C})$, we define

$$\Pi_l(\mathfrak{C}) = \left\{ \pi \in \Pi_{l+1}(\mathfrak{C}) : \forall j, k \in C_q \in \mathcal{C}^l, \forall i \in N, \pi(j) < \pi(i) < \pi(k) \Rightarrow i \in C_q \right\}.$$

In particular, permutations in $\Pi_1(\mathfrak{C})$ are those in which the players in the same coalition on any level always appear together.

Given $\pi \in \Pi$, $i \in N$, we denote by $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$ the set of *predecessors* of i under π. We term as *levels structure value* (Winter, 1989) the solution concept $\Psi : LTU(N) \to \mathbb{R}^N$ given by

$$\Psi_i(N, v, \mathfrak{C}) = \frac{1}{|\Pi_1(\mathfrak{C})|} \sum_{\pi \in \Pi_1(\mathfrak{C})} [v(P_i^\pi \cup \{i\}) - v(P_i^\pi)]$$

for all $i \in N$.

This solution concept generalizes the Owen value (1977) for $h = 2$ with coalition structure $\mathcal{C}^1$ and the Shapley value for $h = 1$.

A simple and powerful characterization for the levels structure value is as follows (Calvo, Lasaga and Winter, 1996): the levels structure value is the only solution concept on $LTU(N)$ which satisfies efficiency and balanced contributions.

Efficiency. For any game $(N, v, \mathfrak{C}) \in LTU(N)$, we have

$$\sum_{i \in N} \Psi_i(N, v, \mathfrak{C}) = v(N).$$

Balanced contributions. For any $(N, v, \mathfrak{C}) \in LTU\,(N)$ and any $S, T \in \mathcal{C}^l$ with $0 \le l < h$ such that $S, T \subset R \in \mathcal{C}^{l+1}$, $S \neq T$, we have

$$\begin{aligned} & \sum_{i \in S} \Psi_i(N, v, \mathfrak{C}) - \sum_{i \in S} \Psi_i(N \setminus T, v_{-T}, \mathfrak{C}_{-T}) \\ = \; & \sum_{i \in T} \Psi_i(N, v, \mathfrak{C}) - \sum_{i \in T} \Psi_i(N \setminus S, v_{-S}, \mathfrak{C}_{-S}). \end{aligned}$$

Furthermore, the levels structure value also satisfies additivity and quotient game property (Winter, 1989).

Additivity. For any $(N, v, \mathfrak{C})$, $(N, w, \mathfrak{C}) \in LTU\,(N)$, we have

$$\Psi(N, v + w, \mathfrak{C}) = \Psi(N, v, \mathfrak{C}) + \Psi(N, w, \mathfrak{C})$$

with $(N, v + w)$ the TU game defined on N by $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$.

Quotient game property. For any $(N, v, \mathfrak{C}) \in LTU\,(N)$, we have

$$\sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) = \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$$

for $C_q \in \mathcal{C}^1$.

3 The levels bidding mechanism

Given a cooperative game (N, v), Pérez-Castrillo and Wettstein (2001) designed a non-cooperative game, called the *bidding mechanism*. In the bidding mechanism, players bid for the right to propose a payoff, which should be accepted by all the other players. Otherwise the proposer leaves the game. Pérez-Castrillo and Wettstein proved that the payoff of any subgame perfect Nash equilibrium (henceforth, SPNE) of this mechanism with pure strategies always coincides with the Shapley value of the cooperative game (N, v). Like Pérez-Castrillo and Wettstein, we will not analyze mixed strategies.

Our mechanism is played in several rounds. In each round, coalitions in each coalition structure play a bidding mechanism in order to obtain the resources of their own coalition. In other words, they bid for the right to propose a payoff. In the first round, coalitions in $\mathcal{C}^1$ play sequentially. Assume they play in the order $C_1, \ldots, C_m$. Each player $i \in C_1$ simultaneously announces a bid $b_j^i \in \mathbb{R}$ for each player $j \in C_1 \setminus \{i\}$. The bid b_j^i represents the amount that player i is willing to pay to player j in order to be chosen as proposer. Note that a bid may be positive or negative. A negative bid is interpreted as the amount that player i is willing to accept from player j in exchange for being chosen as proposer.

The net bid for player i is the difference between what player i offers to other players and what other players offer him. The player with the highest net bid (say, player α) is chosen as proposer. Player α pays b_j^α to each $j \in C_q \setminus \{i\}$ and makes an offer. This offer is an amount $y_j^\alpha \in \mathbb{R}$ to be paid to each $j \in C_1 \setminus \{\alpha\}$. If all the players in $C_1 \setminus \{\alpha\}$ accept the offer, player α pays y_j^α to each $j \in C_1 \setminus \{\alpha\}$, he becomes the representative of C_1, and the turn passes to the next coalition C_2. Coalition C_2 should then choose a representative following the same procedure, and so on. If some player rejects an offer from a proposer (say, proposer $\alpha_q \in C_q$), all bids and offers are cancelled except the bids in C_q, and player α_q leaves the game. The mechanism is then played again with one player less.

The first round ends when each coalition has its own representative. This representative has the resources of all the members of his coalition (or a subcoalition, if some players quit). The representatives are themselves divided into coalitions, given by the coalition structure $\mathcal{C}^2$. The second round is then played, following the same procedure as before, with the coalitions in $\mathcal{C}^2$, and so on. Since the last coalition structure $\mathcal{C}^h$ has a single coalition, after round h there exists a unique representative i, who obtains the value of the grand coalition minus the bids and offers previously made by him. The rest of the players obtain the sum of the bids and offers in which they participated.

When the levels structure is trivial, there is a single round and the mechanism coincides with that of Pérez-Castrillo and Wettstein.

We will now describe the *levels bidding mechanism (LBM)* more formally. We proceed by double induction on h (degree of $\mathfrak{C}$) and n (number of players).

For $h = 1$, the players play a single round. This round comprises the bidding mechanism (Pérez-Castrillo and Wettstein, 2001) associated with the game (N, v).

Assume that we know the rules of the LBM when the levels structure has degree $h-1$, and it comprises $h-1$ rounds.

If there is only one player i, he obtains $v(\{i\})$. Let us now assume that we know the rules of the LBM when played by $n-1$ players. For a set of players $N = \{1, \ldots, n\}$ and a levels structure $\mathfrak{C} = (\mathcal{C}^0, \mathcal{C}^1, \ldots, \mathcal{C}^h)$ with $\mathcal{C}^1 = \{C_1, \ldots, C_m\}$, the LBM proceeds as follows:

Round 1

The players of any coalition $C_q \in \mathcal{C}^1$ play the bidding mechanism trying to obtain the resources of C_q. Formally, if there is only one player i, then this player has his resources. Assume now that we know the rules when played by $|C_q| - 1$ players. For $|C_q| > 1$ the mechanism proceeds as follows:

Stage 1. Each player $i \in C_q$ makes bids $b^i_j \in \mathbb{R}$ for every $j \in C_q \setminus \{i\}$. For each $i \in C_q$, we take $B^i = \sum_{j \in C_q \setminus \{i\}} b^i_j - \sum_{j \in C_q \setminus \{i\}} b^j_i$. Assume that $\alpha_q = argmax_i\{B^i\}$. In the case of a non-unique maximizer, α_q is randomly chosen from among the maximizing indices.

Stage 2. Player α_q, called the *proposer*, makes an offer $y^{\alpha_q}_i$ to each player $i \in C_q \setminus \{\alpha_q\}$.

Stage 3. In turn, the players of $C_q \setminus \{\alpha_q\}$ either accept or reject the offer. If a rejection is encountered, we say the offer is rejected. Otherwise, we say the offer is accepted.

The coalitions of $\mathcal{C}^1$ play sequentially in the order $C_1, \ldots, C_m$ until either we find $C_{q_0} \in \mathcal{C}^1$ and $\alpha_{q_0} \in C_{q_0}$ such that the offer of α_{q_0} is rejected, or for any $C_q \in \mathcal{C}^1$ the offer of α_q is accepted.

In the first case, player α_{q_0} pays $b^{\alpha_{q_0}}_i$ to each player $i \in C_{q_0} \setminus \{\alpha_{q_0}\}$ and leaves the non-cooperative game obtaining $v(\{\alpha_{q_0}\}) - \sum_{i \in C_{q_0} \setminus \{\alpha_{q_0}\}} b^{\alpha_{q_0}}_i$. All players other than α_{q_0} proceed to play the LBM with $(N', v', \mathfrak{C}')$ where $N' = N \setminus \{\alpha_{q_0}\}$, $v' = v_{-\alpha_{q_0}}$, and $\mathfrak{C}' = \mathfrak{C}_{-\alpha_{q_0}}$. Any player $i \in C_{q_0} \setminus \{\alpha_{q_0}\}$ obtains as his final payoff the sum of the bids received, $b^{\alpha_{q_0}}_i$, and the payoff outcome of the mechanism corresponding to (N', C', v'). Any player $i \in N \setminus C_{q_0}$ obtains as his final payoff the payoff outcome of the mechanism corresponding to (N', C', v').

In the second case, for any $C_q \in \mathcal{C}^1$, player α_q pays $b^{\alpha_q}_i + y^{\alpha_q}_i$ to every $i \in C_q \setminus \{\alpha_q\}$ and becomes the *representative* of coalition C_q. This means that player α_q goes to Round 2 with all the resources of C_q. Moreover, the payoff obtained by this player in this round is $p^1_{\alpha_q} = -\sum_{i \in C_q \setminus \{\alpha_q\}} (b^{\alpha_q}_i + y^{\alpha_q}_i)$. Any other player $i \in C_q \setminus \{\alpha_q\}$ leaves the non-cooperative game obtaining a final payoff of $b^{\alpha_q}_i + y^{\alpha_q}_i$.

After finishing Round 1, for any $C_q \in \mathcal{C}^1$ we can find the representative (denoted by r_q) for this coalition.

Rounds 2 through h

The representatives play the LBM associated with the quotient game $(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1)$, where each r_q plays the role of $[C_q]$. These rounds are well defined by induction on

h. For any representative r_q, we denote by $p^2_{r_q}$ the payoff obtained by r_q (or $[C_q]$) in these rounds.

The final payoff obtained by any representative r_q is the sum of the payoffs obtained in all the rounds, *i.e.* $p^1_{r_q} + p^2_{r_q}$.

Note that the LBM terminates in a finite number of moves.

Remark 3.1 *Assume that in Round 1 the offer of player α_q is accepted for any $q < q_0$, but the offer of α_{q_0} is rejected. A new subgame begins, therefore, which coincides with the LBM associated with $\left(N \setminus \{\alpha_{q_0}\}, v_{-\alpha_{q_0}}, \mathfrak{C}_{-\alpha_{q_0}}\right)$. Moreover, when all the offers in Round 1 are accepted, another subgame begins, which is equivalent to the LBM associated with $(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1)$.*

Before characterizing the SPNE outcomes of the levels bidding mechanism we need the following result:

Proposition 3.1 *Given a triple $(N, v, \mathfrak{C}) \in LTU(N)$ such that (N, v) is zero-monotonic, $j \in C_q \in \mathcal{C}^1 \in \mathfrak{C}$ and $\{j\} \subsetneq C_q$ then*

$$\sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) \geq \sum_{i \in C_q \setminus \{j\}} \Psi_i(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}) + v(\{j\}).$$

Proof. We take $\mathfrak{C}/\mathcal{C}^1 = ([\mathcal{C}^1], \ldots, [\mathcal{C}^h])$ levels coalition structure. Assume $\mathcal{C}^1 = \{C_1, \ldots, C_m\}$ and $M = \{1, 2, \ldots, m\}$. Let $\mathfrak{Q} = \left(Q^1, \ldots, Q^h\right)$ be the levels structure on M which equals $\mathfrak{C}/\mathcal{C}^1$ except for the name of the players, i.e.

$$\{q_1, q_2, \ldots, q_k\} \in Q^l \Leftrightarrow \{[C_{q_1}], [C_{q_2}], \ldots, [C_{q_k}]\} \in [\mathcal{C}^l] \qquad 1 \leq l \leq h.$$

We define the following games on M. For all $R \subset M$,

$$\begin{aligned}
u(R) &= v\left(\bigcup_{r \in R} C_r\right) \\
w_1(R) &= \begin{cases} v\left(\bigcup_{r \in R} C_r \setminus \{j\}\right) & \text{if } q \in R \\ v\left(\bigcup_{r \in R} C_r\right) & \text{if } q \notin R \end{cases} \\
w_2(R) &= \begin{cases} v(\{j\}) & \text{if } q \in R \\ 0 & \text{if } q \notin R \end{cases} \\
w &= w_1 + w_2.
\end{aligned}$$

Note that the game u on M equals the quotient game $v/\mathcal{C}^1$ on $\mathcal{C}^1$. Thus, their levels structure values are the same, namely,

$$\Psi_q(M, u, \mathfrak{Q}) = \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$$

by the quotient game property,

$$\Psi_q(M, u, \mathfrak{Q}) = \sum_{i \in C_q} \Psi_i (N, v, \mathfrak{C}).$$

Analogously, the game w_1 on M equals the quotient game $v_{-j}/\mathcal{C}^1_{-j}$ on $\mathcal{C}^1_{-j}$. Thus,

$$\Psi_q(M, w_1, \mathfrak{Q}) = \sum_{i \in C_q \setminus \{j\}} \Psi_i (N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}).$$

Finally, the levels structure value of q for the game w_2 is

$$\Psi_q(M, w_2, \mathfrak{Q}) = v(\{j\}).$$

By applying the zero-monotonicity of v, we get $\Psi_q(M, u, \mathfrak{Q}) \geq \Psi_q(M, w, \mathfrak{Q})$. By the additivity of the levels structure value,

$$\begin{aligned}
\sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) &= \Psi_q(M, u, \mathfrak{Q}) \geq \Psi_q(M, w, \mathfrak{Q}) = \Psi_q(M, w_1 + w_2, \mathfrak{Q}) \\
&= \Psi_q(M, w_1, \mathfrak{Q}) + \Psi_q(M, w_2, \mathfrak{Q}) \\
&= \sum_{i \in C_q \setminus \{j\}} \Psi_i (N \setminus \{j\}, v_{-\alpha}, \mathfrak{C}_{-j}) + v(\{j\}).
\end{aligned}$$

■

In order to cope with the technical problems of ties, we need an additional assumption in regard to the SPNE. These problems of ties appear when players are indifferent as to two or more strategies yielding the same payoff. In Section 4 we will study an example of a game where the associated LBMs have SPNE outcomes whose payoff is different from the levels structure value.

Vidal-Puga and Bergantiños (2003) modified their mechanism, so that the player who rejects an offer and the proposer whose offer is rejected must pay a small penalty $\varepsilon > 0$.

In this article, we will not move in that direction. Moldovanu and Winter (1994) assume that a player prefers agreements which involve larger rather than smaller coalitions (provided his final payoff is the same in both agreements). Hart and Mas-Colell (1996) assumed that players "break ties in favor of quick termination of the game"[2]. In this paper we make both assumptions.

As a consequence of our assumptions, we can define a tie-breaking rule satisfying the following conditions:

- If a player is indifferent to accepting or rejecting an offer from a proposer, he always accepts the offer.

[2]However, tie-breaking rules are not needed in Hart and Mas-Colell's model.

- If a proposer $\alpha \in C_q$ is indifferent to offering b^α or $\widetilde{b}^\alpha$ with b^α likely to be rejected by some player $i \in C_q \setminus \{\alpha\}$ and $\widetilde{b}^\alpha$ likely to be accepted by each player in $C_q \setminus \{\alpha\}$, he always offers $\widetilde{b}^\alpha$.

In the rest of the section, by SPNE we mean an SPNE satisfying this tie-breaking rule.

A similar approach by means of the tie-breaking rule for SPNE can be found in Navarro and Perea (2001). In their model, a player is required to choose prices, propose offers and accept or reject offers[3]. If a player is indifferent to accepting or rejecting an offer, he is supposed to accept. If, under certain circumstances, a player is indifferent to proposing Δ or $\widetilde{\Delta}$ with $\Delta < \widetilde{\Delta}$, he is supposed to propose $\widetilde{\Delta}$. If a player is indifferent to choosing between price p or $\widetilde{p}$ with $p < \widetilde{p}$, he is supposed to choose price p.

Theorem 3.1 *The LBM implements the levels structure value in SPNE.*

Proof. Althought the structure of this proof is similar to that of the main result by Vidal-Puga and Bergantiños (2003), the computations are different.

We proceed by double induction on h and n. For $h = 1$, the mechanism coincides with that by Pérez-Castrillo and Wettstein (2001). Thus, we assume that the players play according to a strategy profile described in Pérez-Castrillo and Wettstein (2001) when they construct, for any zero-monotonic game, an SPNE that yields the Shapley value of this game as a payoff outcome. It is easy to check that this SPNE satisfies the tie-breaking rule. So, the mechanism implements the levels structure value.

Assume the result is true for levels structures of degree $h-1$ or less.

We now prove the result when the degree is h. If there is only one player it is trivial. Assume that if there are at most $n-1$ players the LBM implements the levels structure value in SPNE and, moreover, that all the Round 1 offers are accepted in equilibrium. We now prove that the same holds when there are n players.

We first prove that the levels structure value is indeed an equilibrium outcome. We explicitly construct an SPNE which yields the levels structure value as an SPNE outcome.

We consider the following strategies.

Round 1. First, we define the strategies in the LBM associated with any $C_q \in \mathcal{C}^1$.

Stage 1. For any $i \in C_q$, $b_j^i = \Psi_j(N, v, \mathfrak{C}) - \Psi_j(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i})$ for any $j \in C_q \setminus \{i\}$.

Stage 2. Player α_q, the proposer, offers $y_j^{\alpha_q} = \Psi_j\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ to every $j \in C_q \setminus \{\alpha_q\}$.

[3] These offers are differences in payoffs to be received at the end of the mechanism.

Stage 3. A player $i \in C_q \setminus \{\alpha_q\}$ accepts the offer of α_q if and only if $y_j^{\alpha_q} \geq \Psi_j\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ for every $j \in C_q \setminus \{\alpha_q\}$.

If some offer is rejected, for instance, the offer of α_{q0}, we go to the subgame where all players other than α_{q0} play this mechanism in $\left(N \setminus \{\alpha_{q0}\}, v_{-\alpha_{q0}}, \mathfrak{C}_{-\alpha_{q0}}\right)$. We assume that players in $N \setminus \{\alpha_{q0}\}$ play according to the strategy profiles of some SPNE with associated payoff $\Psi\left(N \setminus \{\alpha_{q0}\}, v_{-\alpha_{q0}}, \mathfrak{C}_{-\alpha_{q0}}\right)$ (by induction hypothesis on n we can find such an SPNE).

Rounds 2 through h**.** We assume that the representatives play according to the strategies of some SPNE with associated payoff $\Psi\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. Again, by induction hypothesis on h, we can find such an SPNE.

Proving that these strategies satisfy the tie-breaking rule is quite straightforward.

First, we prove that according to these strategies any player $i \in N$ receives as payoff the levels structure value $\Psi_i(N, C, v)$. Note that for any $C_q \in \mathcal{C}^1$ the offer of α_q is accepted. Then player α_q goes to Round 2 as the representative of C_q.

Given $C_q \in \mathcal{C}^1$ and $i \in C_q \setminus \{\alpha_q\}$, the payoff obtained by player i is $b_i^{\alpha_q} + y_i^{\alpha_q} =$

$$
\begin{aligned}
& \Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right) + \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right) \\
= \; & \Psi_i(N, v, \mathfrak{C}).
\end{aligned}
$$

We now compute the payoff for any representative r_q. As v is superadditive we have that $v/\mathcal{C}^1$ is also superadditive. By induction hypothesis on h, we know that the payoff obtained by r_q in Rounds 2 through h $(p_{r_q}^2)$ coincides with the levels structure value of $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. The final payoff obtained, therefore, by r_q is

$$
\begin{aligned}
p_{r_q}^1 + p_{r_q}^2 &= -\sum_{i \in C_q \setminus \{r_q\}} b_i^{r_q} - \sum_{i \in C_q \setminus \{r_q\}} y_i^{r_q} + \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right) \\
&= -\sum_{i \in C_q \setminus \{r_q\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{r_q\}, v_{-r_q}, \mathfrak{C}_{-r_q}\right)\right] \\
&\quad - \sum_{i \in C_q \setminus \{r_q\}} \Psi_i\left(N \setminus \{r_q\}, v_{-r_q}, \mathfrak{C}_{-r_q}\right) + \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)
\end{aligned}
$$

by rearranging terms and applying the quotient game property,

$$
= -\sum_{i \in C_q \setminus \{r_q\}} \Psi_i(N, v, \mathfrak{C}) + \sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) = \Psi_{r_q}(N, v, \mathfrak{C}).
$$

We now prove that these strategies are an SPNE. By induction hypothesis on h, we conclude that in the subgames obtained after Round 2 these strategies induce an SPNE.

By induction hypothesis on n, in all the subgames obtained after the rejection of the offer of some proposer α_q, these strategies induce an SPNE.

We only have to prove that these strategies induce an SPNE in the bidding mechanism associated with any coalition C_q (Round 1).

Stage 3. Assume that player i rejects the offer of α_q. The LBM mechanism of $\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ is played and, by induction hypothesis on n, after rejection player i can obtain at most $c = \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, C_{-\alpha_q}\right)$. Hence, it is optimal for player i to accept any offer which is at least equal to c (notice the tie-breaking rule), and reject any offer which is lower than c.

Stage 2. If player α_q offers to some player $i \in C_q$ less than

$$\Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, C_{-\alpha_q}\right),$$

the offer is rejected and player α_q obtains, therefore, a final payoff of

$$v\left(\{\alpha_q\}\right) - \sum_{i \in C_q \setminus \{\alpha_q\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_l}, \mathfrak{C}_{-\alpha_q}\right)\right].$$

By Proposition 3.1, this payoff is no greater than $\Psi_{\alpha_q}(N, v, \mathfrak{C})$, which means that player α_q does not improve his payoff.

If player α_q offers to any player $i \in C_q \setminus \{\alpha_q\}$ at least $\Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$, the offer is accepted. It is simple to prove that player α_q obtains, at the most, $\Psi_{\alpha_q}(N, v, \mathfrak{C})$.

Stage 1. First, we prove that for any $i \in C_q \in \mathcal{C}^1$, $B^i = 0$.

$$\begin{aligned} B^i &= \sum_{j \in C_q \setminus \{i\}} b^i_j - \sum_{j \in C_q \setminus \{i\}} b^j_i \\ &= \sum_{j \in C_q \setminus \{i\}} \left[\Psi_j(N, v, \mathfrak{C}) - \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right)\right] \\ &\quad - \sum_{j \in C_q \setminus \{i\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right)\right]. \end{aligned}$$

As the levels structure value satisfies balanced contributions, we have that for any $j \in C_q \setminus \{i\}$,

$$\Psi_i\left(N, v, \mathfrak{C}\right) - \Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right) = \Psi_j\left(N, v, \mathfrak{C}\right) - \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right)$$

and hence $B^i = 0$.

Assume that player $i \in C_q$ makes a different bid b^*. If $B^{*i} < 0$, the proposer will be another player of C_q. In that case player i cannot increase his payoff.

If $B^{*i} > 0$, he becomes the proposer but must pay $\sum_{j \in C_q \setminus \{i\}} b_j^{*i}$ to the other players of $C_q \setminus \{i\}$. It is straightforward to prove that player i can obtain, at most, a final payoff of

$$\Psi_i(N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus \{i\}} b_j^{*i} + \sum_{j \in C_q \setminus \{i\}} b_j^i$$

which is lower than $\Psi_i(N, v, \mathfrak{C})$.

If $B^{*i} = 0$ and player i is not the proposer, using similar arguments to those used when $B^{*i} < 0$, we can conclude that player i does not increase his payoff. If $B^{*i} = 0$ and player i is the proposer, using similar arguments to those used when $B^{*i} > 0$ we can conclude that player i does not increase his payoff.

We will now prove that the payoff in all SPNE outcomes coincides with the levels structure value. We will do this in several steps.

Step A. At every SPNE outcome, and for every $C_q \in \mathcal{C}^1$, the offer from the proposer α_q to each player $i \in C_q \setminus \{\alpha_q\}$ is $y_i^{\alpha_q} = \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ and every $i \in C_q \setminus \{\alpha_q\}$ accepts this offer.

Assume that in each coalition $C_q \in \{C_1, \dots, C_{m-1}\}$, the offer from a proposer $\alpha_q \in C_q$ is accepted. Considering the subgame starting with the last coalition C_m, let $\alpha_m \in C_m$ be the proposer in C_m, let y^{α_m} be an offer from α_m, and let the order of reply for the players in $C_m \setminus \{\alpha_m\}$ be $i_1, \dots, i_k$.

Claim 1. At each SPNE, the strategies of the players in $C_m \setminus \{\alpha_m\}$ must satisfy the following statements:

(i) If $y_i^{\alpha_m} \geq \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$ for every $i \in C_m \setminus \{\alpha_m\}$, then every $i \in C_m \setminus \{\alpha_m\}$ accepts y^{α_m}.

(ii) If $y_j^{\alpha_m} < \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$ for some $j \in C_m \setminus \{\alpha_m\}$, then some player in $C_m \setminus \{\alpha_m\}$ rejects y^{α_m}.

(i) Consider the strategy of the last player i_k. Assuming that his decision node is reached, if he accepts the offer y^{α_m} then he receives $b_{i_k}^{\alpha_m} + y_{i_k}^{\alpha_m}$, whereas if he rejects y^{α_m} then by the induction hypothesis he obtains $b_{i_k}^{\alpha_m} + \Psi_{i_k}\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. Hence, at any SPNE,

- if $y_{i_k}^{\alpha_m} > \Psi_{i_k}\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, then i_k accepts the offer because it is optimal;
- if $y_{i_k}^{\alpha_m} = \Psi_{i_k}\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, then i_k accepts the offer because of the tie-breaking rule.

Repeating the same argument in reverse, we can show that players $i_{k-1}, \dots, i_1$ accept the offer.

(ii) Supposing, to the contrary, that there exists $j \in C_m \setminus \{\alpha_m\}$ with $y_j^{\alpha_m} < \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, but all the players in $C_m \setminus \{\alpha_m\}$ accept the offer y^{α_m}. In this case, player j receives $b_j^{\alpha_m} + y_j^{\alpha_m}$. However, if player j deviates and rejects the offer, then he obtains $b_j^{\alpha_m} + \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, which is more than $b_j^{\alpha_m} + y_j^{\alpha_m}$. Hence, the strategies of the players in $C_m \setminus \{\alpha_m\}$ cannot constitute an SPNE.

Claim 2. At every SPNE outcome, each $i \in C_m \setminus \{\alpha_m\}$ accepts the offer from the proposer α_m.

Supposing, to the contrary, that at some SPNE outcome, there exists $i \in C_m \setminus \{\alpha_m\}$ who rejects the offer y^{α_m}. In this case, the proposer obtains

$$e = v\left(\{\alpha_m\}\right) - \sum_{i \in C_m \setminus \{\alpha_m\}} b_i^{\alpha_m}.$$

Supposing that the proposer α_m proposes $z_i^{\alpha_m} = \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$ to every $i \in C_m \setminus \{\alpha_m\}$. By Claim 1 (i), every $i \in C_m \setminus \{\alpha_m\}$ accepts z^{α_m}. Hence, player α_m is the representative of coalition C_m in Round 2. Now, in Rounds 2 through h, there are m players $\{\alpha_1, \ldots, \alpha_m\}$, where, for any coalition $C_q \in \mathcal{C}^1$, α_q is the representative of coalition C_q. As the representatives are playing an SPNE for the LBM associated with $(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1)$, by induction hypothesis on h we know that the payoff obtained by player α_m in Rounds 2 through h is $\Psi_{[C_m]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$, which, by the quotient game property, equals $\sum_{i \in C_m} \Psi_i\left(N, v, \mathfrak{C}\right)$. The final payoff of player α_m is, therefore,

$$\widetilde{e} = \sum_{i \in C_m} \Psi_i(N, v, \mathfrak{C}) - \sum_{i \in C_m \setminus \{\alpha_m\}} \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right) - \sum_{i \in C_m \setminus \{\alpha_m\}} b_i^{\alpha_m}.$$

By Proposition 3.1 we know that

$$\sum_{i \in C_m} \Psi_i(N, v, \mathfrak{C}) - \sum_{i \in C_m \setminus \{\alpha_m\}} \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right) \geq v\left(\{\alpha_m\}\right).$$

Thus, $e \leq \widetilde{e}$.

• If $e < \widetilde{e}$, to offer y^{α_m} cannot be an SPNE strategy of the proposer α_m, which is a contradiction.

• If $e = \widetilde{e}$, then α_m is indifferent to offering y^{α_m} or z^{α_m}. By Claim 1 (i), offer z^{α_m} is accepted by each $i \in C_m \setminus \{\alpha_m\}$. By the tie-breaking rule α_m must propose z^{α_m} better than y^{α_m}, which is a contradiction.

Claim 3. At each SPNE, and for each $i \in C_m \setminus \{\alpha_m\}$, we have $y_i^{\alpha_m} = \Psi_i\left(N \setminus \{\alpha_m\}, C_{-\alpha_m}, v_{-\alpha_m}\right)$.

Let y^{α_m} be the offer from α_m at an SPNE. By Claim 2, y^{α_m} must be accepted by every $i \in C_m \setminus \{\alpha_m\}$. It follows, therefore, from Claim 1 (ii) that for every $i \in$

$C_m \setminus \{\alpha_m\}$, $y_i^{\alpha_m} \geq \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. Suppose that for some $j \in C_m \setminus \{\alpha_m\}$, $y_j^{\alpha_m} > \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. For each $i \in C_m \setminus \{\alpha_m\}$, we define $w_i^{\alpha_m} = \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. Assuming that the proposer α_m deviates and offers w^{α_m}. Then, by Claim 1 (i), each $i \in C_m \setminus \{\alpha_m\}$ accepts w^{α_m}. Moreover, since

$$\sum_{i \in C_m \setminus \{\alpha_m\}} w_i^{\alpha_m} = \sum_{i \in C_m \setminus \{\alpha_m\}} \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right) < \sum_{i \in C_m \setminus \{\alpha_m\}} y_i^{\alpha_m},$$

the proposer α_m obtains a greater payoff by offering w^{α_m} than by offering y^{α_m}. Hence, to offer y^{α_m} cannot be an SPNE strategy, which is a contradiction.

Repeating the same arguments for coalitions $C_{m-1}, \ldots, C_1$, we can prove Step A.

Step B. Assume that we are in Stage 1 of Round 1 of the LBM associated with $C_q \in \mathcal{C}^1$. For any SPNE, $B^i = 0$ for any $i \in C_q$.

It is not difficult to prove that $\sum_{i \in C_q} B^i = 0$. We take

$$X = \left\{ i \in C_q : B^i = \max_{j \in C_q} B^j \right\}.$$

If $X = C_q$, the result holds because $\sum_{i \in C_q} B^i = 0$.

If $X \neq C_q$, a contradiction results from proving that player $i \in X$ has a deviation which improves his final payoff. We take $j \in C_q \setminus X$ such that $B^j \geq B^k$ for any $k \in C_q \setminus X$. Assume that player i makes a new bid b'^i, where $b'^i_k = b^i_k + \delta$ if $k \in X \setminus \{i\}$, $b'^i_j = b^i_j - |X|\delta$, and $b'^i_k = b^i_k$ if $k \in C_q \setminus (X \cup \{j\})$.

For any $k \in C_q$, we compute B'^k assuming that $b'^k = b^k$ for any $k \in C_q \setminus \{i\}$. Then $B'^k = B^k - \delta$ if $k \in X$, $B'^j = B^j + |X|\delta$, and $B'^k = B^k$ if $k \in C_l \setminus (X \cup \{j\})$.

Since $B^j < B^i$, we can find $\delta > 0$ satisfying $B^j + |X|\delta < B^i - \delta$. Moreover, $X' = \left\{k \in C_q : B'^k = \max_{h \in C_q} B'^h\right\} = X$. This means that any player of X is the proposer with the same probability under b^i and b'^i. When player i is not the proposer, which happens with probability $\frac{|X|-1}{|X|}$, he obtains, by Step A, the same by making a bid b^i or b'^i. But if player i is the proposer, which happens with probability $\frac{1}{|X|}$, he obtains, by Step A, δ units more with b'^i than with b^i.

Step C. Assume that we are in Stage 1 of Round 1 of the LBM associated with $C_q \in \mathcal{C}^1$. Then, at each SPNE, the payoff for any player $i \in C_q$ is the same regardless of who is chosen as the proposer.

By Step B, we know that $B^i = 0$ for any $i \in C_q$.

Assume that some player i strictly prefers to be (not to be) the proposer. In that case player i can improve his payoff by slightly increasing (decreasing) one of his bids b^i_j. But this is impossible in an SPNE.

Step D. In any SPNE outcome for the LBM any player $i \in N$ obtains as his final payoff his levels structure value.

Assume that players are playing according to some SPNE. Given $i \in C_q \in \mathcal{C}^1$, we denote by p_i the final payoff obtained by player i in this SPNE.

By Step B, any player of C_q is the proposer with probability $\frac{1}{|C_q|}$.

If player i is the proposer, we know, by Step A, that his final payoff is

$$\sum_{j \in C_q} \Psi_j(N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus \{i\}} \Psi_j(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}) - \sum_{j \in C_q \setminus \{i\}} b_j^i.$$

If $j \in C_q \setminus \{i\}$ is the proposer then the final payoff of player i is, by Step A,

$$b_i^j + \Psi_i(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}).$$

By Step C, we know that

$$\begin{aligned} |C_q| p_i &= \sum_{j \in C_q} \Psi_j(N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus \{i\}} \Psi_j(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}) - \sum_{j \in C_q \setminus \{i\}} b_j^i \\ &\quad + \sum_{j \in C_q \setminus \{i\}} \left[b_i^j + \Psi_i(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}) \right]. \end{aligned}$$

By Step B, we know that $-\sum_{j \in C_q \setminus \{i\}} b_j^i + \sum_{j \in C_q \setminus \{i\}} b_i^j = -B^i = 0$.

Hence, $|C_q| p_i =$

$$\sum_{j \in C_q \setminus \{i\}} \left[\Psi_i(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}) - \Psi_j(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}) \right] + \sum_{j \in C_q} \Psi_j(N, v, \mathfrak{C}).$$

Since the levels structure value satisfies the property of balanced contributions, we have

$$\begin{aligned} |C_q| p_i &= \sum_{j \in C_q \setminus \{i\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_j(N, v, \mathfrak{C}) \right] + \sum_{j \in C_q} \Psi_j(N, v, \mathfrak{C}) \\ &= (|C_q| - 1) \Psi_i(N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus \{i\}} \Psi_j(N, v, \mathfrak{C}) + \sum_{j \in C_q} \Psi_j(N, v, \mathfrak{C}) \\ &= |C_q| \Psi_i(N, v, \mathfrak{C}). \end{aligned}$$

Therefore, $p_i = \Psi_i(N, v, \mathfrak{C})$. ■

4 Conclusions

In this paper we have developed a bidding mechanism that implements the levels structure value of every game with a levels structure of cooperation. The mechanism is a generalization of the bidding model described by Pérez-Castrillo and Wettstein (2001).

In equilibrium, we have imposed the condition that players prefer larger to smaller coalitions. The next example shows that this condition is necessary. Note that players from coalition $\{1,2\}$ are indifferent to leaving the game and staying in it (they obtain 0 anyway). Players in $\{3,4\}$, however, are sensitive to player 1 or player 2 leaving the game.

Consider $(N, v, \mathfrak{C})$ with $h = 2$, where $N = \{1,2,3,4\}$, $\mathfrak{C} = \{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2\}$, $\mathcal{C}^1 = \{\{1,2\}, \{3,4\}\}$. Moreover, v is the characteristic function associated with the weighted majority game where the quota is 3 and the weights are 1, 1, 1, and 2 respectively. This means that $v(S) = 1$ if and only if S contains some of the following subsets: $\{1,2,3\}$, $\{1,4\}$, $\{2,4\}$, or $\{3,4\}$.

It is straightforward to prove that

$$\begin{aligned}
\Psi(N, v, \mathfrak{C}) &= \left(0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right) \\
\Psi(N \smallsetminus \{1\}, v_{-1}, \mathfrak{C}_{-1}) &= \left(-, 0, \tfrac{1}{4}, \tfrac{3}{4}\right) \\
\Psi(N \smallsetminus \{2\}, v_{-2}, \mathfrak{C}_{-2}) &= \left(0, -, \tfrac{1}{4}, \tfrac{3}{4}\right) \\
\Psi(N \smallsetminus \{3\}, v_{-3}, \mathfrak{C}_{-3}) &= \left(\tfrac{1}{4}, \tfrac{1}{4}, -, \tfrac{1}{2}\right) \\
\Psi(N \smallsetminus \{4\}, v_{-4}, \mathfrak{C}_{-4}) &= \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, -\right).
\end{aligned}$$

The LBM in this example is as described as follows: Players 1 and 2 simultaneously choose bids b_2^1 and b_1^2, respectively. If $b_2^1 > b_1^2$, then player 1 proposes y_2^1. If $b_2^1 < b_1^2$, then player 2 proposes y_1^2. If $b_2^1 = b_1^2$, then a randomly chosen player (1 or 2) proposes y_2^1 or y_1^2. Let α_1 be the proposer and β_1 be the other player. Player β_1 can either accept or reject. Assume first that player β_1 accepts. Then, players 3 and 4 simultaneously choose bids b_4^3 and b_3^4, respectively. Let α_2 be the player with the highest bid and let β_2 be the player with the lowest bid (as before). Player α_2 proposes $y_{\beta_2}^{\alpha_2}$ and player β_2 either accepts or rejects. If player β_2 accepts, then player α_l pays $b_{\beta_l}^{\alpha_l} + y_{\beta_l}^{\alpha_l}$ to player β_l for $l = 1, 2$. Players β_1 and β_2 give their resources to α_1 and α_2 (respectively) and leave the game. Players α_1 and α_2, with their new resources, repeat the procedure.

If player β_2 rejects, player α_2 pays $b_{\beta_2}^{\alpha_2}$ to player β_2 and leaves the game. Players in $N \smallsetminus \{\alpha_2\}$ should then repeat the procedure from the beginning.

Now assume that player β_1 rejects. In this case, player α_1 pays $b_{\beta 1}^{\alpha_1}$ to player β_1 and leaves the game. Players in $N \smallsetminus \{\alpha_1\}$ repeat the procedure from the beginning.

The mechanism when there is a single player in one of the coalitions (C_1 or C_2) is similar to the above. However, the only player in the singleton becomes representative for sure, but only with his own resources.

We now define an SPNE whose payoff outcome is $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

Round 1. First, we describe the strategies of players 1 and 2. The bids are $b_2^1 = b_1^2 = 0$. The proposer α is randomly chosen between 1 and 2. Moreover, $y_j^\alpha = 0$ and player j accepts the offer of α if and only if α offers him something strictly positive (hence, under these strategies, the proposal is rejected).

We now describe the strategies of players 3 and 4. In the subgame obtained after the offer of α is accepted, the strategies of players 3 and 4 coincide with the strategies whose payoff outcome is the levels structure value. We know that these strategies exist by Theorem 3.1. In the subgame obtained after the offer of α is rejected, the strategies of players 3 and 4 coincide with the strategies whose payoff outcome is the levels structure value of $(N \setminus \{\alpha\}, v_{-\alpha}, \mathfrak{C}_{-\alpha})$.

Round 2. We assume the representatives play according to the strategies described in Pérez-Castrillo and Wettstein (2001), which implement the levels structure value.

It is not difficult to confirm that these strategies are an SPNE. However, they do not satisfy the tie-breaking rule.

According to these strategies, the offer of player α is rejected, which means that player α obtains a final payoff of $v(\{\alpha\}) = 0$. Players of $N \setminus \{\alpha\}$, therefore, obtain as their final payoff $\Psi(N \setminus \{\alpha\}, v_{-\alpha}, \mathfrak{C}_{-\alpha})$. This means that the final payoff induced by these strategies is $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

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