A monotonic and merge-proof rule in minimum cost spanning tree situations

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Abstract

We present a new model for cost-sharing in minimum cost spanning tree problems to allow planners to identify how many agents merge. Under this new framework, in contrast to the traditional model, there are rules that satisfy the property of *merge-proofness*. Furthermore, strengthening this property and adding some others, such as *population monotonicity* and *solidarity*, makes it possible to define a unique rule that coincides with the weighted Shapley value of an associated cost game.

Keywords: Minimum cost spanning tree problems, cost sharing, core selection, cost-monotonicity, merge-proofness, weighted Shapley value.

JEL Classification Codes: C71, D61, D63, D7.

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1 Introduction

Minimum cost spanning tree ($MCST$) problems study situations in which a group of agents located at different geographical points want some particular good or service that can only be provided by a common supplier (the *source*). Agents can be served through connections, which entail a cost, and do not care whether they are directly or indirectly connected to the source.

In this paper, we study the cost-sharing aspect, which determines which rule should be used to share the cost of building an optimal network among the agents (this is normally a tree). One approach is to study the desirable properties that are satisfied by the different rules, since these properties can help a planner compare different rules and decide which is preferable in a specific case.

We focus on three important classes of properties. The first, based on the property of *Core Selection*, states that no group of nodes should subsidize others by paying more than the cost of connecting themselves to the source. This property is highly relevant in literature from economics and although, generally speaking, the core may be empty, it is always non-empty in a game associated with an $MCST$ problem. In fact, most of the rules proposed in the literature satisfy Core Selection, with the notable exception of the rule derived from the Shapley value of the associated game (Kar, 2002). A stronger version of Core Selection is *Population Monotonicity*, which requires that the cost allocated to any node will not decrease if new nodes join the society.

The second, based on the property of *Cost Monotonicity*, states that the cost allocated to a node will not increase if the cost of a link involving this node goes down, *caeteris paribus.* A violation of this desirable property could act as a disincentive for agents to reduce the costs of constructing links (Dutta and Kar, 2004). A stronger version of cost monotonicity requires that the cost allocated to any node will not increase if the cost of any link—regardless of whether it involves this link—decreases, *caeteris paribus.* This strong version would also prevent agents sabotaging links.

The third class is based on the properties of *Split-* and *Merge-proofness*. The former states that no node should have an incentive to split into two or more different nodes, whereas the latter states that two or more nodes should not have an incentive to merge into a single node. These properties are relevant in situations where the identity of the nodes is ambiguous. Hence, for example, different departments on a university campus may already be connected by an internal network. In the event they wish to be connected to a wider supply network, should they be considered as a single node (the campus) or as several different nodes (the departments) connected at zero cost? Other examples include the different shops at a mall, apartments in a building, or houses in a residential area.

In the context of $MCST$ problems, the folk rule (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007a) satisfies Split-proofness. Moreover, it is not difficult to derive a split-proof rule from a cost-monotonic one. However, this is not the case with Merge-proofness. Under domain restrictions,[1] the Bird rule (Bird, 1976) satisfies Merge-proofness (Özsoy, 2006; Athanassoglou and Sethuraman, 2008; Gómez-Rúa and Vidal-Puga, 2011). However, in the most general circumstances, no rule satisfies Merge-proofness, as shown by Example 1.1:

Example 1.1 (Özsoy (2006)) *Assume three agents located at nodes* 1*,* 2 *and* 3*. The connection cost between each agent and the source is* 1*, and the connection cost between any pair of agents is* 0*. The minimal cost is* 1*. Hence, any rule* Φ *should satisfy* $\Phi_1 + \Phi_2 + \Phi_3 = 1$*. Now let us assume w.l.o.g.* $\Phi_3 \geq \max\{\Phi_1, \Phi_2\}$*. If players* 1 *and* 3 *join and appear as agent* 1 *alone, the planner would face the same problem as if players* 2 *and* 3 *join and appear as agent* 2 *alone. Here a merge-proof rule should assign player* 1 *(say* Φ'_1*) at least as much as players* 1 *and* 3 *in the original problem, whereas it should assign player* 2 *(say* Φ'_2*) at least as much as players* 2 *and* 3 *in the original problem. Hence, we have* $\Phi_1 + \Phi_3 \leq \Phi'_1$ *and* $\Phi_2 + \Phi_3 \leq \Phi'_2$*, which implies* $\Phi_1 + \Phi_2 + 2\Phi_3 \leq \Phi'_1 + \Phi'_2$*. Furthermore, since* $\Phi_1 + \Phi_2 + \Phi_3 = 1$ *and*

[1] For example, assuming all the costs are different.

$\Phi'_1 + \Phi'_2 = 1$, *we can deduce* $\Phi_3 \leq 0$, *which we know to be impossible because* $\Phi_3 \geq \max\{\Phi_1, \Phi_2\}$ *and* $\Phi_1 + \Phi_2 + \Phi_3 = 1$.

The key issue with this example is that the planner has no way to know whether agent 3 has merged with agent 1 or agent 2. This assumption is necessary in situations where the agents may use multiple replicas without being detected, such as the users of a web page. However, this may not be a reasonable assumption in many other situations. In the $MCST$ model, such examples include departments of a campus or apartments in a building. If the planner knows which mergers may have taken place, it is not difficult to derive a merge-proof rule.[2] On the other hand, it is not clear whether a merge-proof rule could also satisfy Core Selection and Cost Monotonicity.[3]

In this paper, we model the $MCST$ situation in such a way that the planner has some awareness of which mergers take place. If some agents merge and present themselves to the planner in this way, the planner must solve a situation in which each node has an associated weight representing the number of agents that belong to this node. We refer to this new problem as an $MCST$ *situation.* An $MCST$ situation generalizes the classical $MCST$ problem.

Under this model, all the rules presented in the literature fail to satisfy at least one of the properties. However, we propose a new rule that satisfies all of them, including the stronger versions. This rule is the weighted Shapley value of a particular cost game.[4] We also propose characterization for this rule using these properties and the additional properties of *Efficient Merging*, *Piece-wise Additivity*, *Symmetry* and and *Positivity.*

The first of these additional properties, Efficient Merging, relates to the properties of Split- and Merge-proofness. Although Merge-proofness prevents

[2]For example, charging all the cost to the merging agents.

[3]Charging all the cost to the merging agents will clearly not satisfy Core Selection.

[4]In the context of pricing traffic demand in a spanning network, Moulin (2014) also finds the weighted Shapley value of a cooperative game to satisfy so-called routing-proofness. This property is related to split-proofness, preventing agents gaining advantage by claiming to be several different users along a path

agents creating an inefficient network, this concern is no longer relevant when the nodes that merge are already the closest ones (as was the case in Example 1.1). Efficient Merging states that these nodes should experience no harm by merging in advance in these cases. The second, Piece-wise Additivity, is a weaker version of additivity and states that when the same network is optimal for two different cost matrices, cost-sharing is additive in the cost function. Finally, Symmetry states that symmetric nodes should pay the same and Positivity states that each node should pay at least zero.

The remainder of this paper is organized as follows: in Section 2 we present the model, in Section 3 we describe some desirable properties of the rules, in Section 4 we define a new rule, prove that it satisfies all these properties and characterize it with some of them, and in Section 5 we present our conclusions.

2 The model

Let $\mathbb{N}_+$ be the set of positive natural numbers. Let $N \subset \mathbb{N}_+$, usually $N = \{1, \ldots, n\}$, be a finite set of nodes, and let 0 be a special point called the *source*. Let $N_0 = N \cup \{0\}$.

A *minimum cost spanning tree situation*, or simply a *situation*, is a triple (N_0, c, ω) where c is a cost function $c : N_0 \times N_0 \to \mathbb{R}_+$ that assigns a non-negative cost to each pair $(i, j) \in N_0 \times N_0$, and $\omega \in \mathbb{N}_+^N$ reflects the number of agents belonging to each node. We assume $c(i, i) = 0$ for all $i \in N_0$ and $c(i, j) = c(j, i)$ for all $i, j \in N_0$.

With some abuse of notation, given $S \subset N$, we write (S_0, c, ω) instead of (S_0, c_S, ω_S), where $c_S : S_0 \times S_0 \to \mathbb{R}_+$ and $\omega_S \in \mathbb{N}_+^S$ are the restrictions to S of c and ω, respectively.

We denote as $E^{N_0} = \{(i, j) : i, j \in N_0, \, i \neq j\}$ the set of *edges* in N_0. A graph g in N_0 is a subset of E^{N_0}. The *cost* of a graph g in N_0 is defined as $c\,(g) = \sum_{(i,j)\in g} c\,(i, j)$. We denote as G^N the set of graphs in (N_0, c, ω).

A *path* in N_0 is a sequence $(i_0, ..., i_k)$ of different nodes in N_0. In particular, we say that $(i_0, ..., i_k)$ is a *path from* i_0 to i_k. We say that a path $(i_0, ..., i_k)$ is *in* a graph g in N_0 if $(i_{l-1}, i_l) \in g$ for all $l = 1, ..., k$.

A *spanning graph* in N_0 is a graph g in N_0 such that for all $i, j \in N_0$, there exists a path in N_0 from i to j. We denote as SG^{N_0} the set of spanning graphs in N_0.

A *rule* Φ is a function that assigns to each (N_0, c, ω) a vector $\Phi(N_0, c, \omega) \in \mathbb{R}^N$ satisfying $\sum_{i \in N} \Phi_i(N_0, c, \omega) = \min_{g \in SG^{N_0}} c(g)$.

A *spanning tree* in N_0 is a graph t in N_0 such that for all $i, j \in N_0$, there exists a unique path in N_0 from i to j. We denote as ST^{N_0} the set of spanning trees in N_0.

Since $c(i, j) \geq 0$ for all $(i, j) \in E^{N_0}$, it is clear that we can replace SG^{N_0} with ST^{N_0} in the definition of the rule. Such a *minimal cost spanning tree* is called a *minimal tree*, abbreviated to *mt*. A minimal tree always exists but it is not necessarily unique. Let $MT(N_0, c)$ (abbreviated MT^c) be the set of minimal trees in (N_0, c, ω). Let the cost associated with any *mt* on (N_0, c, ω) as $m(N_0, c, \omega)$. Note, however, that $m(N_0, c, \omega)$ does not depend on ω.

For any (N_0, c, ω), a *connected component* is a maximal subset of N_0 where all the nodes can be connected at zero cost. Hence, for any two nodes i, j in the same connected component, there exists a path $(i_0, \ldots, i_k)$ from i to j such that $c(i_{l-1}, i_l) = 0$ for all $l = 1, \ldots, k$. Clearly, the connected components determine a partition $\mathbb{P}$ of N_0 that includes exactly one set P_0 of nodes connected to the source at zero cost. Let us assume $0 \in P_0$ so that $P_0 \neq \emptyset$.

3 Properties of the rules

In this section we describe the properties that we consider a cost-sharing rule should satisfy. Most are well known from the classical model of $MCST$ problems and have been adapted to this new context and supplemented with an additional property. The formal definitions are provided below. Let Φ be a generic rule.

Core Selection: For each $S \subset N$, we have

$$\sum_{i \in S} \Phi_i \left(N_0, c, \omega\right) \leq \min_{t \in ST^{S_0}} c\left(t\right).$$

This property states that no subset of nodes can find it cheaper to create their own network without the others.

Population Monotonicity: For each $i, j \in N$, we have

$$\Phi_i \left(N_0, c, \omega\right) \leq \Phi_i \left(N_0 \setminus \{j\}, c, \omega\right).$$

This property states that if the population of nodes decreases, none is better off. Conversely, if the population of nodes increases, none is worse off.

It is straightforward to check that Population Monotonicity implies Core Selection.

Cost Monotonicity: For each $i \in N$, $\Phi_i(N_0, c, \omega)$ is non-decreasing on $c(i, j)$ for all $j \in N_0 \setminus \{i\}$.

This property states that if a connection cost increases for node i and the rest of the connection costs remain the same, then node i is not better off.

The following property is a stronger version of Cost Monotonicity.

Solidarity: $\Phi(N_0, c, \omega)$ is non-decreasing on $c(i, j)$ for all $i, j \in N_0$.

This property states that if a connection cost increases and the rest of the connection costs remain the same, then no node is better off.

Solidarity clearly implies Cost Monotonicity.

Positivity: $\Phi(N_0, c, \omega)$ only takes non-negative values.

This property states that no node can be compensated by connecting to the source.

The following properties consider the possibility that a group of nodes $S \subset N$ merge in advance to be treated as a single node $s \in S$. The result is a new problem, called a *reduced problem*, defined as follows:

Definition 3.1 *Given a situation* (N_0, c, ω) *and* $s \in S \subset N$, *the* reduced problem (N_0^s, c^s, ω^s) *is defined as* $N_0^s = (N_0 \setminus S) \cup \{s\}$, $c^s(i,j) = c(i,j)$ *for all* $i, j \in N_0 \setminus S$, $c^s(i,s) = \min_{j\in S} c(i,j)$ *for all* $i \in N_0 \setminus S$, $\omega_i^s = \omega_i$ *for all* $i \in N \setminus S$, *and* $\omega_s^s = \sum_{i\in S} \omega_i$.

Merge-proofness: For each $S \subset N$ and $g \in SG^S$, we have

$$\sum_{i\in S} \Phi_i (N_0, c, \omega) \leq \Phi_s (N_0^s, c^s, \omega^s) + c(g).$$

This property states that no group of agents have incentives to merge in advance, assuming cost $c(g)$, to be treated as a single node.

Strong Merge-proofness: For each $S \subset N$, and $i \in N \setminus S$, we have

$$\Phi_i (N_0^s, c^s, \omega^s) \leq \Phi_i (N_0, c, \omega).$$

This property states that if a group of agents located at S merge in advance to be treated as a single node (s), no other node (i) will be worse off in the reduced problem.

Strong Merge-proofness implies Merge-proofness (Gómez-Rúa and Vidal-Puga, 2011).

The following property considers the case in which one particular node splits into several nodes, producing a new situation with additional nodes.

Split-proofness: For each $s \in S \subset N$, we have

$$\Phi_s \left(N_0^s, \left(c^{0S}\right)^s, \omega^s\right) \leq \sum_{i\in S} \Phi_i \left(N_0, c^{0S}, \omega\right)$$

where c^{0S} is a cost function satisfying $c^{0S}(i,j) = 0$ for all $i, j \in S$ and $c^{0S}(i,j) = c(i,s)$ for all $i \in N_0 \setminus S, j \in S$.

This property states that no node (s) has incentives to split into several nodes (S).

Efficient Merging: If there exist two nodes $s, s' \in N$ such that $c(s, s') = \min_{i,j \in N_0} c(i, j)$, then

$$\Phi_s(N_0^s, c^s, \omega^s) + c(s, s') \leq \Phi_s(N_0, c, \omega) + \Phi_{s'}(N_0, c, \omega)$$

where $S = \{s, s'\}$.

This property states that if the closest nodes (s and s') are formed by agents, then they should find it optimal to merge, avoiding disincentives for the creation of an optimal network.

Proposition 3.1 shows that Strong Merge-proofness and Efficient Merging imply Split-proofness.

Proposition 3.1 *If a rule satisfies Strong Merge-proofness and Efficient Merging, then it also satisfies Split-proofness.*

Proof. Let $s \in S \subset N$. We must prove, under Strong Merge-proofness and Efficient Merging, that $\Phi_s\left(N_0^s, \left(c^{0S}\right)^s, \omega^s\right) \leq \sum_{i \in S} \Phi_i\left(N_0, c^{0S}, \omega\right)$. Let us proceed by induction on $|S|$, the cardinality of S. For $S = \{s\}$, the result is trivial. Suppose now the result holds for $|S| < \alpha$ with $\alpha > 1$, and assume $|S| = \alpha$. Let $s' \in S \setminus \{s\}$ and $S' = S \setminus \{s\}$. Taking, with some abuse of notation, $S = \{s, s'\}$, it is clear that $\Phi_s\left(N_0^s, \left(c^{0S}\right)^s, \omega^s\right) = \Phi_s\left(\left(N_0^{s'}\right)^s, \left(c^{0S}\right)^s, \left(\omega^{s'}\right)^s\right)$. Under Efficient Merging, this is less than or equal to $\Phi_s\left(N_0^{s'}, c^{0S}, \omega^{s'}\right) + \Phi_{s'}\left(N_0^{s'}, c^{0S}, \omega^{s'}\right)$. Under Strong Merge-proofness, we have $\Phi_s\left(N_0^{s'}, c^{0S}, \omega^{s'}\right) \leq \Phi_s\left(N_0, c^{0S}, \omega\right)$. Hence, it suffices to prove $\Phi_{s'}\left(N_0^{s'}, c^{0S}, \omega^{s'}\right) \leq \sum_{i \in S'} \Phi_i\left(N_0, c^{0S}, \omega\right)$. It is straightforward to check that $\left(N_0^{s'}, c^{0S}, \omega^{s'}\right) = \left(N_0^{s'}, \left(c^{0S}\right)^{s'}, \omega^{s'}\right)$. Under the induction hypothesis, $\Phi_{s'}\left(N_0^{s'}, \left(c^{0S}\right)^{s'}, \omega^{s'}\right) \leq \sum_{i \in S'} \Phi_i\left(N_0, c^{0S}, \omega\right)$. ■

For the next property, given an order $\sigma : \{1, \ldots, |E^{N_0}|\} \longrightarrow E^{N_0}$, let $\mathsf{C}_\sigma = \left\{x \in \mathbb{R}_+^{E^{N_0}} : 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(|E^{N_0}|)}\right\}$ as the cone in $\mathbb{R}^{E^{N_0}}$ such that the ordering of the coordinates is given by σ. A real value function F with domain $\mathbb{R}_+^{E^{N_0}}$ is *piece-wise additive* if for any σ its restriction to C_σ is additive, i.e. $F(x + y) = F(x) + F(y)$ for all $x, y \in \mathsf{C}_\sigma$.

Piece-wise Additivity: Φ is piece-wise additive as a function with domain $\mathbb{R}_+^{E^{N_0}}$. Namely,

$$\Phi(N_0, c + c', \omega) = \Phi(N_0, c, \omega) + \Phi(N_0, c', \omega)$$

for all $c, c' \in \mathsf{C}_\sigma$.

This property provides a vector structure for $\Phi(N_0, c, \omega)$. The main advantage of a piece-wise additive cost sharing rule is that it is entirely determined by its value over the $\left|E^{N_0}\right|$-coordinate vectors whose coordinates take exactly two values, one of them positive and the other zero (compare page 302 in Hougaard et al. (2010)).

The last property defines symmetric nodes. Two nodes $i, j \in N$ are said to be *symmetric* in (N_0, c, ω) if $\omega_i = \omega_j$ and $c(i, k) = c(j, k)$ for all $k \in N_0 \setminus \{i, j\}$.

Symmetry: Symmetric nodes pay the same cost. Namely,

$$\Phi_i(N_0, c, \omega) = \Phi_j(N_0, c, \omega)$$

for all $i, j \in N$ symmetric nodes in (N_0, c, ω).

4 A monotonic and merge-proof rule

Definition of the rule

For each (N_0, c, ω), let us consider the transfer utility (TU) cost game (N, v^+) defined as $v^+(S) = m(S_0, c^+)$ for all $S \subset N$, where $c^+(i, j) = c(i, j)$ for all $i, j \in S$, and $c^+(i, 0) = \min_{j \in N_0 \setminus S} c(i, j)$ for all $i \in S$.

This TU cost game was first defined by Bergantiños and Vidal-Puga (2007b). It follows an "optimistic" interpretation of the worth of a coalition of players, since it assumes the rest of the players are already connected and it is possible to reach the to the source through them.

We define $\Psi(N_0, c, \omega)$ as the weighted Shapley value of (N, v^+) (Kalai and Samet, 1987) with weights given by ω. Hence,

$$\Psi(N_0, c, \omega) = Sh^\omega\left(N, v^+\right).$$

We also provide two additional interpretations of Ψ. As a weighted Shapley rule, not only does Ψ satisfy Piece-wise Additivity (Bergantiños et al., 2010), it also satisfies a stronger version of this property: Ψ is piece-wise linear as a function with domain $\mathbb{R}^{E^{N_0}}$. This allows us to characterize Ψ by providing its value for any c with $c(i,j) \in \{0,1\}$ for all i, j, as shown in Proposition 4.1.

Proposition 4.1 *Let c^{01} be a cost function with $c^{01}(i,j) \in \{0,1\}$ for all $i, j \in N$. Then,*

$$\Psi_i\left(N_0, c^{01}, \omega\right) = \begin{cases} 0 & \text{if } i \in P_0 \\ \frac{\omega_i}{\sum_{j \in R} \omega_j} & \text{if } i \in R \in \mathbb{P} \setminus \{P_0\} \end{cases}$$

for all $i \in N$.

Proof. For each (N_0, c^{01}, ω), the total cost is $m\left(N_0, c^{01}, \omega\right) = |\mathbb{P}| - 1$, where $\mathbb{P}$ is the partition of N_0 in connected components. It is clear that the TU cost game (N, v^+) associated with c^{01} is given by $v^+ = \sum_{R \in \mathbb{P} \setminus \{P_0\}} u_R$, where u_R is the unanimity game with carrier R. In other words, $u_T(S) = 1$ if $T \subset S$ and $u_T(S) = 0$ otherwise. Moreover, by definition of the weighted Shapley value (Kalai and Samet, 1987),

$$Sh_i^\omega(N, u_R) = \begin{cases} 0 & \text{if } i \notin R \\ \frac{\omega_i}{\sum_{j \in R} \omega_j} & \text{if } i \in R \end{cases}$$

for all $i \in N$. Since $\mathbb{P}$ is a partition of N_0, by additivity of Sh^ω,

$$Sh_i^\omega\left(N, v^+\right) = \begin{cases} 0 & \text{if } i \in P_0 \\ \frac{\omega_i}{\sum_{j \in R} \omega_j} & \text{if } i \in R \in \mathbb{P} \setminus \{P_0\} \end{cases}$$

for all $i \in N$. ■

For the second interpretation, we need some additional notation. Given any $t \in MT^c$ and $i, j \in N_0$, let $\overline{c}(i,j)$ denote the maximum cost in the (unique) path from i to j in t. In particular, $\overline{c}(i,j) = 0$ for all $i \in N_0$. This cost function $\overline{c}$ determines the *irreducible matrix* first defined by Bird (1976) in the context of $MCST$ problems. Even though the path depends on t, it is possible to show that $\overline{c}(i,j)$ is independent of the chosen t (Aarts and Driessen, 1993).

Let Π_0^ω denote the set of orderings of nodes in N_0 with 0 as the first element, such that for each $\pi \in \Pi_0^\omega$, node $i \in N$ appears ω_i times. Hence:

$$\Pi_0^\omega = \left\{\pi : \Omega \longrightarrow N_0 : \pi(1) = 0 \text{ and } \left|\pi^{-1}(i)\right| = \omega_i \text{ for all } i \in N\right\}$$

where $\Omega = \left\{0, 1, \ldots, \sum_{i \in N} \omega_i\right\}$ and $\pi^{-1}(i) = \{l \in \Omega : \pi(l) = i\}$.

Given $\pi \in \Pi_0^\omega$, we define $\Psi^\pi(N_0, c, \omega) \in \mathbb{R}^N$ inductively as follows:

$$\Psi^\pi_{\pi(l)}(N_0, c, \omega) = \min_{l'=1,\ldots,l-1} \overline{c}\left(\pi(l'), \pi(l)\right)$$

for all $l = 1, \ldots, \sum_{i \in N} \omega_i$ such that $\pi(l) \neq \pi(l')$ for all $l' < l$.

Then, Ψ is the average of these vectors.

Proposition 4.2 *For each situation* (N_0, c, ω)*,*

$$\Psi(N_0, c, \omega) = \frac{1}{|\Pi_0^\omega|} \sum_{\pi \in \Pi_0^\omega} \Psi^\pi(N_0, c, \omega). \tag{1}$$

Proof. Let c^{01} be a cost function with $c^{01}(i,j) \in \{0,1\}$ for all i, j. Since the weighted Shapley value is linear in the characteristic function, it suffices to prove that Ψ can be written as in (1) for every such c^{01}. Given $i = \pi(l) \in R \in \mathbb{P}$, it is clear that $\Psi_i^\pi(N_0, c^{01}, \omega) = 0$ if there exists some $l' < l$ such that $\pi(l') \in R$, and $\Psi_i^\pi(N_0, c^{01}, \omega) = 1$ otherwise. In particular, $\Psi_i^\pi(N_0, c^{01}, \omega) = 0$ if $i \in P_0$, since in this case, $R = P_0$ and $\pi(0) = 0$, and hence the result follows from Proposition 4.1. Now assume $R \neq P_0$. Let $\Pi^l \subset \Pi_0^\omega$ be the set of orderings in which node i is the first node in R. Then,

$$\frac{1}{|\Pi_0^\omega|} \sum_{\pi \in \Pi_0^\omega} \Psi_i^\pi(N_0, c, \omega) = \frac{1}{|\Pi_0^\omega|} \left|\Pi^l\right| = \frac{\omega_i}{\sum_{j \in R} \omega_j}.$$

and the result follows from Proposition 4.1. ■

Main characterization

Let us now prove Ψ satisfies all the relevant properties (Theorem 4.1 and Proposition 4.3) and is characterized by them (Theorem 4.2).

Theorem 4.1 Ψ *satisfies Core Selection, Population Monotonicity, Cost Monotonicity, Solidarity, Merge-proofness, Strong Merge-proofness, Split-proofness, Efficient Merging, Piece-wise Additivity, and Symmetry.*

Proof. We have already seen that Ψ satisfies Piece-wise Additivity. Moreover, it follows from its definition that it also satisfies Symmetry. On the other hand, the weighted Shapley values of the cost game (N, v^+) are obligation rules (Bergantiños and Lorenzo-Freire, 2008) and obligation rules satisfy Population Monotonicity and Solidarity (Bergantiños and Kar, 2010). Since Population Monotonicity implies Core Selection, and Solidarity implies Cost Monotonicity, Ψ satisfies those properties also.

Let us now check that Ψ satisfies (Strong) Merge-proofness. Note that the maximum cost of a path between any pair of nodes cannot increase when some other nodes merge. Hence, for each $\pi \in \Pi_0^\omega$, Ψ^π satisfies Strong Merge-proofness. Since Ψ is a weighted average of these vectors, and the weights aggregate when two or more nodes merge, we deduce that Ψ also satisfies Strong Merge-proofness. Since Strong Merge-proofness implies Merge-proofness, Ψ also satisfies this property.

Let us also check that Ψ satisfies Efficient Merging. Let $s, s' \in N$ be one of the closest pairs of nodes. Then, for each $\pi \in \Pi_0^\omega$ satisfying $\min\{l \in \Omega : \pi(l) = s\} < \min\{l \in \Omega : \pi(l) = s'\}$, we have $\Psi_s^\pi(N_0, c, \omega) = \Psi_s^\pi(N_0^s, c^s, \omega^s)$ and $\Psi_{s'}^\pi(N_0, c, \omega) = c(i, j)$. Let Π be the subset of these orderings. Analogously, let $\Pi' = \Pi_0^\omega \setminus \Pi$, so that $\Pi \cap \Pi' = \emptyset$ and $\Pi \cup \Pi' = \Pi_0^\omega$.

Hence,

$$\begin{aligned}
\Psi_s\left(N_0, c, \omega\right) &= \frac{1}{\left|\Pi_0^\omega\right|}\sum_{\pi\in\Pi_0^\omega}\Psi_s^\pi\left(N_0, c, \omega\right) \\
&= \frac{1}{\left|\Pi_0^\omega\right|}\left(\sum_{\pi\in\Pi}\Psi_s^\pi\left(N_0, c, \omega\right) + \sum_{\pi\in\Pi'}\Psi_s^\pi\left(N_0, c, \omega\right)\right) \\
&= \frac{1}{\left|\Pi_0^\omega\right|}\left(\sum_{\pi\in\Pi}\Psi_s^\pi\left(N_0^s, c^s, \omega^s\right) + \sum_{\pi\in\Pi'}c\left(s, s'\right)\right)
\end{aligned}$$

analogously,

$$\Psi_{s'}\left(N_0, c, \omega\right) = \frac{1}{\left|\Pi_0^\omega\right|}\left(\sum_{\pi\in\Pi'}\Psi_s^\pi\left(N_0^s, c^s, \omega^s\right) + \sum_{\pi\in\Pi}c\left(s, s'\right)\right)$$

so that $\Psi_s\left(N_0, c, \omega\right) + \Psi_{s'}\left(N_0, c, \omega\right) = \Psi_s\left(N_0^s, c^s, \omega^s\right) + c\left(s, s'\right)$. The "greater or equal" part of this equality constitutes the proof of Efficient Merging.

From Proposition 3.1, we can deduce that Ψ satisfies Split-proofness. ■

Moreover, Ψ also satisfies Positivity, as deduced from the next result:

Proposition 4.3 *Solidarity, Strong Merge-proofness, Efficient Merging and Symmetry imply Positivity.*

Proof. Define $\left(N_0^*, c^*, \omega^*\right)$ as follows: $N^* = \bigcup_{i\in N} S^i$ with $\left|S^i\right| = \omega_i$ for all $i \in N$, and $S^i \cap S^j = \emptyset$ for all $i, j \in N$; $c^*\left(s^i, s^j\right) = c(i, j)$ for all $s^i \in S^i$, $s^j \in S^j$ with $i \neq j$, $c^*\left(s, s'\right) = 0$ for all $s, s' \in S^i$, and $c^*\left(s, 0\right) = c(i, 0)$ for all $s \in S^i$; and $\omega_s^* = 1$ for all $s \in N^*$. From Strong Merge-proofness and Efficient Merging, it is straightforward to check that $\Psi_i\left(N_0, c, \omega\right) = \sum_{s\in S^i}\Psi_s\left(N_0^*, c^*, \omega^*\right)$ for all $i \in N$. Under Solidarity, $\Psi\left(N_0^*, c^*, \omega^*\right) \geq \Psi\left(N_0^*, c^0, \omega^*\right)$, where $c^0\left(s, s'\right) = 0$ for all $s, s' \in N_0^*$. Under Symmetry, $\Psi\left(N_0^*, c^0, \omega^*\right) = (0, \ldots, 0)$ and hence $\Psi\left(N_0, c, \omega\right) \geq (0, \ldots, 0)$. ■

We now present our main result:

Theorem 4.2 Ψ *is the only rule that satisfies Population Monotonicity, Solidarity, Strong Merge-proofness, Efficient Merging, Piece-wise Additivity, and Symmetry.*

Proof. We already know that Ψ satisfies all these properties. Let Φ be a rule that satisfies them. Under Proposition 4.3, Φ also satisfies Positivity. We will prove that Φ is unique for each (N_0, c, ω). We proceed by induction on the number of nodes. If $|N| = 1$, the result is trivial. Now let us assume the result is true when there are less than $|N|$ nodes.

Under Strong Merge-proofness and Efficient Merging, and by using the same reasoning of the proof of Proposition 4.3, it suffices to prove the result assuming $\omega_i = 1$ for all $i \in N$.

Under Piece-wise Additivity, it suffices to prove the result assuming that c only takes two values: 0 and some $x \in \mathbb{R}_+$ since every (N_0, c, ω) can be expressed as the sum of these situations, all of them in the same cone C_σ for some σ satisfying $c(\sigma(l)) \leq c(\sigma(l'))$ iff $l \leq l'$.

Under Population Monotonicity and the induction hypothesis, we can assume there exists a spanning graph g in N such that $c(i,j) = 0$ for all $(i,j) \in g$. Suppose, on the contrary, there exist two groups of nodes $S, T \subset N$ such that $S \cup T = N$ and $c(i,j) = x$ for all $i \in S$ and $j \in T$. Population Monotonicity implies that $\Phi_i(S_0, c, \omega) \geq \Phi_i(N_0, c, \omega)$ for all $i \in S$ and $\Phi_i(T_0, c, \omega) \geq \Phi_i(N_0, c, \omega)$ for all $i \in T$. Moreover, it is straightforward to check that $m(N_0, c, \omega) = m(S_0, c, \omega) + m(T_0, c, \omega)$. Hence, given $i \in S$ (the case $i \in T$ is analogous),

$$\begin{aligned}
\Phi_i(N_0, c, \omega) &= m(N_0, c, \omega) - \sum_{j \in S \setminus \{i\}} \Phi_j(N_0, c, \omega) - \sum_{j \in T} \Phi_j(N_0, c, \omega) \\
&\geq m(S_0, c, \omega) + m(T_0, c, \omega) - \sum_{j \in S \setminus \{i\}} \Phi_j(S_0, c, \omega) - \sum_{j \in T} \Phi_j(T_0, c, \omega) \\
&= \Phi_i(S_0, c, \omega) \geq \Phi_i(N_0, c, \omega)
\end{aligned}$$

and so $\Phi_i(N_0, c, \omega) = \Phi_i(S_0, c, \omega)$, which is unique by induction hypothesis.

Under Positivity, it suffices to prove the result assuming $c(i, 0) = x$ for all $i \in N$. Suppose, on the contrary, $c(i, 0) = 0$ for some $i \in N$. Hence, $m(N_0, c, \omega) = 0$ since $g \cup \{(i, 0)\}$ is a spanning graph with cost 0. Under Positivity, $\Phi(N_0, c, \omega) \geq (0, \ldots, 0)$ but since $\sum_{j \in N} \Phi_j(N_0, c, \omega) = m(N_0, c, \omega) = 0$, we can conclude that $\Phi(N_0, c, \omega) = (0, \ldots, 0)$.

Clearly, under these assumptions we have $m(N_0, c, \omega) = x$. Assume w.l.o.g. $N = \{1, \ldots, n\}$. Let c^{0x} be the cost function defined as $c^{0x}(i,0) = x$ for all $i \in N$ and $c^{0x}(i,j) = 0$ otherwise. For each $i \in N$, let $S \subset N$ be the set of nodes whose (unique) path to the source in g uses node i (including node i itself), and let $T = N \setminus S$. Both sets S and T can be connected at zero cost. Hence, under Efficient Merging and Strong Merge-proofness, and given $s \in S$ and $s' \in T$, we have

$$\sum_{j \in S} \Phi_j (N_0, c, \omega) = \Phi_s (T_0, c, \omega) = \Phi_s \left(T_0, c^{0x}, \omega\right) = \sum_{j \in S} \Phi_j \left(N_0, c^{0x}, \omega\right)$$

where $T = \{s, s'\}$. Under Symmetry, $\Phi_i (N_0, c^{0x}, \omega) = \frac{x}{n}$ for all $i \in N$. Hence, $\sum_{j \in S} \Phi_j(N_0, c, \omega) = \frac{x|S|}{n}$. We can now proceed by induction on $|S|$ in order to prove that $\Phi(N_0, c, \omega) = \left(\frac{x}{n}, \ldots, \frac{x}{n}\right)$. For $|S| = 1$, we have $S = \{i\}$ and hence $\Phi_i (N_0, c, \omega) = \frac{x}{n}$. Assume now $\Phi_i (N_0, c, \omega) = \frac{x}{n}$ when $|S| < \alpha$ and suppose $|S| = \alpha$. Then,

$$\Phi_i (N_0, c, \omega) = \sum_{j \in S} \Phi_j (N_0, c, \omega) - \sum_{j \in S \setminus \{i\}} \Phi_j (N_0, c, \omega) = \frac{x|S|}{n} - \sum_{j \in S \setminus \{i\}} \frac{x}{n} = \frac{x}{n}.$$

■

Notice, from the proof of Theorem 4.2, that we can replace Solidarity by Positivity in the characterization result. In either case, the properties are independent, as we shall now show.

Independence of the properties

We present six reasonable rules, each of them satisfying all the properties used in Theorem 4.2 except one.

The folk rule satisfies all the properties except Strong Merge-proofness.

Given $\pi \in \Pi_0^\omega$, Ψ^π satisfies all the properties except Symmetry.

Let F^e be defined as

$$F_i^e(N_0, c, \omega) = \left(\overline{c}(i,0) - \sum_{S \ni i : 0 \notin S \subset N, \delta_S > 0} (1 - e_i(\overline{c}, S)) \, \delta_S \right) \omega_i$$

for all $i \in N$, where $\delta_S = \min_{i\in S, j\in N_0\setminus S} \overline{c}(i,j) - \max_{i,j\in S} \overline{c}(i,j)$ determines the extra cost agents in S should face after they get connected, and e is the normalized extra cost function that assigns each irreducible cost function $\overline{c}$ a vector in the simplex Δ^S. See Bergantiños and Vidal-Puga (2015) for a detailed interpretation of these terms.

Let $M(\overline{c}, S) = \{i \in S : \overline{c}(i,j) \leq \overline{c}(k,j) \text{ for all } j,k \in S\}$ be the set of nodes that are closer under $\overline{c}$ within S. When e is defined as

$$e_i(\overline{c}, S) = \begin{cases} \frac{1}{|S|+1} + \frac{1}{(|S|+1)(|M(\overline{c},S)|)} & \text{if } i \in M(\overline{c}, S) \\ \frac{1}{|S|+1} & \text{otherwise} \end{cases}$$

then F^e satisfies all the properties except Piece-wise Additivity.

The following rules are all piece-wise linear, and hence it is enough to define them for any c^{01} with $c^{01}(i,j) \in \{0,1\}$ for all i,j (as in (4.1)).

Given $\alpha \in (0,1)$, let Φ^α be defined by

$$\Phi_i^\alpha\left(N_0, c^{01}, \omega\right) = \begin{cases} 0 & \text{if } P_0 = N_0 \\ \frac{\alpha\omega_i}{\sum_{j\in R\setminus\{0\}} \omega_j} & \text{if } R = P_0 \neq N_0 \\ \frac{\omega_i}{\sum_{j\in R} \omega_j} & \text{if } R \neq P_0 = \{0\} \\ \frac{\omega_i}{\sum_{j\in R} \omega_j} - \frac{\alpha\omega_i}{\sum_{j\in N_0\setminus P_0} \omega_j} & \text{if } R \neq P_0 \neq \{0\} \end{cases}$$

for all $i \in R \in \mathbb{P}$. This rule satisfies all the properties except Population Monotonicity.

For any (N_0, c^{01}, ω) and $i \in R \in \mathbb{P}$, let $\Lambda_i \subset N$ be the set of nodes $j \in N$ such that there exists a path between i and j with zero cost. Let $\lambda_i = |\{j \in \Lambda_i : c^{01}(j,0) = 0\}|$ be the number of nodes in Λ_i with zero cost to the source. It is not difficult to check that $R = P_0$ if and only if $\lambda_i > 0$.

Let Φ^λ be defined by

$$\Phi_i^\lambda\left(N_0, c^{01}, \omega\right) = \begin{cases} \frac{\omega_i}{\sum_{j\in\Lambda_i} \omega_j} & \text{if } \lambda_i = 0 \\ \frac{\omega_i}{\sum_{j\in\Lambda_i} \omega_j - 1} & \text{if } \lambda_i = 1 \text{ and } c^{01}(i,0) = 1 \\ \frac{\omega_i - 1}{\sum_{j\in\Lambda_i} \omega_j - 1} - 1 & \text{if } \lambda_i = 1 \text{ and } c^{01}(i,0) = 0 \\ 0 & \text{if } \lambda_i > 1 \end{cases}$$

for all $i \in N$. This rule satisfies all the properties except Solidarity.

Let Φ^2 be defined by

$$\Phi_i^2\left(N_0, c^{01}, \omega\right) = \begin{cases} 0 & \text{if } i \in P_0 \\ \frac{\omega_i^2}{\sum_{j\in\Lambda_i}\omega_j^2} & \text{if } i \notin P_0 \end{cases}$$

for all $i \in N$. This rule satisfies all the properties but Efficient Merging.

5 Conclusions

In the classical model of $MCST$ problems, it is assumed that the planner cannot distinguish the agents belonging to each node. In many situations, such as those in which the agents are represented by geographical points, it seems reasonable that the planner can identify how many agents may belong to the same node. This assumption is extremely reasonable in the specific case of $MCST$ problems, where it is common knowledge that all the nodes want to be connected to the source. Based on these considerations, we studied three classes of properties for a rule to satisfy. These classes are related to Core Selection, Cost Monotonicity and Split- and Merge-proofness, respectively. While there is no rule satisfying Merge-proofness in the classical model, here we propose a rule that satisfies the three classes of properties. We also provide a characterization using Symmetry, Piece-wise Additivity and several variations of Core Selection, Cost Monotonicity and Split- and Merge-proofness. Symmetry is an important property from the point of view of equality. However, we do not claim that Piece-wise Additivity is an essential property, merely that it provides a linear structure to the solution and, as such, allows us to pick a single reasonable rule.

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