Bargaining and membership

María Gómez-Rúa* Juan Vidal-Puga
University of Vigo University of Vigo

January 24, 2013

Abstract

In coalitional games in which the players are partitioned into groups, we study the incentives of the members of a group to leave it and become singletons. In this context, we model a non-cooperative mechanism in which each player has to decide whether to stay in her group or to exit and act as a singleton. We show that players, acting myopically, always reach a Nash equilibrium.

Keywords: Bargaining, coalitional games, coalition structure, Owen value, Nash equilibrium

1 Introduction

Endogenous formation of coalitions has been widely studied in the game theory literature. For example, Chatterjee et al. [14] and Okada [26] study coalition formation models in which players can agree on payoff division at the time they form a coalition.

*Corresponding author: Facultade de CC. Económicas e Empresariais. Campus Lagoas-Marcosende, s/n. Universidade de Vigo. Vigo (Pontevedra)- SPAIN. Tel: +34986813506; fax: +34986812401. E-mail: mariarua@uvigo.es

In these models, the coalitions are formed along with the final payoff of their members. An alternative approach is to assume that the final payoff is given by the coalition structure. For example, Hart and Kurz [21] and Bloch [8] present models of endogenous formation of coalitions in two stages: in the first stage, players decide the coalition structure. In the second stage, the final payoff is given according to the chosen coalition structure. In Hart and Kurz's model, the final payoff is given by the Owen value (Owen [28]). A similar model is given by Aumann and Myerson [5], where players decide how to connect through a graph, and the final payoff is given by the Myerson value (Myerson [25]) depending on the particular graph.

On the other hand, there are many situations in which the coalition structure is given *a priori*. For example, consider the members of a Parliament. Even though all have the same rights, they do not act independently, since they belong to different political parties. Other examples include wage bargaining between firms and labor unions, tariff bargaining between countries, and bargaining between the member states of a federated country. Broadly speaking, these coalitions negotiate among them as single agents. The fundamental feature is that the coalition structure is exogenously given by the problem, which means that players do not choose which coalition they belong to.

In this paper, we take an intermediate approach between the endogenous and the exogenous coalition structure models. We assume that there exists a prior coalition structure (exogenous), but players inside *a priori* union may have the chance to act as singletons (endogenous). For example, consider the parties with representation in the European Parliament. Some of these parties may decide, prior to the discussion of an issue, to collude and defend a common policy. By doing so, they join forces and act as a single party.

Usually, this cooperation is useful because the colluded party is stronger than its individual parties. It may happen, however, that this cooperation is not beneficial, as the "joint-bargaining paradox" of Harsanyi [20] shows. The paradox is that an individual can be worse off bargaining as a member of a coalition than bargaining alone. Chae and Heidhues [12, p. 47] justify this paradox as follows: *Treatening a group as a single bargainer reduces multiple "rights to talk" to a single right and thereby benefits the outsiders.* See also Chae and Moulin [13] and Vidal-Puga [36] for a study of the Harsanyi paradox from an axiomatic and a cooperative point of view, respectively.

Supranational parties such like the EPP-ED[1] or the Socialist Group usually do not act as single agents, because its members are not committed to follow the same policies on the same issues. Instead, these supranational associations provide a common working environment is which cooperation agreements are easier to settle, but only if they are beneficial for everyone.

In this framework, we define a mechanism[2] in two stages: in the first stage, players simultaneously announce whether they stay or exit their coalition. The decision to stay is interpreted as the agreement to act as a single player in the second stage. The players who decide to leave their coalition act as singletons. Thus, a new coalitional structure derives from players' decisions. In the second stage, the final payoff is given by the Owen value.

A similar mechanism is presented by Thoron [32] based on a model defined by d'Aspremont et al. [4] in the context of cartel formation in oligopolist markets. In those papers, however, firms are identical (only the cartel membership can distinguish them) an the total worth to the shared depends on the actual cartel size. As opposed, the model presented in this paper allows for all the player heterogeneity that a coalitional game can provide. Furthermore, the *total* worth to be shared, as given by the Owen value, is always efficient and independent of the actual coalition structure.

A different approach to coalitional games is consider in Arin et al. [3], where a noncooperative allocation procedure for coalitional games with veto player is studied.

In games with coalition structure, the Owen value is a relevant solution concept. It has been supported axiomatically (Owen [28], Hart and Kurz

[1]European People's Party (Christian Democrats) and European Democrats.

[2]We use the term *mechanism* instead of *non-cooperative game* to avoid confusion with coalitional games.

[21, 22], Winter [40], Calvo et al. [9], Hamiache [17, 18], Peleg and Sudhölter [30], Albizuri and Zarzuelo [2], Gómez-Rúa and Vidal-Puga [16] and also non-cooperatively (Vidal-Puga and Bergantiños [37]). It has been applied to cost allocation problems (Vázquez-Brage et al. [34], Fragnelli and Iandolino [15]), political situations (Carreras and Owen ([10, 11], Vázquez-Brage et al. [33], Ono and Muto [27]), and differential information economies (Krasa et al. [23]). Moreover, it has been successfully generalized to several levels of cooperation (Winter [38]), games without transfer utility (Winter [39], Bergantiños and Vidal-Puga [7], Bergantiños et al. [6]), generalized coalition configurations (Albizuri et al. [1]) and generalized characteristic functions (Sánchez and Bergantiños [31]). In Vidal-Puga [35] it is also shown that the Owen value arises in equilibrium of a mechanism that models the bargaining among heterogeneous groups.

Hence, it seems justifiable to assume that, once the coalition structure is formed, the final payoff is given by the Owen value. Notice that this assumption is also made by Hart and Kurz [21].

In Sections 2 and 3 we present the notation and the model of coalition formation. We are interested in finding the stability of the resulting coalition structure. We focus on the incentives of each player to stay or leave her group. These incentives are given by the difference between what they get by changing their strategies and what they get by not doing so. In Section 3, we show that these differences are independent of the order in which players move. As a consequence, there are no cycles. Players, acting myopically, can reach a Nash equilibrium. In Section 4, we study a possible generalization of the model. In Section 5, we present some concluding remarks.

2 Preliminaries

We consider a *coalitional game* as a pair (N, v) with a finite set of *players* $N = \{1, 2, ..., n\}$ and a *characteristic function* $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Following usual practice, we often refer to "the game v" instead of "the coalitional game (N,v)".

Given two games v, w, let $v+w$ define the game $(v+w)(S)=v(S)+w(S)$ for all $S\subseteq N$.

Given a scalar α and a game v, let αv define the game $(\alpha v)(S)=\alpha v(S)$ for all $S\subseteq N$.

Given a nonempty *coalition* $T\subseteq N$, we define the *unanimity game* (N,u_T) with *carrier* T as the coalitional game given by

$$u_T(S)=\begin{cases}1 & \text{if } T\subseteq S\\ 0 & \text{otherwise.}\end{cases}$$

According to Harsanyi [19], unanimity games form a basis for the space of coalitional games, i.e. $v=\sum_{\emptyset\neq T\subseteq N}\lambda_T(v)\,u_T$ where the *Harsanyi dividends* $\lambda_T(v)$ are given by $\lambda_T(v)=\sum_{S\subseteq T}(-1)^{|T|-|S|}v(S)$ for all $T\subseteq N$.

A *coalition structure* over N is a partition $P=\{S_1,...,S_p\}$ on the set of players N. The *quotient game* of v over P is the coalitional game $(P,v/P)$ defined as $(v/P)(B)=\sum_{S_q\in B}v(S_q)$for all $B\subseteq P$. Thus, v/P is the game played by the coalitions in P.

We denote the set of all games (N,v,P) over N with coalition structure as $CG(N)$.

A *value* is a function $\Psi:CG(N)\to\mathbb{R}^N$ that assigns to each cooperative game with coalition structure (N,v,P) a vector in $\mathbb{R}^N$, so that $\Psi_i(N,v,P)$ represents the payoff assigned to player $i\in N$. With a slight abuse of notation, we say that $\Psi_i(N,v,P)$ is the *value* of player i.

Let Π be the set of permutations of the elements of N. We say that $\pi\in\Pi$ is *compatible* with P if the members of the same coalition keep together. We denote the set of all permutations compatible with P as $\Pi_P\subseteq\Pi$. Namely, $\pi\in\Pi_P$ if and only if it satisfies:

$$\forall i,j\in S_q\in P,\forall k\in N\qquad \pi(i)<\pi(k)<\pi(j)\Longrightarrow k\in S_q.$$

Given $\pi\in\Pi$, the set of *predecessors* of i with respect to π is defined as

$$Pr(i,\pi):=\{j\in N:\pi(j)<\pi(i)\}.$$

The *Owen* (*coalitional*) *value* (Owen [28]) is defined as follows:

$$\Phi_i(N,v,P) := \frac{1}{|\Pi_P|}\sum_{\pi\in\Pi_P}\left[v(Pr(i,\pi)\cup\{i\}) - v(Pr(i,\pi))\right].$$

When the game is clear, we use $\Phi(P)$ instead of the more cumbersome $\Phi(N,v,P)$.

We consider the Owen value as a solution of the game. A characterization of the Owen value is as follows. The Owen value is the only function $\Phi : CG(N) \to \mathbb{R}^N$ satisfying the following axioms:

1. Efficiency: $\sum_{i\in N}\Phi_i(P) = v(N)$ for each $(N,v,P)\in CG(N)$.

2. Symmetry in each union:

   $$v(S\cup\{i\}) = v(S\cup\{j\}), \forall S\subseteq N\backslash\{i,j\} \Longrightarrow \Phi_i(P) = \Phi_j(P)$$

   for all $i,j\in S_q\in P$.

3. Symmetry in the quotient game:

   $$(v/P)(B\cup\{S_q\}) = (v/P)(B\cup\{S_r\}), \forall B\subseteq P\backslash\{S_q,S_r\} \Longrightarrow \sum_{i\in S_q}\Phi_i(P) = \sum_{i\in S_r}\Phi_i(P)$$

   for all $S_q, S_r\in P$.

4. Null player:

   $$v(S\cup\{i\}) = v(S), \forall S\subseteq N\backslash\{i\} \Longrightarrow \Phi_i(P) = 0$$

   for all $i\in N$.

5. Additivity:

   $$\Phi(N,v+w,P) = \Phi(N,v,P) + \Phi(N,w,P)$$

   for all $(N,v,P),(N,w,P)\in CG(N)$.

Given a unanimity game u_T with carrier $T\subseteq N$, Property 4 implies that $\Phi_i(P) = 0$ for all $i\notin T$.

3 The model

Let (N, v, P) be a game with coalition structure. Fix $S_q \in P$. We consider the following mechanism in two stages for players in S_q:

First stage Simultaneously, each player in S_q announces whether she wants to stay or to exit the coalition. Given the announcements of each player, a coalition structure is formed. The players who announced to exit act as singletons.

Second stage Each player receives her Owen value.

Thus, the set of strategies for each player is $\{in, out\}$, where '*in*' means "to stay" and '*out*' means "to exit". We work only with pure strategies. Let $\gamma(i) \in \{in, out\}$ be the strategy of player i. Let $\gamma = (\gamma(i))_{i \in S_q}$ be a strategy profile. We denote the resulting coalition structure as P_γ, namely

$$P_\gamma := \left\{ \{i\}_{i \in S_q : \gamma(i) = in} \right\} \cup \{\{i\}\}_{i \in S_q : \gamma(i) = out} \cup \{S_r\}_{r \neq q}.$$

In particular, if $\gamma(i) = in$ for all $i \in S_q$, then $P_\gamma = P$.

The final payoff for the players is given by the Owen value under this coalition structure $\Phi(P_\gamma)$.

Example 1 *Let*[3] $P = \{123|45|6\}$ *and* $S_q = \{1, 2, 3\}$. *Assume* $\gamma(1) = \gamma(2) = in$ *and* $\gamma(3) = out$. *Then,* $P_\gamma = \{12|3|45|6\}$. *Assume* $\gamma'(1) = in$ *and* $\gamma'(2) = \gamma'(3) = out$. *Then,* $P_{\gamma'} = \{1|2|3|45|6\}$. *Assume* $\gamma''(1) = \gamma''(2) = \gamma''(3) = out$. *Then,* $P_{\gamma''} = \{1|2|3|45|6\}$.

A strategy profile γ is a *Nash equilibrium* if, for all $i \in S_q$, $\Phi_i(P_\gamma) \geq \Phi_i(P_{\gamma'})$ where γ' is defined as $\gamma'(j) = \gamma(j)$ for all $j \in S_q \setminus \{i\}$ and $\gamma'(i) \neq \gamma(i)$.

Consider the strategy profile γ given by $\gamma(i) = out$ for all $i \in S_q$. This γ is clearly a Nash equilibrium, because the coalition structure does not

[3] For simplicity, we write $\{123|45|6\}$ instead of $\{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$, and so on.

change by the individual deviation of a player. We name this specific γ as an *inessential equilibrium.* Analogously, we name a Nash equilibrium in which there exists some $i \in S_q$ with $\gamma(i) = in$ as an *essential equilibrium.*

Assume that players begin playing γ with $\gamma(i) = in$ for some i, and change their strategies myopically. This means that they sequentially change their strategies only if the payoff in the new coalition structure is larger for them. More precisely, when players change their strategies myopically there is an ordered list of strategy profiles $\Im = [\gamma = \gamma_0, \gamma_1, ..., \gamma_m]$ (and therefore a sequence of coalition structures $P_{\gamma_0}, P_{\gamma_1}, ..., P_{\gamma_m}$) such that γ_l differs from γ_{l-1} $(l = 1, ..., m)$ only in the strategy chosen by a player $i_l \in S_q$ and, moreover, $\Phi_{i_l}\left(P_{\gamma_l}\right) > \Phi_{i_l}\left(P_{\gamma_{l-1}}\right)$.

Following Monderer and Shapley [24], we say that such a $\Im$ is an *improvement path.*

Proposition 2 *An inessential equilibrium cannot be reached following this myopic behavior.*

Proof. The proof is straightforward. Assume that, after player $i_m \in S_q$ changes her strategy, an inessential equilibrium is reached. This means that player i_m was the only player in S_q choosing '*in*' and thus the resulting coalition structure does not change (and neither player i's payoff does). This contradicts that $\Phi_{i_m}\left(P_{\gamma_m}\right) > \Phi_{i_m}\left(P_{\gamma_{m-1}}\right)$. ■

Given a strategy profile γ, we say that P_γ *derives* from P, and it is a *derived coalition structure.* We say that two strategy profiles γ and γ' are *adjacent* through $i \in S_q$, and we write $\gamma \sim_i \gamma'$, if $\gamma(j) = \gamma'(j)$ for all $j \in S_q \backslash \{i\}$ and $\gamma(i) \neq \gamma'(i)$. We then call player i the *link* between γ and γ'. We say that γ and γ' are *adjacent*, and we write $\gamma \sim \gamma'$, if there exists a link $i \in S_q$ such that $\gamma \sim_i \gamma'$. Two derived coalition structures P_γ and $P_{\gamma'}$ are *adjacent through* i if their respective strategy profiles γ and γ' are adjacent through i. Also, P_γ and $P_{\gamma'}$ are *adjacent* if there exists a link i such that P_γ and $P_{\gamma'}$ are adjacent through i. We denote these as $P_\gamma \sim_i P_{\gamma'}$ and $P_\gamma \sim P_{\gamma'}$, respectively.

Example 3 *Let* $P = \{123\}$*,* $P_1 = \{12|3\}$*, and* $P_2 = \{1|2|3\}$*. Then,* P*,* P_1 *and* P_2 *derive from* P*. Moreover,* P *and* P_1 *are adjacent. Player* 3 *is the link between* P *and* P_1*. Similarly,* P_1 *and* P_2 *are adjacent, and they have two possible links, player* 1 *or player* 2*. However,* P *and* P_2 *are not adjacent.*

Notice that two adjacent derived coalition structures may be equal, as the next example shows.

Example 4 *Let* $P = \{12\}$*,* $\gamma(1) = \gamma(2) = out$*,* $\gamma'(1) = out$*,* $\gamma'(2) = in$*. Then,* $P_\gamma \sim P_{\gamma'}$ *and* $P_\gamma = P_{\gamma'} = \{1|2\}$*. However,* $\gamma \neq \gamma'$*.*

A *path* over P is an ordered list of strategy profiles $\Im = [\gamma_0, \gamma_1, ..., \gamma_m]$ such that $\gamma_{l-1} \sim \gamma_l$ for all $l = 1, ..., m$. We say that $\Im$ has *length* m. If $\gamma_m = \gamma_0$, we say that $\Im$ is a *closed path*. We say that $\Im$ is a *simple closed path* if, in addition, $\gamma_j \neq \gamma_k$ for every $1 \leq j \neq k \leq m$. Let $[i_1, i_2, ..., i_m]$ be the list of links between the strategy profiles, i.e. $\gamma_{l-1} \sim_{i_l} \gamma_l$ for all $l = 1, ..., m$. Let $[P_0, P_1, ..., P_m]$ be the list of coalition structures derived from $\Im$, i.e. $P_l = P_{\gamma_l}$ for all $l = 0, 1, ..., m$.

Definition 5 *Given a value* Ψ*, a closed path* $\Im = [\gamma_0, \gamma_1, ..., \gamma_m]$ *is a* cycle *for* Ψ *if* $\Psi_{i_l}(P_{l-1}) < \Psi_{i_l}(P_l)$ *for all* $l = 1, 2, ..., m$*, where* $P_l = P_{\gamma_l}$ *is the coalition structure derived from* γ_l *and* i_l *is the link between* γ_{l-1} *and* γ_l*, for all* $l = 1, 2, ..., m$*.*

Example 6 *Let* $P = \{123\}$ *and* $v(\{1,2,3\}) = 30$*. Let* Ψ *be a value such that* $\Psi(P) = (10, 10, 10)$*. If the coalition structure is* $P_\gamma = \{12|3\}$*, the players get* $\Psi(P_\gamma) = (4, 11, 15)$*. If* $P_\gamma = \{1|23\}$*, they get* $\Psi(P_\gamma) = (11, 4, 15)$*. If* $P_\gamma = \{13|2\}$*, they get* $\Psi(P_\gamma) = (15, 4, 11)$*. If* $P_\gamma = \{1|2|3\}$*, they get* $\Psi(P_\gamma) = (10, 10, 10)$*. Then, every coalition structure belongs to a cycle*[4]*. Moreover, the only Nash equilibrium is the inessential equilibrium (see Figure 1).*

[4] We thank María Montero for proposing this example.

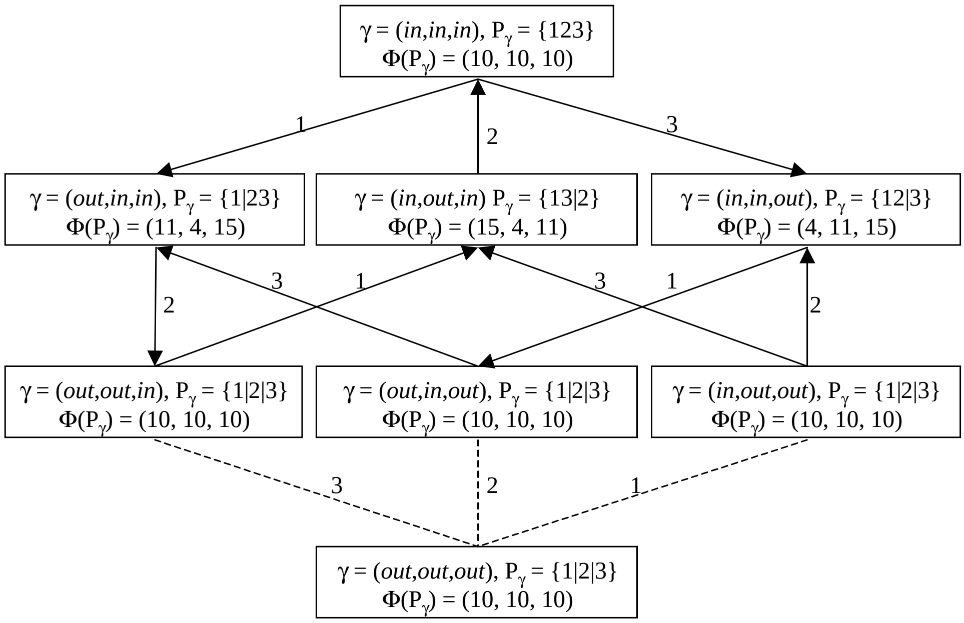

Figure 1: The arrows represent the adjacent strategy profiles. The number next to each arrow indicates the link. Each arrow points to the strategy profile that increases the payoff of the link. Notice that the inessential equilibrium (out, out, out) is not reachable by the arrows (see Proposition 2).

We study the existence of cycles for the Owen value. Hence, from now on, when we say cycle, we mean cycle for Φ.

The existence of cycles may indicate an instability in the mechanism, as the next lemma shows:

Lemma 7 *If the only Nash equilibrium is the inessential equilibrium, then there exists a cycle.*

Proof. Assume the only Nash equilibrium is the inessential equilibrium and there are no cycles. Let γ_0 be a strategy profile that is not the inessential equilibrium. Then, there exists a player $i_1 \in S_q$ who benefits from changing her strategy $\gamma_0(i_1)$. Let γ_1 be the adjacent strategy profile (i.e. $\gamma_0 \sim_{i_1} \gamma_1$) and let P_0 and P_1 be their respective coalition structures (i.e. $P_0 = P_{\gamma_0}$ and $P_1 = P_{\gamma_1}$). By Proposition 2, γ_1 is not the inessential equilibrium. Moreover, $\Phi_{i_1}(P_0) < \Phi_{i_1}(P_1)$. Since γ_1 is not a Nash equilibrium, there exists another player $i_2 \in S_q$ who benefits from changing $\gamma_1(i_2)$. Let γ_2 be the adjacent strategy profile and let P_2 be its derived coalition structure. Then, γ_2 is not the inessential equilibrium, and $\Phi_{i_2}(P_1) < \Phi_{i_2}(P_2)$. The process is repeated with all the players who are willing to change their strategies. Since there exist no cycles, we cannot come back to a previous strategy profile. So, there should be a strategy profile γ_m (which is not the inessential equilibrium) in which no player can improve her payoff by changing her strategy, i.e. γ_m is a Nash equilibrium. This contradiction proves the result. ■

Definition 8 *Given a path* $\Im = [\gamma_0, \gamma_1, ..., \gamma_m]$, *the* differential *of* $\Im$ *in* v *is the number:*

$$\delta(\Im, v, P) := \sum_{l=1}^{m} [\Phi_{i_l}(P_l) - \Phi_{i_l}(P_{l-1})] \tag{1}$$

where $P_l = P_{\gamma_l}$ *is the coalition structure derived from* γ_l, *and* i_l *is the link between* γ_{l-1} *and* γ_l, *for all* $l = 1, 2, ..., m$.

For simplicity, we write $\delta(\Im, v)$ *instead of* $\delta(\Im, v, P)$.

Notice that each term in (1) represents the amount by which a player i_l improves her payoff when the strategy profile changes from γ_{l-1} to γ_l, which is the change that she is capable to do.

Lemma 9 *The differential* $\delta(\Im, v)$ *is additive on* v*, i.e.*

$$\delta(\Im, v + w) = \delta(\Im, v) + \delta(\Im, w)$$

for all $\Im$ *and all games* v, w*.*

Proof. Immediate from the additivity of the Owen value. ■

The following result has been presented by Monderer and Shapley [24]:

Theorem 10 *(Monderer and Shapley, [24]) The following claims are equivalent and characterize a potential game:*

- $\delta(\Im, v) = 0$ *for every finite closed paths* $\Im$*.*
- $\delta(\Im, v) = 0$ *for every finite simple closed paths* $\Im$ *of length 4.*

Proposition 11 *The differential of any closed path is* 0*.*

Proof. Let $\Im = [\gamma_0, \gamma_1, ..., \gamma_m]$ be a closed path with links $[i_1, ..., i_m]$. Let $[P_0, P_1, ..., P_m]$ be their associated coalition structures. We proceed by induction on m. First, note that m should be an even number, because each link i_l should change her strategy $\gamma(i_l)$ an even number of times, so that the strategy profile goes back to its original position, i.e. $\gamma_0 = \gamma_m$.

For $m = 2$, the result is trivial, because $i_1 = i_2$ and $\Phi_{i_2}(P_1) - \Phi_{i_2}(P_0) = -(\Phi_{i_1}(P_0) - \Phi_{i_1}(P_1))$.

For $m = 4$, we have $\Im = [\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4]$ and three cases: a) $i_1 = i_2$, $i_3 = i_4$; b) $i_1 = i_3$, $i_2 = i_4$; and c) $i_1 = i_4$, $i_2 = i_3$. In cases a) and c), we have two closed paths of length 2, so the differential is 0. We prove the result for case b) (Figure 2). Assume without loss of generality that in γ_0 both players play '*in*'.

Assume we are in a unanimity game u_T, and both players belong to the carrier T. In particular, this implies $|S_q \cap T| \geq 2$. Let p_0 be the number of coalitions in P_0 with nonempty intersection with T.

We distinguish two cases:

Case 1. $|S_q \cap T| > 2$.

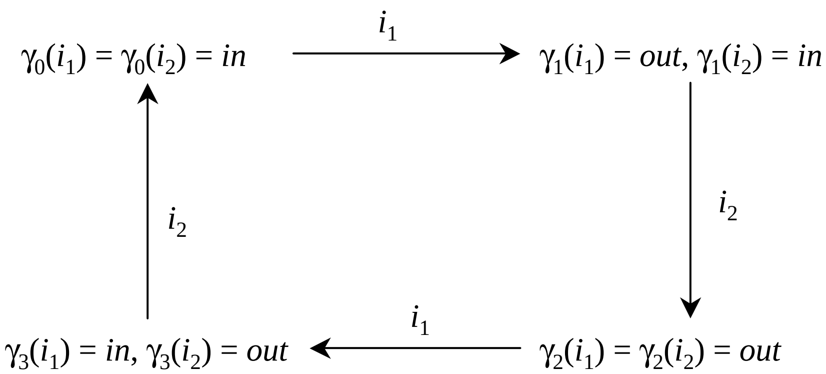

Figure 2: $\mathfrak{T} = [\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_0]$ is a closed path of length 4.

Then, it is well-known (Owen [29, p. 307]) that the Owen values for i_1 and i_2 in P_0 are

$$\Phi_{i_1}(P_0) = \Phi_{i_2}(P_0) = \frac{1}{p_0 |S_q \cap T|}.$$

Analogously,

$$\begin{array}{ll} \Phi_{i_1}(P_1) = \dfrac{1}{p_0+1} & \Phi_{i_2}(P_1) = \dfrac{1}{(p_0+1)\left(|S_q \cap T| - 1\right)} \\ \Phi_{i_1}(P_2) = \dfrac{1}{p_0+2} & \Phi_{i_2}(P_2) = \dfrac{1}{p_0+2} \\ \Phi_{i_1}(P_3) = \dfrac{1}{(p_0+1)\left(|S_q \cap T| - 1\right)} & \Phi_{i_2}(P_3) = \dfrac{1}{p_0+1}. \end{array}$$

Thus,

$$\begin{aligned} \delta\left(\Im, u_T\right) &= \left[\phi_{i_1}(P_1) - \phi_{i_1}(P_0)\right] + \left[\phi_{i_2}(P_2) - \phi_{i_2}(P_1)\right] \\ &\quad + \left[\phi_{i_1}(P_3) - \phi_{i_1}(P_2)\right] + \left[\phi_{i_2}(P_0) - \phi_{i_2}(P_3)\right] \\ &= 0. \end{aligned}$$

Case 2. If $|S_q \cap T| = 2$. All the assignments are equal than before, except

$$\Phi_{i_1}(P_2) = \frac{1}{p_0+1} \quad \Phi_{i_2}(P_2) = \frac{1}{p_0+1}$$

from where it is not difficult to check that $\delta\left(\Im, u_T\right) = 0$.

When one of the players does not belong to the carrier (say, player i_1), then $\Phi_{i_1}(P_\gamma) = 0$ for any γ and we distinguish two cases:

Case 1 $|S_q \cap T| > 1$. Then,

$$\begin{aligned} \Phi_{i_2}(P_0) &= \Phi_{i_2}(P_1) = \frac{1}{p_0 |S_q \cap T|} \\ \Phi_{i_2}(P_2) &= \Phi_{i_2}(P_3) = \frac{1}{p_0 + 1} \end{aligned}$$

Case 2. $|S_q \cap T| = 1$. Then,

$$\Phi_{i_2}(P_\gamma) = \frac{1}{p_0}$$

for all γ.

Thus, in both cases we have again that $\delta(\Im, u_T) = 0$.

In case that both agents do not belong to the carrier, $\Phi_{i_j}(P_\gamma) = 0$ for any γ and $j = 1, 2$. Then it is trivial that $\delta(\Im, u_T) = 0$.

For a general game $v = \sum_{T \subseteq N} \lambda_T(v) u_T$, the additivity property of the differential implies

$$\delta(\Im, v) = \sum_{T \subseteq N} \lambda_T(v) \delta(\Im, u_T) = 0.$$

Applying now Theorem 10, we conclude that $\delta(\Im, v) = 0$ for any closed path $\Im$. ■

An important consequence of Proposition 11 is that there are no cycles.

Corollary 12 *There exist no cycles in the mechanism.*

Proof. Assume there is a cycle $\Im$. Then, $\delta(\Im, v)$ is positive, which contradicts Proposition 11. ■

As another consequence of Proposition 11, we have the following definition:

Definition 13 *Given two strategy profiles γ,γ', the* differential of γ' with respect to γ *is the differential of any path from γ to γ'.*

This differential is well-defined: Assume there are two paths from γ to γ', i.e. $\Im = \{\gamma, \gamma_1, \gamma_2, ..., \gamma_m = \gamma'\}$ and $\Im' = \{\gamma, \gamma'_1, \gamma'_2, ..., \gamma'_{m'} = \gamma'\}$. Then, the closed path $\Im'' = \{\gamma, \gamma_1, \gamma_2, ..., \gamma_m, \gamma'_{m'-1}, ..., \gamma'_1, \gamma\}$ has its differential 0 and

$$0 = \delta(\Im'', v) = \delta(\Im, v) - \delta(\Im', v).$$

Thus, $\delta(\Im, v) = \delta(\Im', v)$.

Theorem 14 *Players, acting myopically, always reach a Nash equilibrium.*

Proof. Fix some strategy profile γ. Suppose that there exists a player $i \in S_q$ who benefits from changing her strategy $\gamma(i)$. Let γ' be the adjacent strategy profile (i.e. $\gamma \sim_i \gamma'$) and let P_γ and $P_{\gamma'}$ be their respective coalition structures. Then, $\phi_i(P_\gamma) < \phi_i(P_{\gamma'})$ and the differential of γ' with respect to γ is positive. Suppose that in the new strategy profile there exists another player $j \in S_q$ who benefits from changing her strategy $\gamma'(j)$. Let γ'' be the adjacent strategy profile and let $P_{\gamma''}$ be its respective coalition structure. Then, $\phi_j(P_{\gamma'}) < \phi_j(P_{\gamma''})$ and the differential of γ'' with respect to γ is again positive. We repeat the process with all the players who are willing to change their strategy. Since the differential is always positive, coming back to a previous strategy profile is not possible. So, there should be a strategy profile γ_m in which no player can improve her payoff by changing her strategy, i.e. γ_m is a Nash equilibrium. ■

Note that Theorem 14 cannot be deduced from Corollary 2.2 in Monderer and Shapley [24] that establishes that "Every finite ordinal potential game possesses a pure-strategy equilibrium".

Theorem 15 *There exists an essential Nash equilibrium.*

Proof. It is an immediate consequence of Lemma 7 and Corollary 12. ■

4 The mechanism with all the coalitions

In the previous section, it was assumed that only the players of a fixed coalition S_q have the chance to exit the coalition. When a coalition negotiate a common behavior among their members (i.e. decide which of them act as a single player), it is natural to assume that the players do so independently of the other coalitions.

However, one may wonder what happens when all the coalitions play simultaneously. Thus, we study the following modification of the mechanism:

First stage Simultaneously, each player in N announces whether she wants to stay or to exit her coalition. Given the announcements of each player, a coalition structure is formed. The players who announced to exit act as singletons.

Second stage Each player receives her Owen value.

Thus, the set of strategies for each player i is again $\gamma(i) \in \{in, out\}$. Let $\gamma = (\gamma(i))_{i \in N}$ be a strategy profile. The derived coalition structure P_γ is given by

$$P_\gamma := \bigcup_{S_q \in P} \left\{ \{i\}_{i \in S_q : \gamma(i) = in} \right\} \cup \{\{i\}\}_{\gamma(i) = out}.$$

The definitions of a path, a closed path, a link, and the differential of a closed path are analogous to those of Section 3. Let γ be a Nash equilibrium. Then, γ is a *inessential equilibrium* if there exists a coalition $S_q \in P$ such that $\gamma(i) = out$ for all $i \in S_q$. Notice that, in this case, there are more than one possible inessential equilibrium.

Proposition 16 *The differential of a closed path is not always zero.*

Proof. Let $N = \{1, 2, 3, 4, 5\}$ and consider the unanimity game (N, u_N). Let $P = \{123|45\}$ and let $\gamma_0 = (in, in, in, in, in)$, $\gamma_1 = (out, in, in, in, in)$, $\gamma_2 = (out, in, in, out, in)$, $\gamma_3 = (in, in, in, out, in)$, and $\gamma_4 = \gamma_0$. The associated coalition structures are $P_0 = P$, $P_1 = \{1|23|45\}$, $P_2 = \{1|23|4|5\}$, $P_3 = \{123|4|5\}$ and $P_4 = P$, respectively. Then, it is straightforward to check that:

$$\begin{aligned}
\Phi_1(P_0) &= \frac{1}{6}, \Phi_4(P_0) = \frac{1}{4} \\
\Phi_1(P_1) &= \frac{1}{3}, \Phi_4(P_1) = \frac{1}{6} \\
\Phi_1(P_2) &= \frac{1}{4}, \Phi_4(P_2) = \frac{1}{4} \\
\Phi_1(P_3) &= \frac{1}{9}, \Phi_4(P_3) = \frac{1}{3} \\
\Phi_1(P_4) &= \frac{1}{6}, \Phi_4(P_4) = \frac{1}{4}
\end{aligned}$$

Let $\mathfrak{T} = [\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4]$ be a closed path. Then, $\delta(\mathfrak{T}, v) = \frac{1}{36} \neq 0$. ■

As the differential is not always zero, a natural question is whether there exist essential equilibria. The next example shows that there exist games whose unique Nash equilibria are the inessential equilibria.

Example 17 *Let $n = 6$ and let v be given by the following table*[5]*:*

S	$v(S)$
$1, 2, 3, 4, 5, 6, 13, 14, 16, 23, 24, 34$	0
$46, 146$	1
$12, 25, 35, 123, 134, 234$	3
$15, 124, 125, 135, 235, 1234$	4
$26, 36, 45, 56, 126, 136, 145, 156, 236, 245, 246, 345, 346, 356, 456, 1246, 1346$	5
$1235, 1345, 2345, 2346$	6
$1236, 1245, 12345$	8
$1256, 1356, 1456, 2356, 12346, 12356$	9
$2456, 3456, 12456, 23456$	10
N	13

This game in monotonic and superadditive[6]*. Moreover, all Nash equilibria are inessential equilibria. For six players, it is tedious to write all the possible*

[5] We write 146 instead of $\{1, 4, 6\}$, and so on.

[6] A game v is *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T$, and *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all S, T with $S \cap T = \emptyset$.

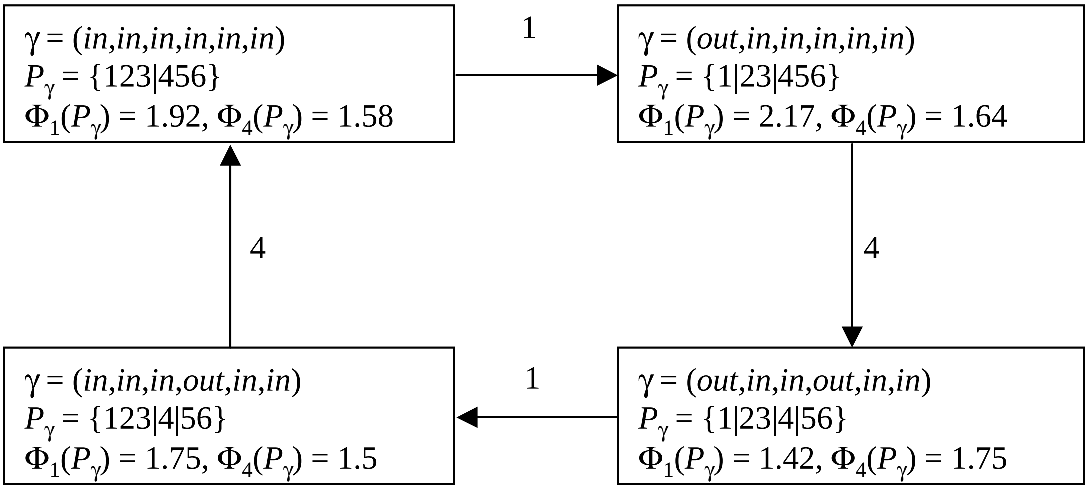

Figure 3: A cycle of length 4.

strategy profiles. In Figure 3, four of these strategy profiles (which form a cycle) are represented.

5 Concluding remarks

In this paper we model situations where players are exogenously divided into coalitions. These coalitions constitute associations where cooperation agreements to act as a single unit are possible, but not obligatory. In particular, players inside a coalition may decide to leave the coalition and act as singletons. Stability is guaranteed as long as two conditions hold: no player who has decided to be a singleton benefits from joining the coalition, and no player who has decided to join the coalition benefits from becoming a singleton.

In this sense, stability is always possible (Theorem 15) when players in other groups have not the option to become singletons. This holds trivially when the coalition structure is trivial, or all the groups but one are singletons. Otherwise, stability may fail (Example 17).

Assume the coalition structure is a singleton, i.e. $P = \{N\}$. A possible way to measure the stability of the grand coalition N is through the differential of the strategy profile "all together" $\gamma(i) = in$ for all $i \in N$ (which induces P) with respect to the inessential equilibrium (which induces the trivial coalition structure $\{\{1\}, ..., \{n\}\}$). Take for example $n = 3$. In the unanimity game u_N, this differential is $\frac{1}{4}$. In this case, players always have incentives to leave the group. In the majority game $w(S) = 1$ if $|S| \geq 2$ and $w(S) = 0$ otherwise, the differential is $-\frac{1}{2}$. In this case, players always have incentives to join the group.

6 Acknowledgments

A previous version of this paper, titled "Negotiating the membership", has benefited from helpful comment from Francesc Carreras. Financial support by the Spanish Ministerio de Ciencia e Innovación through grant ECO2011-23460 and the Xunta de Galicia through grant 10PXIB362299PR is gratefully acknowledged.

References

[1] Albizuri M.J., Aurrekoetxea J. and Zarzuelo J.M. (2006) *Configuration values: Extensions of the coalitional Owen value.* Games and Economic Behavior 57, 1-17.

[2] Albizuri M.J. and Zarzuelo J.M. (2004) *On coalitional semivalues.* Games and Economic Behavior **49**, 221-243.

[3] Arin J., Felktamp V. and Montero M. (2012) *Coalitional games with veto players: myopic and rational behavior.* CeDEx Discussion paper No 2012-11.

[4] d'Aspremont C., Gabszewicz J.J., Jacquemin A. and Weymark J.A. (1983) *On the stability of collusive price leadership*. Canadian Journal of Economics **16**(1), 17-25.

[5] Aumann R.J. and Myerson R. (1988): *Endogenous formation of links between players and coalitions: an application of the Shapley value*. Essays in Honor of Lloyds S. Shapley, ed. by A.E. Roth, Cambridge University Press. Cambridge.

[6] Bergantiños G., Casas B., Fiestras-Janeiro G. and Vidal-Puga J. (2007) *A focal-point solution for bargaining problems with coalition structure*. Mathematical Social Sciences **54**(1), 35-58.

[7] Bergantiños G. and Vidal-Puga J. (2005) *The consistent coalitional value*. Mathematics of Operations Research. Forthcoming.

[8] Bloch F. (1996) *Sequential formation of coalitions in games with externalities and fixed payoff division*. Games and Economic Behavior **14**, 90-123.

[9] Calvo E., Lasaga J. and Winter E. (1996) *The principle of balanced contributions and hierarchies of cooperation*. Mathematical Social Sciences **31**, 171-182.

[10] Carreras F. and Owen G. (1988) *Evaluation of the Catalonian parliament 1980-1984*. Mathematical Social Sciences **15**, 87-92.

[11] Carreras F. and Owen G. (1996) *An analysis of the Euskarian parliament*. Collective decision-making: social choice and political economy. Ed. by N. Schofield and A. Milford. Kluwer.

[12] Chae S. and Heidhues P. (2004) *A group bargaining solution*. Mathematical Social Sciences **48**(1), 37-53.

[13] Chae S. and Moulin H. (2010) *Bargaining among groups: an axiomatic viewpoint*. International Journal of Game Theory **39**(1-2), 71-88.

[14] Chatterjee K., Dutta B., Ray D. and Sengupta K. (1993) *A noncooperative theory of coalitional bargaining.* Review of Economic Studies **60**, 463-477.

[15] Fragnelli V. and Iandolino A. (2004) *A cost allocation problem in urban solid wastes collection and disposal.* Mathematical Methods of Operations Research **59**, 447-463.

[16] Gómez-Rúa M. and Vidal-Puga J. (2010) *The axiomatic approach to three values in games with coalition structure.* European Journal of Operational Research 207(2), 795-806.

[17] Hamiache G. (1999) *A new axiomatization of the Owen value for games with coalition structures.* Mathematical Social Sciences **37**, 281-305.

[18] Hamiache G. (2001) *The Owen value values friendship.* International Journal of Game Theory **29**, 517-532.

[19] Harsanyi J.C. (1959) *A bargaining model for the cooperative n-person game.* In: Tucker and Luce (eds.) Contributions to the theory of games IV. Princeton University Press, 325-355.

[20] Harsanyi J.C. (1977) *Rational behavior and bargaining equilibrium in games and social situations.* Cambridge University Press.

[21] Hart S. and Kurz M. (1983) *Endogenous formation of coalitions.* Econometrica **51**(4), 1047-1064.

[22] Hart S. and Kurz M. (1984) *Stable coalition structures.* Coalitions and collective action. Ed. by M.J. Holler. Würzburg. Physica-Verlag, 235-258.

[23] Krasa S., Temimi A. and Yannelis N.C. (2003) *Coalition structure values in differential information economies: is unity a strength?* Journal of Mathematical Economics **39**, 51-62.

[24] Monderer D. and Shapley L. (1996) *Potential Games*. Games and Economic Behavior **14**, 124-143.

[25] Myerson R.B. (1977) *Graphs and cooperation in games*. Mathematics of Operations Research **2**, 225-229.

[26] Okada A. (1996) *A noncooperative coalitional bargaining game with random proposers*. Games and Economic Behavior **16**, 97-108.

[27] Ono R. and Muto S. (2001) *Coalition cabinets in Japan (1993-1997): A game theory analysis*. International Game Theory Review **2&3**(3), 113-125.

[28] Owen G. (1977) *Values of games with a priori unions*. Essays in mathematical economics and game theory. Ed. by R. Henn R. and O. Moeschlin. Berlin. Springer-Verlag, 76-88.

[29] Owen G. (1995) *Game theory*. Third edition. Academic Press. First edition in 1968. Second edition in 1982.

[30] Peleg B. and Sudhölter P. (2003) *Introduction to the theory of cooperative games*. Kluwer Academic Publishers.

[31] Sánchez E. and Bergantiños G. (1999) *Coalitional values and generalized characteristic functions*. Mathematical Methods of Operations Research **49**, 413-433.

[32] Thoron S. (1998) *Formation of a coalition-proof stable cartel*. Canadian Journal of Economics **31**(1), 63-76.

[33] Vázquez-Brage M., García-Jurado I. and Carreras F. (1996) *The Owen value applied to games with graph-restricted communication*. Games and Economic Behavior **12**, 42-53.

[34] Vázquez-Brage M., Van den Nouweland A. and García-Jurado I. (1997) *Owen's coalitional value and aircraft landing fees*. Mathematical Social Sciences **34**, 273-286.

[35] Vidal-Puga J. (2005) *A bargaining approach to the Owen value and the Nash solution with coalition structure*. Economic Theory **25**(3), 679-701.

[36] Vidal-Puga J. (2012) *The Harsanyi paradox and the "right to talk" in bargaining among coalitions*. Mathematical Social Sciences 64(3), 214-224.

[37] Vidal-Puga J. and Bergantiños G. (2003) *An implementation of the Owen value*. Games and Economic Behavior **44**, 412-427.

[38] Winter E. (1989) *A value for cooperative games with level structure of cooperation*. International Journal or Game Theory **18**, 227-240.

[39] Winter E. (1991) *On non-transferable utility games with coalition structure*. International Journal or Game Theory **20**, 53-63.

[40] Winter E. (1992) *The consistency and potential for values of games with coalition structure*. Games and Economic Behavior **4**, 132-144.