Reassignment-proof rules for land rental problems*

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Abstract

We consider land rental problems where there are several communities that can act as lessors and a single tenant who does not necessary need all the available land. A rule should determine which communities become lessors, how much land they rent and at which price. We present a complete characterization of the family of rules that satisfy reassignment-proofness by merging and spliting, apart from land monotonicity. We also define two parametric subfamilies. The first one is characterized by adding a property of weighted standard for two-person. The second one is characterized by adding consistency and continuity.

Keywords— land rental, non-manipulability, reassignment-proofness, land monotonicity, consistency.

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1 Introduction

The management of land and natural resources is one of the most critical challenges facing developing countries (Kaye and Yahya, 2012; van der Ploeg and Rohner, 2012). In particular, natural resource exploitation is an industrial activity that has recently been generating conflicts between firms and indigenous communities in many countries in Latin America, Africa and Asia. Examples include Mexico (Tetreault, 2015), Peru (Arellano-Yanguas, 2011; Fraser, 2018), Sierra Leone (Akiwumi, 2014), India (Sarkar, 2015, 2017), Vietnam (Nguyen et al., 2018) and Indonesia (Welker, 2009). Another examples appear in Sosa (2011) and Walter and Urkidi (2015). Another two examples, both in Colombia, arise from a restitution problem where two agents have rights over the land (Jaramillo et al., 2014) and from land aggregating for housing and infrastructure (Kominers and Weyl, 2012), respectively.

In these land conflicts, there exist rights over the land for each side. For the case of mining activities, Article 10 of the United Nations Declaration on the Rights of Indigenous People defined Free Prior and Informed Consent (FPIC) as the principle that indigenous communities have the right to give or withhold its consent to proposed projects that may affect the land they customarily own, occupy or otherwise use (UN, 2007). On the other hand, the mining firm has an investment and a concession over those lands, or, even if a concession has not been granted yet, the firm may have a profit opportunity high enough to make it possible to compensate the land owners in a fair way (Helwege, 2015). In order to solve these land conflicts, it is fundamental for the planner (e.g. the government) to have all the relevant information about both sides.

In many situations, land identification and demarcation may be not clear, as in the case of customary land (Gildenhuys, 2005; Azima et al., 2015). This situation can lead to manipulation by merging or splitting of the communities, due to the fact that they may have incentives to strategically misrepresent their identity in order to influence the final outcome to their own advantage. The study of this kind of manipulation is common in the strategy-proofness literature in the context of cost sharing (Moulin and Shenker, 2001; Sprumont, 2005; Gómez-Rúa and Vidal-Puga, 2011; Ju, 2013; Massó et al., 2015), resource allocation (Erlanson and Flores-Szwagrzak, 2015), job scheduling (Moulin, 2007, 2008), indivisible object allocation (Sun and Yang, 2003; Svensson, 2009; Morimoto and Serizawa, 2015), assigning problems (Kojima and Manea, 2010), and taxation problems (Ju and Moreno-Ternero, 2011), among others. Splitting and merging proofness have also been deeply studied in bankruptcy problems where an estate $E > 0$ should be divided among a set of claimants N with claims given by $c \in \mathbb{R}^N$. Several authors (O'Neill, 1982; Moulin, 1987; Chun, 1988; de Frutos, 1999; Ju, 2003; Moreno-Ternero, 2006, 2007; Ju et al., 2007) have showed that merging and splitting proofness in bankruptcy problems leads to a proportional share of the estate. See for example Thomson (2003, 2015a).

In this article, we assume that the government or planner seeks to assign a price and amount of land fairly and efficiently, and at the same time, to guarantee non-manipulability by reassignment-proofness. In particular, our work can be seen as part of the theory of mechanism design applied to land rental (see Sen (2007) for an overview and Sarkar (2017) for a more recent contribution). We assume there is a single tenant who can be a mining firm, and several lessors who can be a group of communities. Each community has some available amount of land c_i with a reservation price r per unit, that for simplicity we consider equal for all of them. The mining firm needs to rent an optimal amount of adjacent land E, which is a completely divisible object[1].

A rule determines, for each land rental problem, a quantity of adjacent land to be rented by each community and a price that the mining firm must pay as a way of compensation.

In order to study rules that guarantee non-manipulability, we propose a version of strategy-proofness such that communities should not find it profitable to re-assign the land among them. For instance, asumme we have two

[1] We use the terms c and E because of their resemblance to bankruptcy problems.

lessors, and the first of them may decide to act as two lessors by spliting her land. A rule which considers a fix price per unit of land and an equalitarian land share will not satisfy reassigment-proofness, because the first lessor finds it profitable to split her land.

Also, we propose a version of land monotonicity that assures fairness, in the sense that an increase in the quantity of available land affects positively the final profits to both sides.

Our first result is a complete characterization of the family of rules that satisfy these properties. A rule belongs to this family of rules if the price does not depend on the available land and each amount of rented land is proportional. By adding a property inspired by "standard for two-person" in Hart and Mas-Colell (1989), we characterize a parametric subfamily. A rule belongs to this parametric subfamily of rules if, aditionally, the price depends on a parameter. Another property is consistency, that states that the rule should behave in a similar way independently of the number of agents involved. This is a classical property in cooperative games (see van den Brink et al. (2013) and Huettner (2015) for two recent applications), and it has also been studied in bankruptcy problems (see Thomson (2008, 2015b) and references herein) and cost sharing problems (see for example Albizuri and Zarzuelo (2007) and Koster (2012)). By adding consistency and continuity we characterize another parametric subfamily of rules. The intersection of both parametric subfamilies singles out two particular rules: one of them optimal for the tenant, where the price coincides with to the tenant's reservation price, and the other optimal for the lessors, where the price coincides with the maximun feasible value.

We organize the paper as follows: In Section 2, we present the model. In Section 3, we study and characterize the family of rules that satisfy land reassignment-proofness and land monotonicity. In Section 4, we characterize the family of rules that also satisfy a weighted version of "standard for two-person". Finally, in Section 5, we characterize the subfamily of rules that satisfy reassignment-proofness, land monotonicity, consistency and continuity.

2 The model

Let $\mathbb{N}_+ = \{1, 2, \dots\}$ be the set of potential lessors. Let $N = \{1, 2, \dots, n\}$ be an arbitrary set of lessors, and let S be an arbitrary subset of N. Given $y \in \mathbb{R}^S$, we write $y(S) = \sum_{i \in S} y_i$. Given $x, y \in \mathbb{R}^S$, we write $x \leq y$ when $x_i \leq y_i$ for all $i \in S$. Moreover 0_S denotes the vector $(0, \dots, 0) \in \mathbb{R}^S$. We denote the set of nonnegative real numbers as $\mathbb{R}_+$, and the set of positive real numbers as $\mathbb{R}_{++}$. We denote the set of rational numbers as $\mathbb{Q}$.

Let $V^N = \{\{i, j\} : i, j \in N\}$ be the set of all unordered pairs $\{i, j\}$ over N. The elements of V^N are called *edges*. A *network* G over N is a subset of V^N. We say that G is a *connected network* when, for all $i, j \in N$, there exists a sequence of different edges $\{\{i_{s-1}, i_s\}\}_{s=1}^e$ that satisfy $\{i_{s-1}, i_s\} \in G$ for all $s \in \{1, 2, \dots, e\}$, $i = i_0$ and $j = i_e$. We denote the set of all connected networks over N as $\mathcal{G}^N$. Given $G \in \mathcal{G}^N$ and $S \subset N$, we denote the restriction of G to S as G_S, i.e. $G_S = \{\{i, j\} \in G : i, j \in S\}$.

A *land rental problem* is a tuple (N_0, μ, c, r, G) where $N_0 = \{0\} \cup N$ is the set of agents with 0 the unique tenant and N the set of lessors, $\mu : \mathbb{R}_+ \to \mathbb{R}$ is a function that assign to each amount of adjacent land the tenant's revenue when that amount is rented, $c \in \mathbb{R}^N_{++}$ is the vector whose coordinates represent the amount of available land for each lessor, $r \in \mathbb{R}_+$ is the reservation price per unit of land for lessors, $G \in \mathcal{G}^N$ identifies the lessors whose land is adjacent. Hence, the aggregate welfare when the tenant rents l units of adjacent land is $\mu(l) + (c(N) - l)r$. We normalise $\mu(0) = 0$, and assume that G is a connected network and that there exists a unique $E \in]0, c(N)]$ such that $\mu(E) + (c(N) - E)r$ is maximum[2] on $[0, c(N)]$.[3] We then denote $K = \mu(E)$ as the optimal welfare that the agents can obtain. This implies that $K > rE$, i.e. there exists benefit of cooperation.

[2]Since $rc(N)$ is constant, this condition is equivalent to $\mu(E) - rE$ be maximum.

[3]This condition holds, for example, when μ is increasing and strictly concave.

Under these conditions, an efficient allocation implies that the amount of rented adjacent land is E and the welfare of the tenant is K. Thus, the only relevant parameters of μ are E and K. Furthermore, for convenience we use N instead of N_0. Henceforth, we would be interested in the "efficient land rental problem", denoted by (N, K, E, c, r, G). Let $\mathcal{L}$ be the set of all land rental problems.

A *feasible agreement* is a pair $(x, p) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ satisfying $x \leq c$ and $\{i \in N : x_i > 0\}$ a connected component in G, where x_i is the land rented by lessor $i \in N$, and p is the price per unit of land. The set of feasible agreements on a land rental problem L is denoted as A^L. Let $\mathcal{A} = \bigcup_{L \in \mathcal{L}} A^L$ be the set of all potential feasible agreements.

Given $(x, p) \in A^L$, the utility for tenant and each lessor $i \in N$ are $u_0(x, p) = \mu(x(N)) - px(N)$ and $u_i(x, p) = (p - r)x_i$, respectively.

We define a *rule* as a function $\psi : \mathcal{L} \longrightarrow \mathcal{A}$ that assigns to each problem $L = (N, K, E, c, r, G) \in \mathcal{L}$ a feasible agreement $(x, p) = \psi(L) \in A^L$, satisfying:

(i) $x(N) = E$;

(ii) for all $\alpha, \beta > 0$, $p(N, \alpha K, \beta E, \beta c, \frac{\alpha}{\beta} r, G) = \frac{\alpha}{\beta} p(N, K, E, c, r, G)$ and $x(N, \alpha K, \beta E, \beta c, \frac{\alpha}{\beta} r, G) = \beta x(N, K, E, c, r, G)$;

(iii) $r \leq p \leq \frac{K}{E}$.

The first condition *(efficiency)* says that the amount of land rented is optimal. The second condition *(scale invariance)* says that the final price and the amount of land rented are independent of changes of scale. The third condition *(individual rationality)* says that the lessors get at least zero (this is implied by $r \leq p$), and under efficiency, the tenant also gets at least zero (this is implied by $p \leq \frac{K}{E}$). Under efficiency, the utility of the tenant can be rewritten as $u_0(x, p) = K - pE$.

There exist two special classes of rules: On the one hand, a rule is *tenant-optimal* when the price is given by $p = r$. In that case, x_i is irrelevant for each $i \in N$, because their payoffs are zero, and so the final payoff allocation is unique. On the other hand, a rule is *lessors-optimal* when the price is given by $p = \frac{K}{E}$. In the latter case, there are many possible payoff allocations when $E < c(N)$, all of them giving zero to the tenant.

3 Land reassignment and monotonicity

Since there may be no official registration and demarcation of the customary land, the lessors can reach an agreement of reallocating it in order to share extra benefits so created under a rule.

Formally, assume $N = (N \setminus S) \cup S$, where $N \setminus S$ is connected in G and represents the set of lessors that rearrange their land, while S is the set of lessors that do not. Hence, a new land problem arises, with $N' = (N' \setminus S) \cup S$ as the new set of lessors, so that $S = N \cap N'$. Moreover, the new connected network G' that determines the adjacent lands should be compatible with G in the sense that $G_S = G'_S$ and, for all $i \in S$,

$$\exists j \in N \setminus S : \{i, j\} \in G \Leftrightarrow \exists j' \in N' \setminus S : \{i, j'\} \in G'.$$

In this case, we say that G and G' are *S-compatible.*

For the planner it is not possible to see this customary land situation, and it may be hard to get the outcome that the rule is supposed to attain. In our context manipulation implies that the lessors will benefit by merging or splitting under reallocating their land. Our aim is to fully identify rules that are free from this concern. We formalise this property as follows.

Reassignment-proofness (RP) Given $(N, K, E, c, r, G), (N', K, E, c', r, G') \in \mathcal{L}$ such that $c_i = c'_i$ for all $i \in S = N \cap N'$, $c(N \setminus S) = c'(N' \setminus S)$, and G and G' are S-compatible, a rule ψ is *reassignment-proof if*

$$\sum_{i \in N \setminus S} u_i(\psi(N, K, E, c, r, G)) = \sum_{i \in N' \setminus S} u_i(\psi(N', K, E, c', r, G')).$$

If the right-hand side of expression is larger than the left-hand side and the problem is (N, K, E, c, r, G), then lessors in $N \setminus S$ can gain by reallocating their land so that the problem becomes (N', K, E, c', r, G'). Analogously, if the left-hand side of expression is larger than the right-hand side and the problem is (N', K, E, c', r, G'), then lessors in $N' \setminus S$ can gain by reallocating their land so that the problem becomes (N, K, E, c, r, G). S is the set of lessors that remain unchanged ($S = \emptyset$ is also possible). This property prevents lessors from having incentives for merging or splitting by reallocating their land.

The following property says that an increase of the available land, leaving K and E unaffected, is (weakly) beneficial for everyone involved.

Land Monotonicity (LM) Given $(N, K, E, c, r, G), (N, K, E, c', r, G') \in \mathcal{L}$ with $c \leq c'$ and $G \subseteq G'$, a rule ψ is *land monotonic* if

(i) $u_0(\psi(N, K, E, c, r, G)) \leq u_0(\psi(N, K, E, c', r, G'))$, and

(ii) for each $i \in N$, $c_j = c'_j$ for all $j \neq i$ implies $u_i(\psi(N, K, E, c, r, G)) \leq u_i(\psi(N, K, E, c', r, G'))$.

Under this property, the tenant will be weakly better off when there are more available land. Furthermore, when only one lessor has more available land and the rest of lessors remain unchanged, this lessor will be weakly better off.

Let $\mathcal{F}$ be the set of functions $f : [0, 1] \to [0, 1]$ with $f(t) \geq t$ for all $t \in [0, 1]$. Now, we consider the family of rules defined by $p = \frac{K}{E} f(\frac{rE}{K})$ for some $f \in \mathcal{F}$ and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. So, we obtain different rules with different functions $f \in \mathcal{F}$. These functions determine the price, whereas the amount of land is always divided proportionally, in line with the known results on invariance under reassignment in cost and surplus sharing (cf. Theorem 1.1 in Moulin (2002)). Figure 1 represents six examples of these functions.

Theorem 3.1 *A rule ψ satisfies RP and LM if and only if there exists $f \in \mathcal{F}$ such that the price is given by $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when $p \neq r$, the assigned amount of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

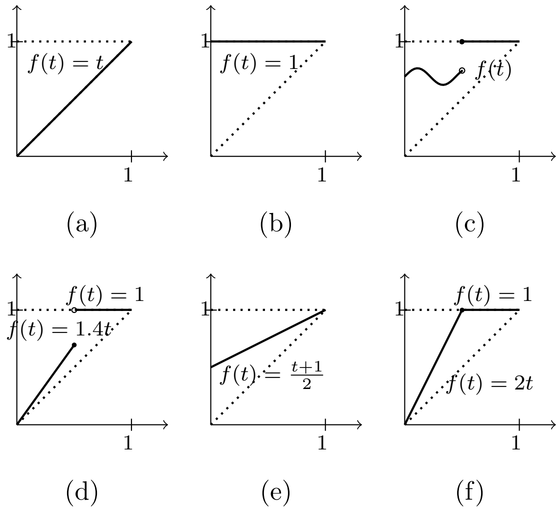

Figure 1: Examples of functions in $\mathcal{F}$ that determine six different rules, including an optimal rule for the tenant (a) and an optimal rule for the lessors (b).

Proof. ($\Leftarrow$) Let ψ be a rule given by $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ for some $f \in \mathcal{F}$ and, when $p \neq r$, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. We will prove that ψ satisfies RP and LM. In order to prove that ψ satisfies RP, let $L = (N, K, E, c, r, G) \in \mathcal{L}$, $L' = (N', K, E, c', r, G') \in \mathcal{L}$ and $S = N \cap N'$ given as the definition of RP. Let $t = \frac{rE}{K} \in [0, 1[$. On the one hand, we have $\sum_{i \in N \setminus S} u_i(\psi(L)) = \sum_{i \in N \setminus S} \left(\frac{K}{E} f(t) - r\right) \frac{c_i E}{c(N)} = K(f(t) - t)\left(1 - \frac{c(S)}{c(N)}\right)$. Analogously, on the other hand, we have $\sum_{i \in N' \setminus S} u_i(\psi(L')) = K(f(t) - t)\left(1 - \frac{c'(S)}{c'(N')}\right)$. Since $c(N \setminus S) = c'(N' \setminus S)$ and $c_i = c'_i$ for all $i \in S$, we have that $c(S) = c'(S)$ and $c(N) = c'(N')$. Hence the last two expressions coincide. We now prove that ψ satisfies LM. Let L and $L' = (N, K, E, c', r, G') \in \mathcal{L}$ given as in the definition of LM. If $c \leq c'$, then, by efficiency, $u_0(\psi(L)) = K - \frac{K}{E} f\left(\frac{rE}{K}\right) E = u_0(\psi(L'))$, hence condition (i) holds. If $c_i \leq c'_i$ and $c(N \setminus \{i\}) > 0$, and $c_j = c'_j$ for all $j \in N \setminus \{i\}$ then $u_i(\psi(L)) = \left(\frac{K}{E} f\left(\frac{rE}{K}\right) - r\right) \frac{c_i E}{c(N)} \leq \left(\frac{K}{E} f\left(\frac{rE}{K}\right) - r\right) \frac{c'_i E}{c'(N)} = u_i(\psi(L'))$ for all $i \in N$, hence condition (ii) also holds.

($\Rightarrow$) Let ψ be a rule that satisfies RP and LM. For simplicity, we write (x, p) instead of $\psi(N, K, E, c, r, G)$, (x', p') instead of $\psi(N', K, E, c', r, G')$ and so on. Furthermore, we write u_i instead of $u_i(x,p)$, u_i' instead of $u_i(x',p')$ and so on. We proceed by series of claims.

Claim 3.1 *If $K = E = 1$ and $N = \{1\}$, then the price p does not depend on c.*

Proof. By LM, if $c_1 \leq c_1'$, then $u_0 \leq u_0'$. By efficiency, $u_0 \leq u_0'$ can be rewritten as $1 - p \leq 1 - p'$, hence $p \geq p'$ (the higher c_1, the higher p). Analogously, $c_1 \leq c_1'$ implies $u_1 \leq u_1'$ and $p \leq p'$ (the higher c_1, the lower p). Therefore, $p = p'$. □

We define $f(t) = p(\{1\}, 1, 1, (1), t, \emptyset)$ for all $t \in [0,1]$. By individual rationality, $t \leq p(\{1\}, 1, 1, (1), t, \emptyset) \leq 1$ for all $t \in [0,1]$, so $f \in \mathcal{F}$.

Claim 3.2 *If $K = E = 1$, then $p = f(r)$.*

Proof. Assume first $1 \notin N$. By RP, $u(N) = u_1(\psi(\{1\}, 1, 1, (c(N)), r, \emptyset))$. Under Claim 3.1 and efficiency, this is equal to $p(\{1\}, 1, 1, (1), r, \emptyset) - r$, hence $u(N) = f(r) - r$. Furthermore, by efficiency, $u(N) = \sum_{i \in N}(p - r)x_i = p - r$. Therefore, we have $p = f(r)$. Assume now $1 \in N$. Let $i \in \mathbb{N}_+ \setminus N$. Under RP, $u_i(\psi(\{i\}, 1, 1, (c(N)), r, \emptyset)) = u_1(\psi(\{1\}, 1, 1, (c(N)), r, \emptyset))$ and we proceed as before. □

Claim 3.3 $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$.

Proof. By scale invariance, $p = \frac{K}{E} p\left(N, 1, 1, \left(\frac{c}{E}\right), \frac{rE}{K}, G\right)$, and under Claim 3.2 we have that $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$. □

Therefore, the price is determined by Claim 3.3. Now we focus on the amount of land x.

Claim 3.4 *If $p \neq r$ and there exist $i, j \in N$ such that $c_i = c_j$, then $x_i = x_j$.*

Proof. Fix $\alpha \in \mathbb{N}_+ \setminus N$. We define $(N^{i\alpha}, K, E, c^{i\alpha}, r, G^{i\alpha}) \in \mathcal{L}$, where $N^{i\alpha} = \{i, \alpha\}$, $c_i^{i\alpha} = c_i$, $c_\alpha^{i\alpha} = c(N \setminus \{i\})$, and $G^{i\alpha} = \{\{i, \alpha\}\}$. Since $N \cap N^{i\alpha} = \{i\}$, then by RP, $u(N \setminus \{i\}) = u^{i\alpha}(N^{i\alpha} \setminus \{i\})$. Under Claim 3.3 and $p \neq r$, we obtain $x(N \setminus \{i\}) = x_\alpha^{i\alpha}$. Furthermore, by efficiency $x(N \setminus \{i\}) + x_i = E$ and

$x_i^{i\alpha} + x_\alpha^{i\alpha} = E$. From these last three equalities we obtain that $x_i = x_i^{i\alpha}$. We define $(N^{j\alpha}, K, E, c^{j\alpha}, r, G^{j\alpha}) \in \mathcal{L}$, where $N^{j\alpha} = \{j, \alpha\}$, $c_j^{j\alpha} = c_i^{i\alpha} = c_i = c_j$, $c_\alpha^{j\alpha} = c_\alpha^{i\alpha}$, and $G^{j\alpha} = \{\{j, \alpha\}\}$. Since $N^{i\alpha} \cap N^{j\alpha} = \{\alpha\}$ and $c_i^{i\alpha} = c_j^{j\alpha}$, by RP, $u_i(x^{i\alpha}, p^{i\alpha}) = u_j(x^{j\alpha}, p^{j\alpha})$. Under Claim 3.3 and $p \neq r$, we obtain $x_i^{i\alpha} = x_j^{j\alpha}$. Since $N^{j\alpha} \cap N = \{j\}$ and $c_\alpha^{j\alpha} = c(N \setminus \{j\})$, by RP, $u_\alpha(x^{j\alpha}, p^{j\alpha}) = u(N \setminus \{j\})$, and under Claim 3.3 and $p \neq r$, we obtain $x_\alpha^{j\alpha} = x(N \setminus \{j\})$. Furthermore, by efficiency we have $x_j^{j\alpha} + x_\alpha^{j\alpha} = E$ and $x_j + x(N \setminus \{j\}) = E$. So, from these last three equalities we obtain $x_j^{j\alpha} = x_j$. Then, from $x_i = x_i^{i\alpha}$, $x_i^{i\alpha} = x_j^{j\alpha}$ and $x_j^{j\alpha} = x_j$ we get that $x_i = x_j$. □

Claim 3.5 *If $N = \{i, j\}$, $p \neq r$ and $c_i, c_j \in \mathbb{Q}$, then $x_i = \frac{c_i E}{ci + c_j}$ and $x_j = \frac{c_j E}{ci + c_j}$.*

Proof. Assume $c_i = \frac{a_i}{b_i}$ and $c_j = \frac{a_j}{b_j}$ where a and b are non-negative integers. Let $N^i, N^j \subset \mathbb{N}_+ \setminus N$ with $N^i \cap N^j = \emptyset$, $|N^i| = a_i b_j$ and $|N^j| = a_j b_i$. We define $(N^{*i}, K, E, c^{*i}, r, G^{*i}) \in \mathcal{L}$ with $N^{*i} = N^i \cup \{j\}$ and $c_k^{*i} = \frac{1}{b_i b_j}$ for all $k \in N^i$, $c_j^{*i} = c_j$, and $G^{*i} = \{\{k, j\} : k \in N^i\}$. Since $N \cap N^{*i} = \{j\}$ and $c^{*i}(N^{*i} \setminus j) = c_i$, by RP, $u_i = u^{*i}(N^{*i} \setminus \{j\})$. Under Claim 3.3, this is equivalent to write $(p - r)x_i = (p - r)x^{*i}(N^{*i} \setminus \{j\})$. Since $p \neq r$, $x_i = x^{*i}(N^{*i} \setminus \{j\})$. We now define $(N^{*ij}, K, E, c^{*ij}, r, G^{*ij}) \in \mathcal{L}$ with $N^{*ij} = N^i \cup N^j$, $c_k^{*ij} = \frac{1}{b_i b_j}$ for all $k \in N^{*ij}$, and $G^{*ij} = \{\{k, k'\} : k \in N^i, k' \in N^j\}$. Since $c^{*i}(N^{*i} \setminus \{j\}) = c^{*ij}(N^{*ij} \setminus N^j)$, $N^{*i} \cap N^{*ij} = N^i$ and $c^{*ij}(N^{*ij} \setminus N^i) = \sum_{l=1}^{a_j b_i} \frac{1}{b_i b_j} = c_j = c_j^{*i}$, by RP, $u_j^{*i} = u^{*ij}(N^{*ij} \setminus N^i)$. Under Claim 3.3, this is equivalent to write $(p - r)x_j^{*i} = (p - r)x^{*ij}(N^{*ij} \setminus N^i)$. Since $p \neq r$, $x_j^{*i} = x^{*ij}(N^{*ij} \setminus N^i) = x^{*ij}(N^j)$. On the one hand, by efficiency, $x_i + x_j = E$ and $x^{*i}(N^{*i} \setminus \{j\}) + x_j^{*i} = E$. Since $x_i = x^{*i}(N^{*i} \setminus \{j\})$ and $x_j^{*i} = x^{*ij}(N^j)$, we obtain $x_j = x^{*ij}(N^j)$. On the other hand, by efficiency, $x^{*i}(N^{*i} \setminus \{j\}) + x_j^{*i} = E$ and $x^{*ij}(N^i) + x^{*ij}(N^j) = E$. Since $x_i = x^{*i}(N^{*i} \setminus \{j\})$ and $x_j^{*i} = x^{*ij}(N^j)$, we obtain $x_i = x^{*ij}(N^i)$. We have $(N^{*ij}, K, E, c^{*ij}, r, G^{*ij})$ with $N^{*ij} \cap N = \emptyset$ and $c^{*ij}(N^{*ij}) = c(N)$. By RP, $u^{*ij}(N^{*ij}) = u(N)$. By $p \neq r$ and Claim 3.3, this is equivalent to write $x^{*ij}(N^{*ij}) = x(N)$. Under Claim 3.4 and efficiency, we obtain that $x_k^{*ij} = \frac{E}{|N^{*ij}|} = \frac{E}{a_i b_j + a_j b_i}$ for all $k \in N^{*ij}$. By efficiency and $x_j = x^{*ij}(N^j)$, we have that $x_i = E - x^{*ij}(N^j)$. Under Claim 3.4, this is equivalent to write $x_i = E - a_j b_i x_k^{*ij}$ for each $k \in N^{*ij}$. Since $x_k^{*ij} = \frac{E}{a_i b_j + a_j b_i}$ for all $k \in N^{*ij}$, we have $x_i = E - \frac{a_j b_i E}{a_i b_j + a_j b_i} = \frac{c_i E}{c_i + c_j}$. Analogously, $x_j = \frac{c_j E}{ci + c_j}$. □

Claim 3.6 *If $N = \{i, j\}$, $p \neq r$ and $c_j \in \mathbb{Q}$, then $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$.*

Proof. Assume first $c_i + c_j = E$. Then, $\frac{c_i E}{c_i + c_j} = c_i$ and $\frac{c_j E}{c_i + c_j} = c_j$. By efficiency, $x_i = c_i$ and $x_j = c_j$. Therefore, $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$. Assume now $c_i + c_j > E$. Let $\{c_i^s\}_{s=1}^{\infty}$ be a decreasing sequence of rational numbers that converges to c_i. For each s, we take $(N, K, E, c^s, r, G) \in \mathcal{L}$ with $c^s = (c_i^s, c_j)$. Under Claim 3.5, we have $x^s = \left(\frac{c_i^s E}{c_i^s + c_j}, \frac{c_j E}{c_i^s + c_j}\right)$. By LM, $u_i(x, p) \leq u_i(x^s, p^s)$. Under Claim 3.3, this is equivalent to write $(p - r)x_i \leq (p - r)x_i^s$. Since $p \neq r$, this is equivalent to $x_i \leq x_i^s$. Under Claim 3.5, $x^s = \left(\frac{c_i^s E}{c_i^s + c_j}, \frac{c_j E}{c_i^s + c_j}\right)$, which is equivalent to write $x_i \leq \frac{c_i^s E}{c_i^s + c_j}$. Hence, $x_i \leq \frac{c_i E}{c_i + c_j}$. Let $\{\widehat{c}_i^s\}_{s=1}^{\infty}$ be an increasing sequence of positive rational numbers that converges to c_i and such that $\widehat{L}^s = (N, K, E, \widehat{c}^s, r, G) \in \mathcal{L}$, where $\widehat{c}^s = (\widehat{c}_i^s, c_j)$. We can find such a sequence because $c_i > 0$ and $c_i + c_j > E$. Under Claim 3.5, we have $\widehat{x}^s = \left(\frac{\widehat{c}_i^s E}{\widehat{c}_i^s + c_j}, \frac{c_j E}{\widehat{c}_i^s + c_j}\right)$. By LM, $\widehat{u}_i^s \leq u_i$. Under Claim 3.3, this is equivalent to write $(p - r)\widehat{x}_i^s \leq (p - r)x_i$. Since $p \neq r$, this is equivalent to $\widehat{x}_i^s \leq x_i$. Under Claim 3.5, $\widehat{x}^s = \left(\frac{\widehat{c}_i^s E}{\widehat{c}_i^s + c_j}, \frac{c_j E}{\widehat{c}_i^s + c_j}\right)$, which is equivalent to write $\frac{\widehat{c}_i^s E}{\widehat{c}_i^s + c_j} \leq x_i$. Hence, $\frac{c_i E}{c_i + c_j} \leq x_i$. Since $x_i \leq \frac{c_i E}{c_i + c_j}$ and $\frac{c_i E}{c_i + c_j} \leq x_i$, we obtain $x_i = \frac{c_i E}{c_i + c_j}$. By efficiency, $x_j = E - x_i$. Since $x_i = \frac{c_i E}{c_i + c_j}$, we deduce $x_j = E - \frac{c_i E}{c_i + c_j} = \frac{c_j E}{c_i + c_j}$.
□

Claim 3.7 *If $N = \{i, j\}$ and $p \neq r$, then $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$.*

Proof. Assume first $c_i + c_j = E$. Then, $\frac{c_i E}{c_i + c_j} = c_i$ and $\frac{c_j E}{c_i + c_j} = c_j$. By efficiency, $x_i = c_i$ and $x_j = c_j$. Therefore, $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$. Assume now $c_i + c_j > E$. Let $\{c_j^s\}_{s=1}^{\infty}$ a decreasing sequence of rational numbers that converges to c_j. For each s, we take $(N, K, E, c^s, r, G) \in \mathcal{L}$ with $c^s = (c_i, c_j^s)$. Under Claim 3.6, we have $x^s = \left(x_i, \frac{c_j^s E}{c_i + c_j^s}\right)$. By LM, $u_j(x, p) \leq u_j(x^s, p^s)$. Under Claim 3.3, this is equivalent to write $(p - r)x_j \leq (p - r)x_j^s$. Since $p \neq r$, this is equivalent to $x_j \leq x_j^s = \frac{c_j^s E}{c_i + c_j^s}$. Hence, $x_j \leq \frac{c_j E}{c_i + c_j}$. Let $\{\widehat{c}_j^s\}_{s=1}^{\infty}$

be an increasing sequence of rational numbers that converges to c_j and such that $\widehat{L}^s = (N, K, E, \widehat{c}^s, r, G) \in \mathcal{L}$, where $\widehat{c}^s = (c_i, \widehat{c}_j^s)$. We can find such a sequence because $c_i > 0$ and $c_i + c_j > E$. Under Claim 3.6, we have $\widehat{x}^s = \left(x_i, \frac{\widehat{c}_j^s E}{c_i + \widehat{c}_j^s}\right)$. By LM, $\widehat{u}_j^s \leq u_j$. Under Claim 3.3, this is equivalent to write $(p-r)\widehat{x}_j^s \leq (p-r)x_j$. Since $p \neq r$, this is equivalent to $\widehat{x}_j^s \leq x_j$, or $\frac{\widehat{c}_j^s E}{c_i + \widehat{c}_j^s} \leq x_j$. Hence, $\frac{c_j E}{c_i + c_j} \leq x_j$. Since $x_j \leq \frac{c_j E}{c_i + c_j}$ and $\frac{c_j E}{c_i + c_j} \leq x_j$, we obtain $x_j = \frac{c_j E}{c_i + c_j}$. By efficiency, $x_i = E - x_j$. Since $x_j = \frac{c_j E}{c_i + c_j}$, we deduce $x_i = E - \frac{c_j E}{c_i + c_j} = \frac{c_i E}{c_i + c_j}$. □

Claim 3.8 *If $p \neq r$, then $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. Let $i \in N$, $j \in \mathbb{N}_+ \setminus N$ and $(N^{ij}, K, E, c^{ij}, r, G^{ij}) \in \mathcal{L}$ with $N^{ij} = \{i, j\}$, $c_i^{ij} = c_i$, $c_j^{ij} = c(N \setminus \{i\})$, and $G^{ij} = \{\{i, j\}\}$. By efficiency, $x_i = E - x(N \setminus \{i\})$. By RP and $p \neq r$, we have $x(N \setminus \{i\}) = x_j^{ij}$, so that $x_i = E - x_j^{ij}$. Under Claim 3.7, $x_j^{ij} = \frac{c_j^{ij} E}{c_i^{ij} + c_j^{ij}}$. Hence, $x_i = E - \frac{c_j^{ij} E}{c_i^{ij} + c_j^{ij}} = E - \frac{c(N \setminus \{i\})E}{c_i + c(N \setminus \{i\})} = \frac{c_i E}{c(N)}$. □

Therefore, the amount of land is determined by Claim 3.8. ■

We denote ψ^f as the rule corresponding to $f \in \mathcal{F}$ that is given by $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$, and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.

4 Weighted standard for two-person

We study a property that is inspired on the so called standard for two-person property by Hart and Mas-Colell (1989). This property follows a "divide the surplus equally" idea for two-person situations. In our context, the two-person case arises when $|N| = 1$, i.e. the only agents are the tenant and a single lessor. Standard for two-person says that both the tenant and the lessor obtain equal benefit. We formalize this property as follows. Let $\mathcal{L}^2$ be the set of land rental problems with a unique lessor.

Standard for 2-person (S2) Given $L = (\{1\}, K, E, c, r, \emptyset) \in \mathcal{L}^2$,

$$u_0(\psi(L)) = u_1(\psi(L)).$$

Next theorem characterizes the unique rule that satisfies RP, LM and S2. The function that determines this rule is represented in Figure 1(e).

Theorem 4.1 *A rule ψ satisfies RP, LM and S2 if and only if the price is given by $p = \frac{K+rE}{2E}$ and the amount of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. ($\Leftarrow$) Let ψ be a rule given by $p = \frac{K+rE}{2E}$ and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. It is straightforward to check that $\psi = \psi^f$ with $f(t) = \frac{1+t}{2}$ for all t and $p = \frac{K+rE}{2E}$. By Theorem 3.1, ψ satisfies RP and LM. So, we just need to prove that $u_0(\psi(\{1\}, K, E, c, r, \emptyset)) = u_1(\psi(\{1\}, K, E, c, r, \emptyset))$. The left side of the equality is equal to $K - \frac{K+rE}{2E}E = \frac{K-rE}{2}$. Analogously, the right side of the equality is equal to $\left(\frac{K+rE}{2E} - r\right) x_1$. By efficiency, $x_1 = E$, and hence we obtain $u_1(\psi(\{1\}, K, E, c, r, \emptyset)) = \frac{K-rE}{2}$. Therefore, the equality holds.

($\Rightarrow$) Let ψ be a rule that satisfies RP, LM and S2. By Theorem 3.1 there exists $f \in \mathcal{F}$ such that $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when, $p \neq r$, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. We need to prove that $\frac{K}{E} f\left(\frac{rE}{K}\right) = \frac{K+rE}{2E}$ or equivalently $f(t) = \frac{1+t}{2}$ for $t = \frac{rE}{K} \in [0,1]$. By S2, we have $u_0\left(\psi(\{1\}, 1, 1, (1), t, \emptyset)\right) = u_1\left(\psi(\{1\}, 1, 1, (1), t, \emptyset)\right)$. This is equivalent to $1 - f(t)x_1 = (f(t) - t)x_1$. By efficiency, $x_1 = 1$, which is equivalent to write $1 - f(t) = f(t) - t$. Hence, $f(t) = \frac{1+t}{2}$. Finally, since $K > rE$ and $c_1 = c(N)$, we deduce $p \neq r$ so $x_1 = \frac{c_1 E}{c(N)} = E$. ■

Next, we generalize the standard for two-person concept in a nonsymmetric way. Notice that S2 determines the final payoffs for two-person problems, forcing both the tenant and the unique lessor to receive the same value. Since tenant and lessor are not symmetric, we can reasonably allow one side of the market to extract a higher value than the other. In our context, since the rules satisfy efficiency, it is enough to fix the relative payoff between both agents. In particular, a rule satisfies the next property when the payoffs are in the same proportion for every single-lessor problem.

Weighted Standard for 2-person (WS2) There exists $\omega \in [0,1]$ such that

$$(1-\omega)u_0\left(\psi(L)\right) = \omega u_1\left(\psi(L)\right)$$

for all $L = (\{1\}, K, E, c, r, \emptyset) \in \mathcal{L}^2$.

Next theorem characterizes the parametric subfamily of rules that satisfy RP, LM and WS2. We can see three examples of functions that determine these rules in Figure 1 (a), (b) and (e), respectively.

Theorem 4.2 *A rule ψ satisfies RP, LM and WS2 if and only if there exists $\omega \in [0,1]$ such that the price is given by $p = \frac{K-(K-rE)\omega}{E}$ and, when $\omega < 1$, the quantity of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. ($\Leftarrow$) Fix $\omega \in [0,1]$. Let ψ be a rule given by $p = \frac{K-(K-rE)\omega}{E}$ and, if $\omega < 1$, then $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. By Theorem 3.1, ψ satisfies RP and LM for $f(t) = 1-(1-t)\omega$ and $p = \frac{K-(K-rE)\omega}{E}$. Fix $L = (\{1\}, K, E, c, r, \emptyset)$. We just need to prove that $(1-\omega)u_0(\psi(L)) = \omega u_1(\psi(L))$. The left side of the equality is equal to $(1-\omega)\left(K - \frac{K-(K-rE)\omega}{E}E\right) = (1-\omega)\omega(K-rE)$. Analogously, the right side of the equality is equal to $\omega(\frac{K-(K-rE)\omega}{E} - r)x_1$. By efficiency, $x_1 = E$, and hence the right hand side of the equality is $\omega(1-\omega)(K-rE)$. Therefore, equality holds.

($\Rightarrow$) Let ψ be a rule that satisfies RP, LM and WS2. Let $\omega \in [0,1]$. By Theorem 3.1, there exists $f \in \mathcal{F}$ such that $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when $p \neq r$, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. This implies $x(N) = E$. It is clear that $\omega < 1$ implies $p \neq r$. To see why, notice that $p = r$ implies $u_1 = 0$, whereas $u_0 + u_1 = K - rE > 0$, so $u_0 > 0$, and by WS2, $(1-\omega)u_0 = \omega u_1 = 0$, so $(1-\omega)u_0 = 0$, which implies $\omega = 1$. We still need to prove that $\frac{K}{E} f\left(\frac{rE}{K}\right) = \frac{K-(K-rE)\omega}{E}$ or, equivalently, $f(t) = 1-(1-t)\omega$ for all $t \in [0,1]$. Let $t = \frac{rE}{K} \in [0,1]$. By WS2 we have $(1-\omega)u_0\left(\psi(\{1\}, 1, 1, (1), t, \emptyset)\right) = \omega u_1\left(\psi(\{1\}, 1, 1, (1), t, \emptyset)\right)$. This is equivalent to $(1-\omega)(1-f(t)x_1) = \omega(f(t)-t)x_1$, which by efficiency is equivalent to write $(1-\omega)(1-f(t)) = (f(t)-t)\omega$. Rearranging terms, we deduce $f(t) = 1-(1-t)\omega$. ■

Notice that, when $\omega = 1$, we obtain an optimal rule for the tenant, and when $\omega = 0$, we obtain an optimal rule for the lessors. Given $\omega \in [0,1]$, we denote ψ^ω as the rule corresponding to the function ψ^f with $f(t) = 1-(1-t)\omega$

for all t and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. In particular, $\psi^{\frac{1}{2}}$ is the rule given in Theorem 4.1.

5 Consistency

Consistency is a well-known principle. Assume that there exists an agreement on what the right price and land share are, and that some lessors take this price and leave. The tenant and the rest of lessors can proceed in two ways: On the one hand, they can keep the previous price and land share. On the other hand, they can recompute the right price and land share following the same principle as before in the new reduced land renting problem. This new reduced land rental problem is defined as $L' = (N', K', E', c', r, G') \in \mathcal{L}$ given by $N' = N \backslash S$ where $S \subset N$ is the set of lessors that leave, $K' = K - px(S)$ is the new maximal profit of the tenant, $E - x(S)$ is the amount of land that the tenant still needs in the new reduced land rental problem, $c' = c_{N \backslash S} \in \mathbb{R}^{N \backslash S}_{++}$ is the vector whose coordinates represent the amount of available land, r is the reservation price, which is equal as in the original land rental problem, and G' identifies the lessors in N' whose land is adjacent, directly or through lessors in S. If this procedure always gives the same result for agents in $N_0 \backslash S$ as before, we say that ψ is consistent.

Consistency For all $(N, K, E, c, r, G) \in \mathcal{L}$ and $S \subset N$ such that G_S is a connected network and $x(S) < E$, a rule ψ is *consistent* if

$$u_i\left(\psi\left(N', K', E', c', r, G'\right)\right) = u_i(\psi(N, K, E, c, r, G))$$

for all $i \in N'_0$, where $N' = N \setminus S$, $K' = K - px(S)$, $E' = E - x(S)$, $c'_i = c_i$ for all $i \in N'$, and

$$G' = G_{N'} \cup \left\{ \{i, j\} \in V^{N'} : \exists k, k' \in S \text{ s.t. } \{i, k\}, \{j, k'\} \in G \right\}.$$

Next proposition characterizes the second parametric subfamily of rules that satisfy RP, LM and consistency. We can see some examples of functions that determine these rules in Figure 1 (a), (b), (d) and (f), respectively.

Proposition 5.1 *A rule ψ satisfies RP, LM and consistency if and only if there exist $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$ such that:*

a) *The price is given as follows:*

a.1) If $r = 0$, then either $p = 0$ or $p = \frac{K}{E}$.

a.2) If $r > 0$ and $rE < \alpha K$, then $p = \frac{\beta}{\alpha} r$.

a.3) If $r > 0$ and $rE = \alpha K$, then $p = \frac{\beta}{\alpha} r$ or $p = \frac{K}{E}$.

a.4) If $r > 0$ and $rE > \alpha K$, then $p = \frac{K}{E}$.

b) *The amount of land when $p \neq r$ is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. ($\Leftarrow$) Let $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$ so that the price and the amount of land are given by a) and b), respectively. Let $f \in \mathcal{F}$ defined as follows: $f(0) \in \{0,1\}$, $f(t) = \frac{\beta}{\alpha} t$ if $0 < t < \alpha$, $f(\alpha) \in \{\beta, 1\}$ if $\alpha > 0$, and $f(t) = 1$ if $t > \alpha$. Then, the price can be written as $p = \frac{K}{E} f(\frac{rE}{K})$. Hence, by Theorem 3.1, ψ satisfies RP and LM. Let $L = (N, K, E, c, r, G)$ and $L' = (N', K', E', c', r, G')$ given as in the definition of consistency. We will prove that $u_i(x,p) = u_i(x',p')$ for all $i \in N_0 \setminus S$, where $(x,p) = \psi(L)$ and $(x',p') = \psi(L')$. Firstly, we prove that $p' = p$. We distinguish the following cases:

Case 1: $r = 0$ and $p = 0$. In this case, $f(0) = 0$. Hence, $p' = 0$ and $p = 0$.

Case 2: $r = 0$ and $p = \frac{K}{E}$. In this case, $f(0) = 1$. Hence, $p' = \frac{K'}{E'}$. Therefore, $p' = \frac{K'}{E'} = \frac{K - \frac{K}{E} x(S)}{E - x(S)} = \frac{K}{E} = p$.

Case 3: $r > 0$, $rE < \alpha K$ and $p = \frac{\beta}{\alpha} r$. Under $a.2$), we know that $p' = \frac{\beta}{\alpha} r$ when $r > 0$ and $rE' < \alpha K'$. Since $r > 0$, it is enough to check that $rE' < \alpha K'$. Equivalently, $r\,(E - x(S)) < \alpha \left(K - \frac{\beta}{\alpha} r x(S)\right)$. Since $rE < \alpha K$, it is enough to check that $rx(S) \geq \beta r x(S)$. This is trivially true when $rx(S) = 0$. Otherwise, it is equivalent to check that $\beta \leq 1$, which is true by definition.

Case 4: $r > 0$, $rE = \alpha K$ and $p = \frac{\beta}{\alpha} r$. In this case, $p = \frac{K}{E}\beta$, so $f\left(\frac{rE}{K}\right) = \beta$. Since $\frac{rE'}{K'} = \frac{r(E-x(S))}{K-\frac{K}{E}\beta x(S)} = \frac{rE(E-x(S))}{K(E-\beta x(S))}$ and $\beta \leq 1$, we have that $\frac{rE'}{K'} \leq \frac{rE(E-x(S))}{K(E-x(S))} = \frac{rE}{K} = \alpha$. Hence, $rE' \leq \alpha K'$. We will show that $f\left(\frac{rE'}{K'}\right) = \frac{\beta}{\alpha}\frac{rE'}{K'}$. We have two sub-cases: First, if $rE' < \alpha K'$, then it holds by $a.2)$ and the fact that $p' = \frac{K'}{E'} f\left(\frac{rE'}{K'}\right)$. Second, if $rE' = \alpha K'$, then $f\left(\frac{rE'}{K'}\right) = f(\alpha) \overset{(rE=\alpha K)}{=} f\left(\frac{rE}{K}\right) = \beta$. Since $rE' = \alpha K'$, we obtain that $f\left(\frac{rE'}{K'}\right) = \frac{\beta}{\alpha}\frac{rE'}{K'}$. Hence, $p' = \frac{K'}{E'} f\left(\frac{rE'}{K'}\right) = \frac{K'}{E'}\frac{\beta}{\alpha}\frac{rE'}{K'} = \frac{\beta}{\alpha} r = p$

Case 5: $r > 0$, $rE = \alpha K$, and $p = \frac{K}{E}$. Since $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and $p = \frac{K}{E}$, we deduce $f\left(\frac{rE}{K}\right) = 1$. Moreover, $\frac{rE'}{K'} = \frac{r(E-x(S))}{K-\frac{K}{E}x(S)} = \frac{rE(E-x(S))}{K(E-x(S))} = \frac{rE}{K} = \alpha$. Hence, $f\left(\frac{rE'}{K'}\right) = f(\alpha)$. Since $rE = \alpha K$ and $f\left(\frac{rE}{K}\right) = 1$, we deduce $f\left(\frac{rE'}{K'}\right) = 1$. Hence, $p' = \frac{K'}{E'} f\left(\frac{rE'}{K'}\right) = \frac{K'}{E'} = \frac{K-\frac{K}{E}x(S)}{E-x(S)} = \frac{K}{E} = p$.

Case 6: $rE > \alpha K$. In this case, $p = \frac{K}{E}$. Under $a.4)$, we know that $p' = \frac{K'}{E'}$ when $rE' > \alpha K'$. Since $\frac{K'}{E'} = \frac{K-\frac{K}{E}x(S)}{E-x(S)} = \frac{K}{E}$, it is enough to check that $rE' > \alpha K'$. This is equivalent to check that $r(E - x(S)) > \alpha\left(K - \frac{K}{E}x(S)\right)$. Equivalently, $r\left(E - x(S)\right) > \alpha K\left(\frac{E-x(S)}{E}\right)$. Since $E - x(S) > 0$, this is equivalent to $rE > \alpha K$, which is true in this case.

We check now that $u_i(\psi(L')) = u_i(\psi(L))$ for all $i \in N_0 \setminus S$. Assume first $i \in N \setminus S$. We need to prove that $(p-r)x'_i = (p-r)x_i$. This is trivially true when $p = r$. Hence, assume $p \neq r$. We need to prove $x'_i = x_i$. Since $c(N) = c(N \setminus S) + c(S)$, then $x'_i = \frac{c_i}{c(N\setminus S)}(E - x(S)) = \frac{c_i}{c(N\setminus S)}\left(E - \frac{c(S)E}{c(N)}\right) = \frac{c_i}{c(N\setminus S)}\left(\frac{c(N)-c(S)}{c(N)}\right)E = \frac{c_i}{c(N\setminus S)}\left(\frac{c(N\setminus S)}{c(N)}\right)E = \frac{c_i}{c(N)}E = x_i$. Assume now $i = 0$. We check that $u_0(\psi(L')) = u_0(\psi(L))$, or $K' - pE' = K - pE$. By definition, $K' - pE' = (K - px(S)) - p(E - x(S)) = K - pE$.

($\Rightarrow$) Let ψ be a rule that satisfies RP, LM and consistency. Under RP and LM, by Theorem 3.1 there exists $f \in F$ such that $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when $p \neq r$, $x_i = \frac{c_i}{c(N)}E$ for all $i \in N$.

Denote $L = (N, K, E, c, r, G)$ and let $S \subset N$ with $E > x(S)$ and $L' = (N', K', E', c', r, G')$ be defined as in the definition of consistency. Hence, we have $u_i(x, p) = u_i(x', p')$ for all $i \in N_0 \setminus S$, where $(x, p) = \psi(L)$ and

$(x', p') = \psi(L')$. In particular, $u_0(x,p) = u_0(x',p')$. By definition, this is equivalent to $K' - p'E' = K - pE$, or $K - px(S) - p'(E - x(S)) = K - pE$. Since $E \neq x(S)$, we deduce $p' = p$.

We will prove the existence of α and β with $\alpha \leq \beta$, such that the price is given as in a). Or, equivalently,

$$\begin{aligned}
&f(0) \in \{0,1\}, \\
&f(t) = \frac{\beta}{\alpha} t \text{ if } t \in]0,\alpha[, \\
&f(\alpha) \in \{\beta, 1\} \text{ when } \alpha > 0, \text{ and} \\
&f(t) = 1 \text{ if } t > \alpha.
\end{aligned} \tag{1}$$

If $f(t) = 1$ for all $t \in [0,1]$, then $\alpha = 0$ and $\beta = 1$ satisfy (1). Hence, we can assume that there exists $\widehat{t}$ such that $f\left(\widehat{t}\right) < 1$. Let $\alpha = Sup\{t : f(t) < 1\}$ and $\beta = Sup\{f(t) : f(t) < 1\}$. Then, $f(t) \geq t$ for all t implies $\alpha, \beta \in [0,1]$ and $\alpha \leq \beta$.

For each $r \in [0,1]$ and $\gamma \in]0,1[$, assume $L = (\{1,2\}, 1, 1, (\gamma, 1-\gamma), r, G)$ and $S = \{2\}$. So, $K' = K - px_2 = 1 - f(r)(1-\gamma)$ and $E' = E - x_2 = \gamma$. Hence, we have $L' = (\{1\}, 1 - f(r)(1-\gamma), \gamma, (\gamma), r, \emptyset)$.

So, $p = \frac{K}{E} f\left(\frac{rE}{K}\right) = f(r)$ and $p' = \frac{K'}{E'} f\left(\frac{rE'}{K'}\right) = \frac{1-f(r)(1-\gamma)}{\gamma} f\left(\frac{r\gamma}{1-f(r)(1-\gamma)}\right)$.

Since $p' = p$, from the last two expressions, we have

$$f\left(\frac{r\gamma}{1 - f(r)(1-\gamma)}\right) = \frac{f(r)\gamma}{1 - f(r)(1-\gamma)} \text{ for all } r \in [0,1] \text{ and } \gamma \in]0,1[. \tag{2}$$

In particular, for $r = 0$, we have $f(0) = \frac{f(0)\gamma}{1-f(0)(1-\gamma)}$ for all γ. If $f(0) \neq 0$, then $\gamma = 1 - f(0)(1-\gamma)$ for all γ, or equivalently, $(1 - f(0))\gamma = 1 - f(0)$ for all γ, which implies that $f(0) = 1$. Hence $f(0) \in \{0,1\}$. This is the first line of (1).

For $t > \alpha$, we have $f(t) = 1$. This is the fourth line of (1).

For each $r \in]0,1[$, we define $F^r(\delta) = \frac{r\delta}{1-f(r)(1-\delta)} \in [0,r]$ for all $\delta \in]0,1]$. If $f(r) < 1$, then F^r is a strictly increasing and continuous function, and its inverse is given by $G^r(t) = \frac{(1-f(r))t}{r-f(r)t} \in]0,1[$ for all $t \in]0,r]$. Given $t \in]0,r]$ and $r \in]0,1[$ such that $f(r) < 1$,

$$\begin{aligned}
f(t) = f\left(F^r\left(G^r(t)\right)\right) = f\left(\frac{rG^r(t)}{1-f(r)(1-G^r(t))}\right) \\
&\overset{(2)}{=} \frac{f(r)G^r(t)}{1-f(r)(1-G^r(t))} \\
&= \frac{f(r)\frac{(1-f(r))t}{r-f(r)t}}{1-f(r)\left(1-\frac{(1-f(r))t}{r-f(r)t}\right)} = \frac{f(r)}{r}t.
\end{aligned}$$

Assume $\alpha > 0$. Then we can fix $r \in]0,1[$ such that $f(r) < 1$. Hence, $\frac{f(t)}{t} = \frac{f(r)}{r}$ for all $t \in]0,r]$. We will prove that $f(t) = \theta t$ for all $t \in]0,\alpha[$, where $\theta = \frac{f(r)}{r}$. For all $t \in]0,\alpha[$, there exists $r' > t$ such that $f(r') < 1$ and $\frac{f(t)}{t} = \frac{f(r')}{r'}$. If $t < r$, we can take $r' = r$, thus $\frac{f(t)}{t} = \theta$. If $t \geq r$, then $r' > r$, thus $\frac{f(r)}{r} = \frac{f(r')}{r'} = \theta$. Hence, $f(t) = \theta t$ for all $t \in]0,\alpha[$. We will prove that $\theta = \frac{\beta}{\alpha}$, or equivalently $r\beta = \alpha f(r)$. We have two cases:

Case I. If $f(\alpha) = 1$, then $\beta = Sup\{f(t) : t \in]0,\alpha[\} = Sup\{\theta t : t \in]0,\alpha[\} = \theta\alpha$. Hence, $\theta = \frac{\beta}{\alpha}$.

Case II. If $f(\alpha) < 1$, then $\frac{f(\alpha)}{\alpha} = \theta$, so that $f(t) = \theta t$ for all $t \in]0,\alpha]$ and $\beta = Sup\{f(t) : t \in]0,\alpha]\} = Sup\{\theta t : t \in]0,\alpha]\} = \theta\alpha$. Hence, $\theta = \frac{\beta}{\alpha}$.

Then, the second line of (1) is satisfied.

From Case I and Case II we can deduce $f(\alpha) \in \{\beta, 1\}$. This is the third line of (1). ■

Given $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, we define $\psi^{\alpha,\beta}$ as the rule corresponding to the function given in Proposition 5.1 with $f(0) = 0$, $f(\alpha) = \beta$ and such that the amount of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.

Next property says that small changes in the land rental problem should not cause large changes in the chosen allocation.

Continuity The price p and the amount of land x are continuous functions on $\mathcal{L}$.

The rules that satisfy RP, LM, consistency and continuity constitute a particular subfamily of rules from the one determined in Proposition 5.1

and it is characterized in the next theorem. We can see some examples of functions that determine these rules in Figure 1 (a), (b) and (f), respectively.

Theorem 5.1 *A rule ψ satisfies RP, LM, consistency and continuity if and only if there exists $\alpha \in [0,1]$ such that:*

a)

$$p = \begin{cases} \frac{r}{\alpha} & \textit{if } rE < \alpha K \\ \frac{K}{E} & \textit{if } rE \geq \alpha K \end{cases}$$

and

b)

$$x_i = \frac{c_i E}{c(N)} \textit{ for all } i \in N.$$

Proof. ($\Leftarrow$) Let $\alpha \in [0,1]$ such that the price and amount of land are given by a) and b) respectively. Part a) can be written as $p = \frac{K}{E} f(\frac{rE}{K})$ with $f \in \mathcal{F}$ given as $f(t) = \frac{t}{\alpha}$ if $t < \alpha$ and $f(t) = 1$ if $t \geq \alpha$. Hence, by Proposition 5.1, ψ satisfies RP, LM and consistency. To prove it that satisfies continuity we still need to check that p is continuous at the points where $rE = \alpha K$. Equivalently, $\lim_{rE \to \alpha K^+} \frac{r}{\alpha} = \frac{K}{E}$, which holds trivially. Moreover, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$ also determines a continuous function.

($\Rightarrow$) Under RP, LM and consistency, by Proposition 5.1 there exist $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$ such that the price is given as in part a) of Proposition 5.1, and the amount of land when $p \neq r$, is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. By adding continuity, we will prove that $p = \frac{r}{\alpha}$ if $rE \leq \alpha K$ and $p = \frac{K}{E}$ if $rE \geq \alpha K$. Moreover, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. In this sense, we have that p is a continuous functions in $]0, \alpha[\cup]\alpha, 1]$. We still need to prove the following cases:

i) If $r = 0$, by continuity, $p = \lim_{t \to 0} \frac{\beta}{\alpha} t = 0 = \frac{t}{\alpha}$.

ii) If $rE = \alpha K$, by continuity, $\frac{\beta}{\alpha} r = \frac{K}{E}$. Then, $\beta = \frac{K\alpha}{rE} = 1$.

We need to prove that $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$ when $p = r$. Let $L^t = (N, K, E, c, r^t, G) \in \mathcal{L}$ with $lim_{t\to\infty} r^t = 0$ and $r^t > 0$ for all t. Therefore, $x_i^t = \frac{c_i E}{c(N)}$ for all $t \in [0, 1]$. Then, under continuity of the land function, $x_i = \lim_{r\to 0} \frac{c_i E}{c(N)} = \frac{c_i E}{c(N)}$. ■

Notice that the functions provided in Theorem 5.1 are those $\psi^{\alpha,\beta}$ with $\beta = 1$. In particular, $\psi^{0,1} = \psi^0$ (rule ψ^ω when $\omega = 0$) is a rule optimal for the tenant and ψ^1 (rule ψ^ω when $\omega = 1$) is a rule optimal for the lessors.

These two rules are the only ones that belong to both parametric subfamilies defined at Theorem 4.2 and Theorem 5.1, respectively. Both rules are characterized in the next proposition. We can see the functions that determine these rules in Figure 1 (a) and (b), respectively.

Proposition 5.2 *ψ^0 and ψ^1 are the only rules that satisfy RP, LM, WS2, consistency and continuity.*

Proof. It is straightforward to check that both ψ^0 and ψ^1 satisfy these properties. Let ψ be a rule that satisfies these properties. We will prove that ψ is either ψ^0 or ψ^1. On the one hand, by Theorem 5.1, there exists $\alpha \in [0, 1]$ such that $p = \frac{K}{E}$ when $rE = \alpha K$. On the other hand, by Theorem 4.2, there exists $\omega \in [0, 1]$ such that $p = \frac{K-(K-rE)\omega}{E}$ for all r. Hence, given $r = \frac{\alpha K}{E}$, we have $\frac{K}{E} = \frac{K-(K-rE)\omega}{E}$, or $(K - rE)\omega = 0$. There are two possibilities: On the one hand, $\omega = 0$ which gives $\psi = \psi^0$. On the other hand, $K = rE$ and $rE = \alpha K$ imply $K = \alpha K$. Since $K > 0$, we deduce $\alpha = 1$ which gives $\psi = \psi^{1,1} = \psi^1$. ■

A summary of the results is presented in Table 1.

Property	ψ^f	ψ^ω	$\psi^{\alpha,1}$	$\psi^{\frac{1}{2}}$	ψ^0	ψ^1
Reassignment-proofness	Yes*	Yes*	Yes*	Yes*	Yes	Yes
Land Monotonicity	Yes*	Yes*	Yes*	Yes*	Yes	Yes
Standard for 2-person	-	-	No	Yes*	No	No
Weighted Standard for 2-person	-	Yes*	-	Yes	Yes	Yes
Consistency	-	-	Yes*	No	Yes	Yes
Continuity	-	Yes	Yes*	Yes	Yes	Yes

Table 1: Summary of the results. Symbol * means that this property, together with others in the same column, characterizes the family/rule.

References

Akiwumi, F. A. (2014). Strangers and Sierra Leone mining: cultural heritage and sustainable development challenges. *Journal of Cleaner Production*, 84:773–782.

Albizuri, M. J. and Zarzuelo, J. M. (2007). The dual serial cost-sharing rule. *Mathematical Social Sciences*, 53(2):150–163.

Arellano-Yanguas, J. (2011). Aggravating the resource curse: Decentralisation, mining and conflict in Peru. *The Journal of Development Studies*, 47(4):617–638.

Azima, A., Sivapalan, S., Zaimah, R., Suhana, S., and Yusof, H. (2015). Boundry and customary land ownership dispute in Sarawak. *Mediterranean Journal of Social Sciences*, 6(4).

Chun, Y. (1988). The proportional solution for rights problems. *Mathematical Social Sciences*, 15(3):231 – 246.

de Frutos, M. A. (1999). Coalitional manipulations in a bankruptcy problem. *Review of Economic Design*, 4(3):255–272.

Erlanson, A. and Flores-Szwagrzak, K. (2015). Strategy-proof assignment of multiple resources. *Journal of Economic Theory*, 159, Part A:137 – 162.

Fraser, J. (2018). Mining companies and communities: Collaborative approaches to reduce social risk and advance sustainable development. *Resources Policy*, Forthcoming.

Gildenhuys, J. A. (2005). Indigenous peoples' rights to minerals and the mining industry: Current developments in South Africa from a national and international perspective. *Journal of Energy & Natural Resources Law*, 23(4):465–481.

Gómez-Rúa, M. and Vidal-Puga, J. (2011). Merge-proofness in minimum cost spanning tree problems. *International Journal of Game Theory*, 40(2):309–329.

Hart, S. and Mas-Colell, A. (1989). Potential, value, and consistency. *Econometrica*, 57(3):589–614.

Helwege, A. (2015). Challenges with resolving mining conflicts in Latin America. *The Extractive Industries and Society*, 2(1):73–84.

Huettner, F. (2015). A proportional value for cooperative games with a coalition structure. *Theory and Decision*, 78(2):273–287.

Jaramillo, P., Kayı, Ç., and Klijn, F. (2014). Asymmetrically fair rules for an indivisible good problem with a budget constraint. *Social Choice and Welfare*, 43(3):603–633.

Ju, B.-G. (2003). Manipulation via merging and splitting in claims problems. *Review of Economic Design*, 8(2):205–215.

Ju, B.-G. (2013). Coalitional manipulation on networks. *Journal of Economic Theory*, 148(2):627 – 662.

Ju, B.-G., Miyagawa, E., and Sakai, T. (2007). Non-manipulable division rules in claim problems and generalizations. *Journal of Economic Theory*, 132(1):1 – 26.

Ju, B.-G. and Moreno-Ternero, J. (2011). Progressive and merging-proof taxation. *International Journal of Game Theory*, 40(1):43–62.

Kaye, J. L. and Yahya, M. (2012). *Land and Conflict: Tool and guidance for preventing and managing land and natural resources conflict*. UN Interagency Framework Team for Preventive Action. Guidance Note.

Kojima, F. and Manea, M. (2010). Incentives in the probabilistic serial mechanism. *Journal of Economic Theory*, 145(1):106 – 123.

Kominers, S. D. and Weyl, E. G. (2012). Holdout in the assembly of complements: A problem for market design. *American Economic Review: Papers & Proceedings*, 102(3):360–65.

Koster, M. (2012). Consistent cost sharing. *Mathematical Methods of Operations Research*, 75(1):1–28.

Massó, J., Nicolò, A., Sen, A., Sharma, T., and Ülkü, L. (2015). On cost sharing in the provision of a binary and excludable public good. *Journal of Economic Theory*, 155:30 – 49.

Moreno-Ternero, J. D. (2006). Proportionality and non-manipulability in bankruptcy problems. *International Game Theory Review*, 8(1):127–139.

Moreno-Ternero, J. D. (2007). Bankruptcy rules and coalitional manipulation. *International Game Theory Review*, 9(2):411–424.

Morimoto, S. and Serizawa, S. (2015). Strategy-proofness and efficiency with non-quasi-linear preferences: a characterization of minimum price Walrasian rule. *Theoretical Economics*, 10(2):445–487.

Moulin, H. (1987). Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory*, 16(3):161–186.

Moulin, H. (2002). Axiomatic cost and surplus sharing. In Kenneth J. Arrow, A. S. and Suzumura, K., editors, *Handbook of Social Choice and Welfare*, volume I, chapter 6, pages 289–357. Elsevier.

Moulin, H. (2007). On scheduling fees to prevent merging, splitting, and transferring of jobs. *Mathematics of Operations Research*, 32(2):266–283.

Moulin, H. (2008). Proportional scheduling, split-proofness, and merge-proofness. *Games and Economic Behavior*, 63:567–587.

Moulin, H. and Shenker, S. (2001). Strategyproof sharing of submodular costs: budget balance versus efficiency. *Economic Theory*, 18(3):511–533.

Nguyen, N., Boruff, B., and Tonts, M. (2018). Fool's gold: Understanding social, economic and environmental impacts from gold mining in quang nam province, vietnam. *Sustainability*, 10(5):1355–.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 2(4):345 – 371.

Sarkar, S. (2015). *Mechanism Design for Land Acquisition.* PhD thesis, TERI University.

Sarkar, S. (2017). Mechanism design for land acquisition. *International Journal of Game Theory*, 46(3):783–812.

Sen, A. (2007). The theory of mechanism design: An overview. *Economic and Political Weekly*, 42(49):8–13.

Sosa, I. (2011). License to operate: Indigenous relations and free prior and informed consent in the mining industry. *Sustainalytics, Amsterdam, The Netherlands.*

Sprumont, Y. (2005). On the discrete version of the Aumann-Shapley cost-sharing method. *Econometrica*, 73(5):1693–1712.

Sun, N. and Yang, Z. (2003). A general strategy proof fair allocation mechanism. *Economics Letters*, 81(1):73–79.

Svensson, L.-G. (2009). Coalitional strategy-proofness and fairness. *Economic Theory*, 40(2):227–245.

Tetreault, D. (2015). Social environmental mining conflicts in Mexico. *Latin American Perspectives*, 42(5):48–66.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, 45(3):249 – 297.

Thomson, W. (2008). Two families of rules for the adjudication of conflicting claims. *Social Choice and Welfare*, 31(4):667–692.

Thomson, W. (2015a). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update. *Mathematical Social Sciences*, 74:41 – 59.

Thomson, W. (2015b). For claims problems, compromising between the proportional and constrained equal awards rules. *Economic Theory*, 60(3):495–520.

United Nations (2007). United Nations Declaration on the Rights of Indigenous Peoples (UNDRIP). Adopted by the General Assembly on 2 October 2007.

van den Brink, R., Funaki, Y., and Ju, Y. (2013). Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare*, 40(3):693–714.

van der Ploeg, F. and Rohner, D. (2012). War and natural resource exploitation. *European Economic Review*, 56(8):1714 – 1729.

Walter, M. and Urkidi, L. (2015). Community mining consultations in Latin America (2002–2012): The contested emergence of a hybrid institution for participation. *Geoforum*, In press.

Welker, M. A. (2009). Corporate security begins in the community: Mining, the corporate social responsibility industry, and environmental advocacy in Indonesia. *Cultural Anthropology*, 24(1):142–179.