One-way and two-way cost allocation in hub network problems*

G. Bergantiños †1 and J. Vidal-Puga ‡1

[1]Universidade de Vigo, Spain

November 2, 2016

Abstract

We study hub problems where a set of nodes send and receive data from each other. In order to reduce costs, the nodes use a network with a given set of hubs. We address the cost sharing aspect by assuming that nodes are only interested in either sending or receiving data, but not both (one-way flow) or that nodes are interested in both sending and receiving data (two-way flow). In both cases, we study the non-emptiness of the core and the Shapley value of the corresponding cost game.

Keywords: hub network, cost allocation, core, Shapley value.

*This work is partially supported by research grants ECO2014-52616-R from the Spanish Ministerio de Economía y Competitividad, GRC 2015/014 from Xunta de Galicia, and 19320/PI/14 from Fundación Séneca de la Región de Murcia.

†Facultade de Ciencias Económicas e Empresariais. 36310 Vigo, Spain. E-mail: gbergant@uvigo.es

‡Facultade de Ciencias Sociais e da Comunicación. 36005 Pontevedra, Spain. E-mail: vidalpuga@uvigo.es

1 Introduction

Hub networks play a fundamental role in modeling telecommunication, transportation, and parcel delivery systems. Assume that there are users located at different geographical nodes who need to send a certain flow of data or goods to each other through costly connections. A planner needs to locate an optimal number of hub facilities at some nodes so that each other node is connected to exactly one hub and all the hubs are connected to one another at a reduced cost (due to economies of scale). Hence, the optimal flow of data/goods between any pair of origin-destination nodes has a length of at most four: It must go from the point of origin to its assigned hub (when the origin is not itself a hub), then to the hub assigned to the destination (if it is a different node) and finally to the destination (again, if it is not itself a hub). This topology is applied to Internet connections (Bailey, 1997), telecommunications between local networks (Greenfield, 2000), satellite communication (Helme and Magnanti, 1989), airline networks (Bryan and O'Kelly, 1999; Yang, 2009), and small package delivery (Sim et al., 2009).

Several classes of hub problems have been studied. We mention some of them. Aykin (1994) considers that hubs have limited capacities and direct connections between nohubs are allowed. Ernst and Krishnamoorthy (1999) consider the case where there are capacity restrictions but they apply only to the traffic arriving at hubs from nohubs. Sasaki and Fukushima (2003) consider the case where there are capacity constraints on hubs and arcs. Labbé et al. (2005) consider the case where each hub has a limited capacity as regard the traffic that passes through it. The main issue addressed in these papers is the study of algorithms for computing optimal ways of sending goods between the nodes in such a way that the total cost is minimized. Of course the location of the hubs plays a relevant role in the minimization problem. See Alumur and Kara (2008) and Farahani et al. (2013) for surveys on this literature.

Another interesting question is how to divide the cost of sending the good from one node to another. This question has been successfully addressed in several kinds of problem. We mention some of them: Guardiola et al. (2009) study production-inventory problems where players share production processes and warehouse facilities, Bergantiños and Kar (2010), Bogomolnaia and Moulin (2010), Dutta and Mishra (2012) and Trudeau (2012, 2014) consider the cost of connecting agents to a source, Moulin (2014) consider users that need to connect a pair of target nodes in a network, and Alcalde-Unzu et al. (2015) consider the cost of cleaning a river. However, few papers have studied this issue in hub problems. We mention three: in all of them the first step is to consider a class of hub problems, the next is to associate a cooperative game with each problem in the class, and the last is to study the core of such problems. If the core is non empty, an allocation in the core could be considered as a nice way of sharing the cost among the agents.

Skorin-Kapov (1998) studies p-hub allocation problems, where p hubs must be optimally allocated. Several cooperative games are considered depending on who the agents are (nodes or pairs of nodes) and what coalitions can do (whether they must use the optimal network for the whole problem or can construct the optimal network of the reduced problem induced by the coalition). The core of such games is studied. Some games have an empty core but others do not. Finally, the nucleolus of such games is considered.

Skorin-Kapov (2001) studies hub-like networks, which involves a p-hub median problem where direct connection between nodes is possible. Moreover, there are savings when the traffic is high. He defines several associated cooperative games where the set of agents are the links. He shows that some of them can have an empty core, in other cases the core is a singleton, and in other cases it has many points.

Matsubayashi et al. (2005) consider the case where the number of hubs to be located is arbitrary; there is a cost of opening a hub and there is a congestion cost associated with nodes (the greater the flow through a node, the greater its cost). They also define an associated cooperative game and study its core. In the cooperative game the players are the nodes and the characteristic function is defined assuming that each coalition cooperatively constructs a network to minimize the total cost associated with hubs in the coalition and communications initiated also by nodes in the coalition. Moreover, each coalition simply assumes that the rest of the nodes do not establish any hub nodes and the coalition can determine the routing of all the traffic generated by the other nodes. Given this, they prove that the core could be empty, but they find a sufficient condition for the non-emptiness of the core and propose an allocation in the core when it is non-empty.

Our also focuses on the cost sharing issue. We consider two cases. In the first case (called one-way flow) we assume, as in Skorin-Kapov (1998, 2001) and Matsubayashi et al. (2005), that nodes are only interested in sending flow. In the second case (called two-way flow) we assume that agents are interested in both sending and receiving flow. Internet connections are a good example of situations covered by this case.

We study the existence of core allocations and, unlike Skorin-Kapov (1998, 2001) and Matsubayashi et al. (2005), we also present and characterize two rules that belong to the core and also satisfy other nice properties.

We now summarize our results for one-way flow. We consider two cooperative games associated with each hub problem and related to those presented by Skorin-Kapov (1998). In the first game we assume that nodes can only use the optimal network for the whole problem. In the second game we consider that nodes can construct their own optimal networks. In both games, when we compute the cost of a coalition we consider only the flow sent by nodes in the coalition.

We study the cores of both games. The core of the first game has many points. In any allocation in the core each node pays the cost of sending its flow and the cost of any hub is divided in any way among the nodes that use such hub. The core of the second game could be empty. In particular, we prove that the Shapley value corresponds to the allocation where each node pays the cost of sending its flow and the cost of any hub is divided equally among the nodes that use it. Thus, the Shapley value belongs to the core. We also provide two axiomatic characterizations of it. The first one uses core selection (the allocation must be in the core) and equal treatment on hubs (if the cost of a hub increases then any pair of agents such that either both need the hub or neither needs the hub are affected in the same way). The second characterization uses positivity (no node can obtain profits), equal treatment on hubs, independence of irrelevant hubs (nodes are not affected by a change in the cost of hubs that they do not need), and independence of irrelevant flows (if the flow between two nodes increases, other agents should not be affected).

We now summarize our results for the two-way flow. The study is similar to the one-way flow. We associate two games. The first game is concave and hence its core is non empty. It consists of the convex hull of the vector of marginal contributions. The second game could have an empty core.

We study the Shapley value of the first game. Since the game is concave it belongs to the core. We prove that the Shapley value corresponds to the allocation where the cost of sending flow between any pair of nodes is divided equally between the two nodes. Moreover, the cost of each hub is divided equally between the nodes that need the hub to send or receive their flows. Finally, we provide two axiomatic characterizations. The first one uses core selection, equal treatment on hubs, and equal treatment on flows (if there is a flow between a pair of nodes and it increases then both nodes must be affected in the same way). The second characterization uses positivity, independence of irrelevant hubs, independence of irrelevant flows, equal treatment on hubs, and equal treatment on flows.

The paper is organized as follows. Section 2 presents the model. Section 3 studies the one-way flow case, where the nodes are only interested in sending or receiving flow, but not both. Section 4 studies the two-way flow case, where the nodes are interested in both sending and receiving flow.

2 The model

We consider situations where a group of agents, located at different points, want to send and receive some specific good, which is sent through a costly network. Besides, we could locate some hubs in the agent's points. All hub agents are connected among them but each non-hub agent is connected to only a hub agent. We now introduce the model formally.

$N = \{1, \ldots, n\}$ is a finite set of nodes (also called agents).

$C = (c_{ij})_{i,j \in N}$ is a cost matrix. For each $i, j \in N$, c_{ij} is the cost of sending a unit of flow from node i to node j. We assume $c_{ii} = 0$, $c_{ij} = c_{ji} \geq 0$ and $c_{ik} \leq c_{ij} + c_{jk}$ for all $i, j, k \in N$.

$F = (f_{ij})_{i,j \in N}$ is the flow matrix. For each $i, j \in N$, f_{ij} represents the amount of flow from node i to node j. We assume $f_{ij} \geq 0$ and $f_{ii} = 0$ for all $i, j \in N$. Notice that we do not assume $f_{ij} = f_{ji}$, *i.e.* the flow is not necessarily symmetric.

$d = (d_i)_{i \in N}$ indicates the cost of maintaining or constructing a hub at each node. We assume $d_i \geq 0$ for all $i \in N$.

$\alpha \in [0, 1]$ is the discounting factor of the cost when flow goes between a pair of hubs. Namely, if i and j are both hubs, then the cost of sending a unit of flow from i to j is αc_{ij} (instead of c_{ij})[1].

The first issue is to locate an optimal number of hubs, selected from the set of nodes. Besides, each non-hub is linked to exactly one hub and all the hubs are connected to each other. The triangular inequation $c_{ik} \leq c_{ij} + c_{jk}$ assures that the optimal path origin-destination uses at most two hubs. When there is a hub in node $i \in N$, we say with some abuse of notation that node i is a hub. Otherwise, we say that node i is a non-hub.

A *hub network* on N is determined by a nonempty set $H \subseteq N$ and a function $h : N \setminus H \longrightarrow H$ such that $h(i)$ is the hub linked to non-hub i. Let $\mathcal{H}$ be the set of all hub networks on N. For notational convenience, we write $h(i) = i$ when $i \in H$, so that h is a function from N onto H. Besides, we

[1] A generalization would be to assume that these costs are given by another cost matrix $C^h = \left(c^h_{ij}\right)_{i,j \in N}$ with $c^h_{ij} \leq c_{ij}$ for all $i, j \in N$. In our case, $C^h = \alpha C$.

also write $\overline{h}$ for the network associated with the function h. Namely

$$\overline{h} = \{\{i, h(i)\} : i \in N \setminus H\}.$$

Thus, given two nodes $i, j \in N$, flow from node i to node j goes first from node i to hub $h(i)$, then to hub $h(j)$ and finally to node j ($i = h(i)$ and/or $h(i) = h(j)$ and/or $h(j) = j$ are possible).

The *cost* of a hub network h is given by

$$\sum_{i\in N}\sum_{j\in N}\left(c_{ih(i)} + \alpha c_{h(i)h(j)} + c_{h(j)j}\right) f_{ij} + \sum_{i\in H} d_i.$$

For simplicity, we denote

$$\lambda_{ij}^h = \left(c_{ih(i)} + \alpha c_{h(i)h(j)} + c_{h(j)j}\right) f_{ij}$$

so that the cost is

$$\sum_{i\in N}\sum_{j\in N}\lambda_{ij}^h + \sum_{i\in H} d_i.$$

A hub network $h \in \mathcal{H}$ where

$$\min_{h\in\mathcal{H}}\left\{\sum_{i\in N}\sum_{j\in N}\lambda_{ij}^h + \sum_{i\in H} d_i\right\}$$

is reached is called *optimal*. Since $\mathcal{H}$ is finite, there is always at least one optimal hub network.

A *hub network problem* is a tuple $P = (N, C, F, d, \alpha, h)$ where h is a hub network.

Notice that we have not assumed that h is an optimal hub network. We know that to compute an optimal hub network is NP hard. Thus, in many practical situations we use heuristics to decide the hub network h to be constructed. Hence, we do not know exactly if such hub network is optimal or not. We make a very weak assumption on h, all hubs are needed in order to send the flow. Namely, for all $k \in H$ there exist $i, j \in N$ with $f_{ij} > 0$ and $k \in \{h(i), h(j)\}$. Even our results could be reformulated for the case in which some hubs are not needed, in order to make easier the reading, we have decided to present it making this assumption.

We now define $c(P)$ as the cost associated with the hub network h. Namely,

$$c(P) = \sum_{i \in N} \sum_{j \in N} \lambda_{ij}^{h} + \sum_{i \in H} d_i. \tag{1}$$

In many cases after finding an optimal (or quasi optimal) hub network, we need to divide the cost of such network among the agents. A *rule* is a function R that assigns to each hub network problem P an allocation $R(P) \in \mathbb{R}^N$ satisfying

$$\sum_{i \in N} R_i(P) = c(P). \tag{2}$$

Our aim is to study the cost allocation problem generated by each hub network problem P. We are interested into studying fair allocations. The idea is to propose desirable properties and try to find a rule satisfying many of them.

We consider two cases depending on the needs of the agents. In the *one-way flow* case, we assume that each agent is only interested in the flow that leaves from it (the case where each agent is only interested in the flow that arrives to it is similar). In the *two-way flow* case, each agent is interested in both the flow that arrives to it and the flow that leaves from it.

A *cost game* is a pair $(N, \hat{c})$ where N is the set of agents and $\hat{c} : 2^N \to \mathbb{R}$ is a cost function satisfying $\hat{c}(\emptyset) = 0$. Each nonempty subset $S \subseteq N$ is called a *coalition*, and $\hat{c}(S)$ denotes the cost of providing the needs of all agents in S. Since $\hat{c}$ depends on N, we write $\hat{c}$ instead of $(N, \hat{c})$.

We say that $\hat{c}$ is *concave* if for all $l \in T \subset S \subseteq N$, we have $\hat{c}(S) - \hat{c}(S \setminus \{l\}) \leq \hat{c}(T) - \hat{c}(T \setminus \{l\})$.

The *core* of a cost game $\hat{c}$ is defined as

$$Core(\hat{c}) = \left\{ y \in \mathbb{R}^N : \sum_{i \in N} y_i = \hat{c}(N) \text{ and } \sum_{i \in S} y_i \leq \hat{c}(S) \, \forall S \subset N \right\}.$$

It is well known that the core may be empty. Nevertheless, when the cost game is concave, the core is non-empty.

The Shapley value (Shapley, 1953) is defined as the allocation $Sh(\hat{c})$ such that

$$Sh_i(\hat{c}) = \sum_{S \subset N \backslash \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \left[\hat{c}(S \cup \{i\}) - \hat{c}(S)\right]$$

for each $i \in N$.

3 One-way flow

In this section, we assume that agents are interested only in the flow that leaves. The case in which they are interested only in the flow that arrives is completely analogous. We first associate to each hub network problem a cost game. Later we study the core and the Shapley value of such game.

For each hub network problem P, we associate the cost game c_P^{of} where for each $S \subseteq N$, $c_P^{of}(S)$ is the cost of sending the flow of all agents in S to all agents through the hub network h. The cost game c_P^{of} models situations where the hub network h (with associated set of hubs H) has already been constructed. Thus, d could be considered as a vector of maintenance costs. Agents in each coalition are only interested in the hubs they need for sending their flow. We now define this cost game formally.

For each $S \subseteq N$, let $H_S^{of} \subseteq H$ denote the set of hubs needed for sending the flow of agents in S. Namely,

$$H_S^{of} = \{k \in H : \exists i \in S, j \in N \text{ with } f_{ij} > 0 \text{ and } k \in \{h(i), h(j)\}\}.$$

Given $i \in N$, we write H_i^{of} instead of $H_{\{i\}}^{of}$. Notice that $H_S^{of} = \bigcup_{i \in S} H_i^{of}$ for all $S \subseteq N$.

Now,

$$c_P^{of}(S) = \sum_{i \in S} \sum_{j \in N} \lambda_{ij}^h + \sum_{i \in H_S^{of}} d_i. \tag{3}$$

When no confusion arises we write $c^{of}(S)$ instead of $c_P^{of}(S)$.

Remark 3.1 *Skorin-Kapov (1998) associates several games with each hub network problem. One of them, c_1 is closely related with c_P^{of}. Actually it could* *be considered as a generalization of c_1. In our model when h has p hubs and $d_i = 0$ for all i, c_P^{of} coincides with c_1. Besides, Skorin-Kapov (1998) proves that the core of c_1 contains the single allocation where each agents pays the cost of sending its flow.*

3.1 The core

In the next theorem we prove that the core of c^{of} is the set of allocations in which each agent pays the cost of sending its flow. Besides, the cost of any hub is divided in any way among the agents that need the hub for sending its flow.

Theorem 3.1 *For each hub network problem P the core of c^{of} is nonempty, and it is given by*

$$Core\left(c^{of}\right) = \left\{ \begin{array}{c} x \in \mathbb{R}^N : \sum_{i\in N} x_i = c(P), x_i = \sum_{j\in N} \lambda_{ij}^h + y_i \forall i \in N \\ \text{where } y \in \mathbb{R}_+^N \text{ and } \sum_{i\in S} y_i \leq \sum_{i\in H_S^{of}} d_i \forall S \subset N \end{array} \right\}.$$

Proof. "$\supseteq$" is obvious.

We now prove "$\subseteq$". Let $x \in Core\left(c^{of}\right)$. Then, for each $i \in N$, $x_i = \sum_{j\in N} \lambda_{ij}^h + y_i$ where $y_i = x_i - \sum_{j\in N} \lambda_{ij}^h$. Since, $x \in Core\left(c^{of}\right)$, for each $S \subset N$,

$$c^{of}(S) = \sum_{i\in S}\sum_{j\in N} \lambda_{ij}^h + \sum_{i\in H_S^{of}} d_i \geq \sum_{i\in S} x_i = \sum_{i\in S}\sum_{j\in N} \lambda_{ij}^{h^N} + \sum_{i\in S} y_i$$

and thus $\sum_{i\in S} y_i \leq \sum_{i\in H_S^{of}} d_i$. It only remains to prove that $y \in \mathbb{R}_+^N$. Suppose not. Let $j \in N$ be such that $y_j < 0$. Thus,

$$\begin{aligned}
\sum_{i\in N\backslash\{j\}} x_i &= \sum_{i\in N} x_i - x_j = c^{of}(N) - x_j \\
&= \sum_{i\in N}\sum_{j\in N} \lambda_{ij}^h + \sum_{i\in H} d_i - \sum_{j\in N} \lambda_{ij}^h - y_j \\
&= \sum_{i\in N\backslash\{j\}}\sum_{j\in N} \lambda_{ij}^h + \sum_{i\in H} d_i - y_j \\
&> \sum_{i\in N\backslash\{j\}}\sum_{j\in N} \lambda_{ij}^h + \sum_{i\in H} d_i
\end{aligned}$$

since $H^{of}_{N\setminus\{j\}} \subseteq H$,

$$> \sum_{i\in N\setminus\{j\}} \sum_{j\in N} \lambda_{ij}^{h^N} + \sum_{i\in H^{of}_{N\setminus\{j\}}} d_i = c^{of}(N \setminus \{j\})$$

which is a contradiction. ■

Skorin-Kapov (1998) also considers the game c_1^*, which is obtained as c_1 but assuming that each coalition can build their optimal network. Namely, instead of using the hubs given by h, each coalition can locate hubs where they want. Skorin-Kapov (1998) proves that the core of c_1^* could be empty.

In our case the same happens. The core of c^{of*} could be empty. We will see it by proving that the core of the following intermediate situation could be also empty.

Assume that the optimal hub network is not unique. Thus, we should decide which one to construct. It could be the case that some agents prefer one over the other (for instance if an agent is a hub the cost of sending their flow should be smaller). Thus, we can define the cost of a coalition as the minimum over all optimal hub networks. Namely, for each $S \subseteq N$,

$$c^*(S) = \min_{h\in\mathcal{H}, h \text{ is optimal}} \left\{ c^{of}_{P(h)}(S) \right\}$$

where $P(h)$ is the hub network problem induced by the optimal hub network h. Next example shows that the core of c^* can be empty.

Example 3.1 *Let $N = \{1,2,3\}$, $c_{ij} = 1$ for all $i,j \in N$, $f_{12} = f_{23} = f_{31} = 1$, $f_{21} = f_{32} = f_{13} = 10$, $\alpha = 1$, and $d_i = 6$ for all $i \in N$. There exist three optimal hub networks $\{h^i\}_{i\in N}$, corresponding to putting a single hub in each node $i \in N$, respectively. Furthermore, each two-node coalition would prefer a different hub location. Coalition $\{1,2\}$ would prefer the hub to be at 1, because*

$$c^{of}_{P(h^1)}(\{1,2\}) = 29 \leq \min\left\{ c^{of}_{P(h^2)}(\{1,2\}), c^{of}_{P(h^3)}(\{1,2\}) \right\};$$

coalition $\{1,3\}$ would prefer to locate the hub at 3, because

$$c^{of}_{P(h^3)}(\{1,3\}) = 29 \leq \min\left\{ c^{of}_{P(h^1)}(\{1,3\}), c^{of}_{P(h^2)}(\{1,3\}) \right\};$$

and coalition $\{2,3\}$ *would prefer to locate the hub at* 2*, because*

$$c^{of}_{P(h^2)}(\{2,3\}) = 29 \leq \min\left\{c^{of}_{P(h^1)}(\{2,3\}), c^{of}_{P(h^3)}(\{2,3\})\right\}.$$

Let x *be a core allocation. Then*

$$\begin{aligned} 100 = 2c^*(N) &= 2(x_1+x_2+x_3) \\ &= (x_1+x_2)+(x_1+x_3)+(x_2+x_3) \\ &\leq c^{of}_{P(h^1)}(\{1,2\}) + c^{of}_{P(h^3)}(\{1,3\}) + c^{of}_{P(h^2)}(\{2,3\}) \\ &= 29+29+29 = 87, \end{aligned}$$

which is a contradiction.

Thus, the core of c^* *is empty.*

3.2 The Shapley value

We now study the Shapley value of c^{of}, which we call the *Shapley rule.* We first give an explicit formula. Later, we provide two axiomatic characterizations.

In the next theorem we prove that in the Shapley rule each agent pays the cost of sending its flow. Besides, the cost of any hub is divided equally among the agents that need the hub for sending their flow.

Theorem 3.2 *For each hub network problem* P *and each* $i \in N$,

$$Sh_i\left(c^{of}_P\right) = \sum_{j\in N}\lambda^h_{ij} + \sum_{j\in H^{of}_i}\frac{d_j}{\left|\left\{k\in N : j\in H^{of}_k\right\}\right|}.$$

Proof. We consider several cost games. Let c^0 be defined as $c^0(S) = \sum_{i\in S}\sum_{j\in N}\lambda^h_{ij}$ for each $S\subseteq N$. For each $j\in N$, let c^j be defined as $c^j(S) = d_j$ if $j\in H^{of}_S$ and $c^j(S)=0$ otherwise. Thus, for each $S\subseteq N$, $c^{of}(S) = c^0(S) + \sum_{j\in N}c^j(S)$. Since the Shapley value is additive on c, we have that for each $i\in N$, $Sh_i\left(c^{of}\right) = Sh_i\left(c^0\right) + \sum_{j\in N}Sh_i\left(c^j\right)$. Since c^0 is an additive game (there exists $a \in \mathbb{R}^N$ such that for each $S \subseteq N$, $c^0(S) = \sum_{j \in S} a_i$) we deduce that $Sh_i(c^0) = \sum_{j \in N} \lambda_{ij}^h$. For each $j \in N$, in the cost game c^j, all agents that need hub j (i.e. all $k \in N$ such that $j \in H_k^{of}$) are symmetric and the agents that do not need hub j are dummy. Thus, for each $j \in N$,

$$Sh_i\left(c^j\right) = \begin{cases} \frac{d_j}{\left|\left\{k \in N : j \in H_k^{of}\right\}\right|} & \text{if } j \in H_i^{of} \\ 0 & \text{otherwise,} \end{cases}$$

from where it is straightforward to check the result. ■

We now define several properties.

The first property says that no agent should obtain profit.

Positivity (Pos) For any hub network problem P and each $i \in N$, we have $R_i(P) \geq 0$.

The second property says that equal agents must pay the same. Consider the following example.

Example 3.2 *Let P be such that $N = \{1, 2, 3\}$, $c_{ij} = 3$ and $f_{ij} = 1$ for all $i, j \in N$,and $d_i = 6$ for all $i \in N$. There are three optimal hub networks. We construct a hub in node i and we join the other nodes to node i. Thus, $c(P) = 30$. Assume that h is the optimal hub network where the hub is at node 1 (the other cases are analogous). Thus, c^{of} is defined as follows:*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c^{of}(S)$	*12*	*15*	*15*	*21*	*21*	*24*	*30.*

Notice that even agents are symmetric in C, F, and d. Nevertheless, in c^{of}, agents 2 and 3 are symmetric but agents 1 and 2 are not.

Since we are dealing with situations where h is given, we should consider such hub network when defining equal nodes. Thus, given a hub network problem P we say that nodes i and j are *equal* when several conditions hold: First, $f_{ik} = f_{jk}$ for all $k \in N \setminus \{i, j\}$. Second, $f_{ij} = f_{ji}$. Third, $i \in H$ if and only if $j \in H$ (namely, node i is a hub if and only if node j is a hub). Forth,

$\{i,k\} \in \overline{h}$ if and only if $\{j,k\} \in \overline{h}$ (namely, if nodes i and j are nonhubs then both are connected to the same hub). Fifth, for each $\{i,k\}, \{j,k\} \in \overline{h}$, $c_{ik} = c_{jk}$.

Equal Treatment of Equals (ETE) For any hub network problem P and each pair of equal nodes $i, j \in N$, we have that $R_i(P) = R_j(P)$.

As in the case of ETE, the next properties are defined considering the hub network h as fixed. The first of them says that we must select an allocation in the core of the problem.

Core Selection (CS) For any hub network problem P, we have that

$$R(P) \in Core\left(c_P^{of}\right).$$

The next property says that if a node does not send any flow, then it pays nothing.

Null Flow (NF) For any hub network problem P and each $i \in N$ such that $f_{ij} = 0$ for all $j \in N \setminus \{i\}$, we have that $R_i(P) = 0$.

The next property says that if the flow leaving node i increases, then node i cannot pay less.

Flow Monotonicity (FM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exist $i, j \in N$ satisfying $f_{ij} \geq f'_{ij}$ and $f_{kl} = f'_{kl}$ otherwise, then $R_i(P) \geq R_i(P')$.

The next property says that if the maintenance cost of a hub increases, then no node requiring such hub could pay less.

Hub Monotonicity (HM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then for each agent i such that $k \in H_i^{of}$, we have that $R_i(P) \geq R_i(P')$.

The next property says that if the cost of a link increases, then the two agents located at its vertices could not pay less.

Cost Monotonicity (CM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C', F, d, \alpha, h)$ such that there exists $i, j \in N$ satisfying $c_{ij} \geq c'_{ij}$ and $c_{kl} = c'_{kl}$ otherwise, then we have that $R_i(P) \geq R_i(P')$ and $R_j(P) \geq R_j(P')$.

Assume that the cost of some hub d_k decreases. It is then clear that if h was an optimal hub network in the original problem it will be also optimal in the new problem. How agents should be affected? The next two properties give an answer to this question.

The first one says that agents that need hub k or do not need hub k are affected in the same way.

Equal Treatment on Hubs (ETH) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then for all pair of agents i, j such that $k \in H_i^{of} \cap H_j^{of}$ or $k \notin H_i^{of} \cup H_j^{of}$, we have that

$$R_i(P) - R_i(P') = R_j(P) - R_j(P').$$

The second one says that agents that do not need hub k are not affected.

Independence of Irrelevant Hubs (IIH) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then we have that $R_i(P) = R_i(P')$ for each agent i such that $k \notin H_i^{of}$.

We now introduce a similar property to IIH but with flows instead of hubs. If node i increases its flow to some other node j, then the other agents should not be affected.

Independence of Irrelevant Flows (IIF) For any pair of hub network problems $P=(N,C,F,d,\alpha,h)$ and $P'=(N,C,F',d,\alpha,h)$ such that there exist $j,k\in N$ satisfying $0<f'_{jk}\leq f_{jk}$ and $f'_{j'k'}=f_{j'k'}$ otherwise, then we have that $R_i(P)=R_i(P')$ for each agent $i\in N\setminus\{j\}$.

There are some relations between these properties,

Proposition 3.1 (a) *CS implies Pos.*
(b) *Pos, IIH and IIF imply CS.*

Proof. (a) Assume $x\in Core\left(c^{of}\right)$. Then, for all $i\in N$,

$$x_i=c^{of}(N)-\sum_{j\in N\setminus\{i\}}x_j\geq c^{of}(N)-c^{of}(N\setminus\{i\})\geq\sum_{j\in N\setminus\{i\}}\lambda^h_{ij}\geq 0.$$

(b) Let R be a rule satisfying Pos, IIH, and IIF. Fix $S\subset N$. Let $\varepsilon>0$ and define $P^{S,\varepsilon}=\left(N,C,F^{S,\varepsilon},d^S,\alpha,h\right)$ as the problem obtained from P by turning all positive flows not used by S into ε and all hub costs not used by S into zero. Formally,

$$f^{S,\varepsilon}_{ij}=\begin{cases}\varepsilon & \text{if } i\notin S \text{ and } f_{ij}>0\\ f_{ij} & \text{otherwise}\end{cases}$$

and

$$d^S_k=\begin{cases}0 & \text{if } k\notin H^{of}_S\\ d_k & \text{otherwise.}\end{cases}$$

Then, $c^{of}_{P^{S,\varepsilon}}(N)\leq\sum_{i\in S}\sum_{j\in N}\lambda^h_{ij}+\sum_{k\in H^{of}_S}d_k+a(P)\varepsilon$ where

$$a\left(P\right)=\left|\{f_{ij}:f_{ij}>0\}\right|\max\left\{\frac{\lambda^h_{ij}}{f_{ij}}:f_{ij}>0\right\}.$$

Now,

$$\begin{aligned}\sum_{i\in S}R_i(P)&\overset{IIH+IIF}{=}\sum_{i\in S}R_i\left(P^{S,\varepsilon}\right)=c^{of}_{P^{S,\varepsilon}}(N)-\sum_{i\in N\setminus S}R_i\left(P^{S,\varepsilon}\right)\\&\overset{Pos}{\leq}c^{of}_{P^{S,\varepsilon}}(N)\leq\sum_{i\in S}\sum_{j\in N}\lambda^h_{ij}+\sum_{k\in H^{of}_S}d_k+a(P)\varepsilon\\&=c^{of}_P(S)+a(P)\varepsilon\end{aligned}$$

which implies $\sum_{i \in S} R_i(P) \leq c_P^{of}(S)$ because $a(P)$ does not depend on ε. ■

CS does not imply neither IIH nor IIF. The rule in which each agent pays the cost of sending its flow and the cost of each hub is paid equally by the agents that use the most expensive hubs among those that use that hub satisfies CS but not IIH. The rule in which each agent pays the cost of sending its flow and the cost of each hub is paid equally by the agents sending more flow through this hub satisfies CS but not IIF.

In the next proposition we prove that the Shapley rule satisfy all the above properties.

Proposition 3.2 *The Shapley rule satisfies Pos, ETE, CS, NF, FM, HM, CM, ETH, IIH and IIF.*

Proof. From Theorem 3.2, we deduce that $Sh\left(c^{of}\right)$ satisfies Pos, FM, HM, and CM. If i and j are equal in P, then it is easy to see that i and j are symmetric in c^{of}. Now, symmetry of the Shapley value implies that $Sh\left(c^{of}\right)$ satisfies ETE. Any $i \in N$ with $f_{ij} = 0$ for all $j \in N \setminus \{i\}$ is a dummy player in c^{of}. Hence, its Shapley value is zero, and so $Sh\left(c^{of}\right)$ satisfies NF. Let P, P' and k be given as in the definition of ETH and IIH. Given $i, j \in N$ such that $k \in H_i^{of} \cap H_j^{of}$, by Theorem 3.2

$$Sh_i\left(c_P^{of}\right) - Sh_i\left(c_{P'}^{of}\right) = \frac{d_k - d'_k}{\left|\left\{l \in N : k \in H_l^{of}\right\}\right|} = Sh_j\left(c_P^{of}\right) - Sh_j\left(c_{P'}^{of}\right)$$

Given $i, j \in N$ such that $k \notin H_i^{of} \cup H_j^{of}$, by Theorem 3.2

$$Sh_i\left(c_P^{of}\right) - Sh_i\left(c_{P'}^{of}\right) = 0 = Sh_j\left(c_P^{of}\right) - Sh_j\left(c_{P'}^{of}\right).$$

Hence $Sh\left(c^{of}\right)$ satisfies ETH. Given $i \in N$ such that $k \notin H_i^{of}$, from Theorem 3.2 we know that $Sh_i\left(c^{of}\right)$ does not depend on d_k, and so $Sh_i\left(c_P^{of}\right) = Sh_i\left(c_{P'}^{of}\right)$ and hence $Sh\left(c^{of}\right)$ satisfies IIH. Let P, P' and i, j, k be given as in the definition of IIF. From Theorem 3.2 we have that $Sh_i\left(c^{of}\right)$ does not depend on f_{jk}. Hence, $Sh\left(c^{of}\right)$ satisfies IIF. From Proposition 3.1, it satisfies CS. ■

We now give two characterizations of the Shapley rule.

Theorem 3.3 (a) *The Shapley rule is the unique rule satisfying CS and ETH.*

(b) *The Shapley rule is the unique rule satisfying Pos, IIH, IIF, and ETH.*

Proof. (a) By Proposition 3.2 the Shapley rule satisfies these properties. We now prove the uniqueness. Let R be a rule satisfying CS and ETH.

Let $P = (N, C, F, d, \alpha, h)$ be any hub network problem. For each $K \subseteq H$, let $P^K = \left(N, C, F, d^K, \alpha, h\right)$ with d^K defined as follows:

$$d_i^K = \begin{cases} 0 & \text{if } i \in H \setminus K \\ d_i & \text{otherwise.} \end{cases}$$

For all $k \in N$, let $N^{k,0} = \left\{i \in N : k \notin H_i^{of}\right\}$, $N^{k,1} = \left\{i \in N : k \in H_i^{of}\right\}$, $n^{k,0} = \left|N^{k,0}\right|$ and $n^{k,1} = \left|N^{k,1}\right|$ for all $k \in N$.

ETH implies that, for each $k \in K$, there exist $x^{k,0}$ and $x^{k,1}$ such that for all $i \in N^{k,0}$,

$$R_i\left(P^K\right) - R_i\left(P^{K\setminus\{k\}}\right) = x^{k,0} \tag{4}$$

and for all $i \in N^{k,1}$

$$R_i\left(P^K\right) - R_i\left(P^{K\setminus\{k\}}\right) = x^{k,1}. \tag{5}$$

Since $N = N^{k,0} \cup N^{k,1}$ and

$$\sum_{i\in N} R_i\left(P^K\right) - \sum_{i\in N} R_i\left(P^{K\setminus\{k\}}\right) = d_k$$

we have that for all $k \in K$,

$$n^{k,0}x^{k,0} + n^{k,1}x^{k,1} = d_k. \tag{6}$$

The equivalence relation in N defined as

$$i \sim j \Leftrightarrow \exists k \in K : i, j \in N^{k,1} \text{ or } i, j \in N^{k,0}$$

determines a partition $\mathcal{P}_K$ of N. It is straightforward to check that the cost game $c_{P^K}^{of}(N) = \sum_{S\in\mathcal{P}_K} c_{P^K}^{of}(N)$. So CS implies that $\sum_{i\in S} R_i\left(P^K\right) =$

$c^{of}_{P^K}(S)$ for all $S \in \mathcal{P}_K$. Moreover, any $\mathcal{P}_L$ with $L \subset K$ is a refinement of $\mathcal{P}_K$, so $\sum_{i\in S} R_i\left(P^L\right) = c^{of}_{P^L}(S)$ for all $S \in \mathcal{P}_K$.

We now consider several cases.

Case 1. Assume that $\mathcal{P}_K$ has at least two components. Given $k \in K$, there exist $S, S' \in \mathcal{P}_K$ such that $k \in S \cap H$ and $S' \subseteq N^{k,0}$. Besides, $c^{of}_{P^{K\setminus\{k\}}}(S') = c^{of}_{P^K}(S')$. Thus,

$$
\begin{aligned}
c^{of}_{P^K}(S') = \sum_{i\in S'} R_i\left(P^K\right) &\overset{(4)}{=} \sum_{i\in S'} R_i\left(P^{K\setminus\{k\}}\right) + |S'|\, x^{k,0} \\
&\overset{CS}{=} c^{of}_{P^{K\setminus\{k\}}}(S') + |S'|\, x^{k,0} = c^{of}_{P^K}(S') + |S'|\, x^{k,0}
\end{aligned}
$$

which implies that $x^{k,0} = 0$.

By (6), $x^{k,1} = \frac{d_k}{n^{k,1}}$ for all $k \in K$. By (5), for each $i \in N$,

$$R_i\left(P^K\right) = R_i\left(P^{K\setminus\{k\}}\right) + \frac{d_k}{n^{k,1}}.$$

Repeating the same argument, we deduce that for each $i \in N$,

$$R_i\left(P^K\right) = R_i\left(P^{\emptyset}\right) + \sum_{k\in H_i^{of}} \frac{d_k}{n^{k,1}}.$$

Since R satisfies CS, under Theorem 3.1, $R_i\left(P^{\emptyset}\right) = \sum_{j\in N} \lambda^h_{ij} + y_i$ for all $i \in N$, where $0 \le y_i \le \sum_{j\in H_i^{of}} d^{\emptyset}_j$. By definition of $d^{\emptyset}$, we have $d^{\emptyset}_j = 0$ for all $j \in H$. Since $H_i^{of} \subseteq H$, we deduce $y_i = 0$ and so

$$R_i\left(P^{\emptyset}\right) = \sum_{j\in N} \lambda^h_{ij}.$$

By Theorem 3.2,

$$R_i\left(P^K\right) = \sum_{j\in N} \lambda^h_{ij} + \sum_{k\in H_i^{of}} \frac{d_k}{n^{k,1}} = Sh_i\left(P^K\right).$$

Case 2. Assume now $\mathcal{P}_K = \{N\}$. We consider several cases.

Case 2.1. Assume $K = \{k\}$. Since R satisfies CS, under Theorem 3.1,

$$\sum_{i\in N^{k,0}} R_i\left(P^{\{k\}}\right) = \sum_{i\in N^{k,0}} \sum_{j\in N} \lambda^h_{ij} + \sum_{j\in N^{k,0}} y_i$$

where $y \in \mathbb{R}_{+}^{N}$ and

$$0 \leq \sum_{i \in N^{k,0}} y_i \leq \sum_{j \in H^{of}_{N^{k,0}}} d_j^{\{k\}} = 0$$

which implies $\sum_{i \in N^{k,0}} y_i = 0$. Thus,

$$\sum_{i \in N^{k,0}} R_i\left(P^{\{k\}}\right) = \sum_{i \in N^{k,0}} \sum_{j \in N} \lambda_{ij}^{h}.$$

On the other hand,

$$\sum_{i \in N^{k,0}} R_i\left(P^{\{k\}}\right) \overset{(4)}{=} \sum_{i \in N^{k,0}} R_i\left(P^{\emptyset}\right) + n^{k,0}x^{k,0} = \sum_{i \in N^{k,0}} \sum_{j \in N} \lambda_{ij}^{h} + n^{k,0}x^{k,0}$$

which implies $x^{k,0} = 0$. So, for each $i \in N^{k,0}$,

$$R_i\left(P^{\{k\}}\right) = \sum_{j \in N} \lambda_{ij}^{h} = Sh_i\left(P^{\{k\}}\right).$$

Under (6), $x^{k,1} = \frac{d_k}{n^{k,1}}$. So, for each $i \in N^{k,1}$,

$$R_i\left(P^{\{k\}}\right) = R_i\left(P^{\emptyset}\right) + \frac{d_k}{n^{k,1}} = \sum_{j \in N} \lambda_{ij}^{h} + \frac{d_k}{n^{k,1}} = Sh_i\left(P^{\{k\}}\right).$$

Case 2.2. Assume now $|K| > 1$. We proceed by induction on $|K|$. Hence, we assume $R\left(P^{K'}\right) = Sh\left(P^{K'}\right)$ when $|K'| < |K|$. We have three cases:

Case 2.2.1. Assume first $n^{k,0} = 0$ for some $k \in K$. By (5), for all $i \in N = N^{k,1}$,

$$R_i\left(P^{K}\right) = R_i\left(P^{K \setminus \{k\}}\right) + x^{k,1}.$$

Hence,

$$\sum_{i \in N} R_i\left(P^{K}\right) = \sum_{i \in N} R_i\left(P^{K \setminus \{k\}}\right) + nx^{k,1}$$

and thus

$$x^{k,1} = \frac{\sum_{i \in N} R_i\left(P^{K}\right) - \sum_{i \in N} R_i\left(P^{K \setminus \{k\}}\right)}{n} = \frac{d_k}{n^{k,1}}.$$

Now for all $i \in N = N^{k,1}$,

$$R_i\left(P^{K}\right) = R_i\left(P^{K \setminus \{k\}}\right) + \frac{d_k}{n^{k,1}}.$$

By induction hypothesis, for all $i \in N$

$$R_i\left(P^K\right) = \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{of}} \frac{d_j^{K \setminus \{k\}}}{n^{k,1}} + \frac{d_k}{n^{k,1}}$$
$$= \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{of}} \frac{d_j^K}{n^{k,1}} = Sh_i\left(P^K\right).$$

Case 2.2.2. Assume now $n^{k,1} = 0$ for some $k \in K$. By (4), $R_i\left(P^K\right) = R_i\left(P^{K\setminus\{k\}}\right) + x^{k,0}$ for all $i \in N = N^{k,0}$. The rest of reasoning is analogous to the previous case and we omit it.

Case 2.2.3. Finally, assume $n^{k,0} > 0$ and $n^{k,1} > 0$ for all $k \in K$. We can assume w.l.o.g. $1, 2 \in K$. Let $i^1 \in N^{1,1}$ and $i^2 \in N^{1,0}$. Since $\mathcal{P}_K = \{N\}$, we know that there exists some $k \in K$ such that either $i^1, i^2 \in N^{k,1}$ or $i^1, i^2 \in N^{k,0}$. Assume w.l.o.g. that either $i^1, i^2 \in N^{2,1}$ or $i^1, i^2 \in N^{2,0}$. For each $k \in \{1, 2\}$ and each $l \in \{1, 2\}$, let $f^l(k) \in \{0, 1\}$ be defined such that $i^l \in N^{k,f^l(k)}$. Hence, we know that $f^1(1) = 1$ (because $i^1 \in N^{1,1}$), $f^2(1) = 0$ (because $i^2 \in N^{1,0}$), and $f^1(2) = f^2(2)$ (because either $i^1, i^2 \in N^{2,1}$ or $i^1, i^2 \in N^{2,0}$).

By induction hypothesis, for any $k \in \{1, 2\}$ and any $l \in \{1, 2\}$,

$$R_{i^l}\left(P^K\right) \overset{(4)(5)}{=} R_{i^l}\left(P^{K\setminus\{k\}}\right) + x^{k,f^l(k)} = Sh_{i^l}\left(P^{K\setminus\{k\}}\right) + x^{k,f^l(k)}$$
$$= \sum_{j \in N} \lambda_{i^l_j}^h + \sum_{j \in H_{i^l}^{of}} \frac{d_j^{K\setminus\{k\}}}{n^{j,1}} + x^{k,f^l(k)}.$$

Thus, for each $l \in \{1, 2\}$,

$$x^{1,f^l(1)} - x^{2,f^l(2)} = \sum_{j \in H_{i^l}^{of}} \frac{d_j^{K\setminus\{1\}} - d_j^{K\setminus\{2\}}}{n^{j,1}} = \frac{d_1}{n^{1,1}} f^l(1) - \frac{d_2}{n^{2,1}} f^l(2).$$

In particular, taking $a = f^1(2) = f^2(2)$ and $l = 1$,

$$x^{1,1} - x^{2,a} = \frac{d_1}{n^{1,1}} - \frac{d_2}{n^{2,1}} a \tag{7}$$

and taking $l = 2$,

$$x^{1,0} - x^{2,a} = -\frac{d_2}{n^{2,1}} a. \tag{8}$$

Equations (6) for $k = 1, 2$ and equations (7)-(8) can be written as a matrix equation as follows:

$$\begin{bmatrix} n^{1,1} & 0 & n^{1,0} & 0 \\ 0 & n^{2,1} & 0 & n^{2,0} \\ 1 & -a & 0 & a-1 \\ 0 & -a & 1 & a-1 \end{bmatrix} \cdot \begin{bmatrix} x^{1,1} \\ x^{2,1} \\ x^{1,0} \\ x^{2,0} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \frac{d_1}{n^{1,1}} - \frac{d_2}{n^{2,1}} a \\ -\frac{d_2}{n^{2,1}} a \end{bmatrix}.$$

The determinant of the left matrix is $\left(an - n^{2,1}\right) n \neq 0$. Hence, the matrix equation has a unique solution given by $x^{k,1} = \frac{d_k}{n^{k,1}}$ and $x^{k,0} = 0$ for all $k \in \{1, 2\}$. Thus,

$$\begin{aligned} R_{i^1}\left(P^K\right) &\overset{(5)}{=} R_{i^1}\left(P^{K\setminus\{1\}}\right) + x^{1,1} = R_{i^1}\left(P^{K\setminus\{1\}}\right) + \frac{d_1}{n^{1,1}} \\ R_{i^2}\left(P^K\right) &\overset{(4)}{=} R_{i^2}\left(P^{K\setminus\{1\}}\right) + x^{1,0} = R_{i^2}\left(P^{K\setminus\{1\}}\right). \end{aligned}$$

By induction hypothesis,

$$\begin{aligned} R_{i^1}\left(P^K\right) &= Sh_{i^1}\left(P^{K\setminus\{1\}}\right) + \frac{d_1}{n^{1,1}} \overset{Th.3.2}{=} \sum_{j\in N} \lambda^h_{i^1 j} + \sum_{k \in H^{of}_{i^1}} \frac{d_k^{K\setminus\{1\}}}{n^{k,1}} + \frac{d_1}{n^{1,1}} \\ &= \sum_{j\in N} \lambda^h_{i^1 j} + \sum_{k \in H^{of}_{i^1}} \frac{d_k^{K}}{n^{k,1}} \overset{Th.3.2}{=} Sh_{i^1}\left(P^K\right) \\ R_{i^2}\left(P^K\right) &= Sh_{i^2}\left(P^{K\setminus\{1\}}\right) \overset{Th.3.2}{=} \sum_{j\in N} \lambda^h_{i^2 j} + \sum_{k \in H^{of}_{i^2}} \frac{d_k^{K\setminus\{1\}}}{n^{k,1}} \\ &= \sum_{j\in N} \lambda^h_{i^2 j} + \sum_{k \in H^{of}_{i^2}} \frac{d_k^{K}}{n^{k,1}} \overset{Th.3.2}{=} Sh_{i^2}\left(P^K\right). \end{aligned}$$

Since i^1, i^2 were taken arbitrarily from $N^{1,1}$ and $N^{1,0}$, respectively, and these two sets form a partition of N, we conclude that $R_i\left(P^K\right) = Sh_i\left(P^K\right)$ for all $i \in N$.

(b) It follows from part (a) and Proposition 3.1. ■

Remark 3.2 *We now prove that the properties used in Theorem 3.3 are independent.*

- *Let R^0 be defined as $R^0(P) = x + Sh\left(c^{of}\right)$ for some $x \in \mathbb{R}^N$ with $\sum_{i \in N} x_i = 0$ and $x_i \neq 0$ for some $i \in N$. R^0 satisfies IIH, IIF and ETH, but fails CS and Pos.*

- *Let $\omega \in \mathbb{R}^N$ be such that $\omega_i > 0$ for all $i \in N$ and $\omega_i \neq \omega_j$ for some $i \neq j$. Let R^1 be defined for each P and each $i \in N$ as follows:*

$$R_i^1(P) = \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{of}} \frac{\omega_i}{\sum_{k \in N : j \in H_k^{of}} \omega_k} d_j.$$

R^1 satisfies CS, Pos, IIH, and IIF, but fails ETH.

- *Let R^2 be defined for each P and $i \in N$ as follows:*

$$R_i^2(P) = \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H} \frac{d_j}{n}.$$

R^2 satisfies Pos, IIF and ETH, but fails CS and IIH.

- *Let R^3 be defined for each P and $i \in N$ as follows:*

$$R_i^3(P) = \frac{\sum_{k \in N} \sum_{j \in N} \lambda_{kj}^h}{n} + \sum_{j \in H_i^{of}} \frac{d_j}{\left|\left\{k \in N : j \in H_k^{of}\right\}\right|}.$$

R^3 satisfies Pos, IIH and ETH,but fails CS and IIF.

4 Two-way flow

In this section we consider the case in which the users are interested in both receiving and sending data. Again, we assume that communication is carried over a (maybe optimal) hub network. We first associate to each hub network problem a cooperative game. Later on we study the core and the Shapley value.

For each hub network problem P we associate the cost game c_h^{tf} where for each $S \subseteq N$, $c_h^{tf}(S)$ is the cost of sending and receiving the flow of all agents in S to and from all agents through h. The cost game c_h^{tf} models situations where an (optimal) hub network h (with associated set of hubs H) has already been constructed. Thus, d could be considered as a vector of maintenance costs. Agents in each coalition are interested in the hubs they need for sending or receiving their flow. Notice that we are applying the same ideas than in the one-way flow. We now define this cost game formally.

For each $S \subseteq N$, let $H_S^{tf} \subseteq H$ denote the set of hubs needed for sending or receiving the flow of agents in S. Namely,

$$H_S^{tf} = H_S^{of} \cup \{k \in H : \exists i \in S, j \in N \text{ with } f_{ji} > 0 \text{ and } k \in \{h(i), h(j)\}\}.$$

Given $i \in N$, we write H_i^{tf} instead of $H_{\{i\}}^{tf}$. Like in the previous section, $H_S^{tf} = \bigcup_{i \in S} H_i^{tf}$ for all $S \subseteq N$.

Now,

$$c_P^{tf}(S) = \sum_{(i,j) \notin (N \backslash S) \times (N \backslash S)} \lambda_{ij}^h + \sum_{i \in H_S^{tf}} d_i. \tag{9}$$

When no confusion arises we write c^{tf} instead of c_P^{tf}.

4.1 The core

In the next theorem we prove that in the core allocations of c^{tf} the cost of sending or receiving flow between two nodes is divided between them. Besides, the cost of any hub is divided among the agents that need the hub for sending or receiving their flow. Before stating the theorem we need some notation.

Let $\Pi = \{\pi : N \longrightarrow N : \pi \text{ biyective}\}$ be the set of orderings of agents in N.

Given $i \in N$ and $j \in H$, $\Pi_{ij} \subset \Pi$ is the set of orderings such that node i is the first that uses hub j, i.e. $\pi(l) = i$ implies $j \notin H_{\pi(l')}^{tf}$ for all $l' < l$.

Theorem 4.1 *For each hub network problem, c^{tf} is concave. Moreover, the core is nonempty and given by the convex hull of the following set of vectors:*

$$\left\{ \left(\sum_{j \in N : \pi^{-1}(j) > \pi^{-1}(i)} \left(\lambda_{ij}^h + \lambda_{ji}^h\right) + \sum_{j \in H_i^{tf} : \pi \in \Pi_{ij}} d_j \right)_{i \in N} \right\}_{\pi \in \Pi}.$$

Proof. We first prove that $\left(N, c^{tf}\right)$ is concave. Let $l \in T \subset S \subseteq N$.

Since for each $S' \subset N$, $H_{S'}^{tf} = \bigcup_{i \in S'} H_i^{tf}$, we have that

$$H_S^{tf} \setminus H_{S\setminus\{l\}}^{tf} \subset H_T^{tf} \setminus H_{T\setminus\{l\}}^{tf}. \tag{10}$$

Then,

$$\begin{aligned} c^{tf}(S') - c^{tf}\left(S' \setminus \{l\}\right) &= \sum_{(i,j)\notin (N\setminus S')\times(N\setminus S')} \lambda_{ij}^h + \sum_{i\in H_{S'}} d_i \\ &- \sum_{(i,j)\notin (N\setminus (S'\setminus\{l\}))\times(N\setminus (S'\setminus\{l\}))} \lambda_{ij}^h - \sum_{i\in H_{S'\setminus\{l\}}^{tf}} d_i \\ &= \sum_{i\in N\setminus S'} \lambda_{il}^h + \sum_{j\in N\setminus S'} \lambda_{lj}^h + \sum_{i\in H_{S'}^{tf}\setminus H_{S\setminus\{l\}}^{tf}} d_i. \end{aligned}$$

Since all terms are non-negative, $N \setminus S \subset N \setminus T$ and (10), we have that

$$c^{tf}(S) - c^{tf}\left(S \setminus \{l\}\right) \leq c^{tf}(T) - c^{tf}\left(T \setminus \{l\}\right)$$

which proves that $\left(N, c^{tf}\right)$ is concave.

It is well known that when the cost game is concave, the core coincides with the Weber set. So it is the convex hull of the vectors of marginal contributions. Notice that the coordinate i of the vector of marginal contributions for $\pi \in \Pi$ is

$$\sum_{j\in N:\pi^{-1}(j)>\pi^{-1}(i)} \left(\lambda_{ij}^h + \lambda_{ji}^h\right) + \sum_{j\in H_i^{tf}:\pi\in\Pi_{ij}} d_j,$$

hence the result holds. ■

Analogously to the one-way case, we consider now an intermediate situation between a fixed hub network and a variable hub network. Assume that the optimal hub network is not unique and the planner should decide which one to construct. We can define the cost of a coalition as the minimum over all optimal hub networks. Namely, for each $S \subseteq N$,

$$c^{**}(S) = \min_{h\in\mathcal{H}, h \text{ is optimal}} \left\{ c_{P(h)}^{tf}(S) \right\}$$

where $P(h)$ is the hub network problem induced by the optimal hub network h. Next example shows that the core of c^{**} can be empty.

Example 4.1 *Let P be such that $N = \{1, 2, \ldots, 6\}$, $\alpha = 1$, $f_{12} = f_{34} = f_{56} = 1$, $f_{ij} = 0$ otherwise. $d_1 = d_2 = d_3 = 1$ and $d_i \geq 4$ otherwise. The cost matrix is given in the following table:*

c_{ij}	2	3	4	5	6
1	2	2	3	3	3
2		1	3	4	3
3			4	3	3
4				3	3
5					4

This hub problem is depicted in Figure 1.

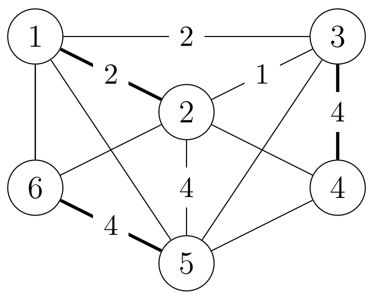

Figure 1: $c_{ij} = 3$ when no specified. Flow goes from 1 to 2, from 3 to 4, and from 5 to 6.

There exist three optimal hub networks $\{h^i\}_{i=1}^3$, corresponding to putting a single hub at either 1, 2 *or* 3*, respectively. The cost of these networks is* 14 *each. Hence, $c^{**}(N) = 14$. Moreover, nodes $1, 2, 3, 4$ can cover their own flow at cost* 7 *when the hub is located at* 2*. Then, $c^{**}(\{1, 2, 3, 4\}) = 7$. Analogously, nodes $1, 2, 5, 6$ can cover their own flow at cost* 9 *when the hub is located at* 1*, so that $c^{**}(\{1, 2, 5, 6\}) = 9$. Analogously, nodes $3, 4, 5, 6$ can cover their own flow at cost* 11 *when the hub is located at* 3*, so that $c^{**}(\{3, 4, 5, 6\}) = 11$. Hence, a core allocation y should satisfy $y_1+y_2+y_3+y_4 \leq 7$, $y_1+y_2+y_5+y_6 \leq 9$, and $y_3+y_4+y_5+y_6 \leq 11$. By adding these inequalities and dividing by* 2*, we deduce that $\sum_{i \in N} y_i \leq 13.5$. Since $c^{**}(N) = 14$, we deduce that the core of c^{**} is empty.*

4.2 The Shapley value

We now study the Shapley value of c^{tf}, which we also call the *Shapley rule*. In the next theorem we prove that in the Shapley rule the cost of sending flow between a pair of agents (λ_{ij}^h) is divided equally between both agents. Besides, the cost of any hub is divided equally among the agents that need the hub for sending or receiving their flow.

Theorem 4.2 *For each hub network problem P and each $i \in N$,*

$$Sh_i\left(c^{tf}\right) = \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2} + \sum_{j \in H_i^{tf}} \frac{d_j}{\left|\left\{k \in N : j \in H_k^{tf}\right\}\right|} \tag{11}$$

Proof. The Shapley value is the average of the vectors of marginal contributions. Thus,

$$Sh_i\left(c^{tf}\right) = \frac{1}{|\Pi|}\left(\sum_{\pi \in \Pi} \sum_{j \in N : \pi^{-1}(j) > \pi^{-1}(i)} \left(\lambda_{ij}^h + \lambda_{ji}^h\right) + \sum_{j \in H_i^{tf}} \sum_{\pi \in \Pi_{ij}} d_j\right).$$

Let $\Pi^{ij} = \{\pi \in \Pi : \pi^{-1}(j) > \pi^{-1}(i)\}$. Clearly, $|\Pi^{ij}| = \frac{|\Pi|}{2}$. Hence,

$$\begin{aligned}
\frac{1}{|\Pi|} \sum_{\pi \in \Pi} \sum_{j \in N : \pi^{-1}(j) > \pi^{-1}(i)} \left(\lambda_{ij}^h + \lambda_{ji}^h\right) &= \frac{1}{|\Pi|} \sum_{j \in N} \sum_{\pi \in \Pi^{ij}} \left(\lambda_{ij}^h + \lambda_{ji}^h\right) \\
&= \frac{1}{|\Pi|} \sum_{j \in N} |\Pi^{ij}| \left(\lambda_{ij}^h + \lambda_{ji}^h\right) \\
&= \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2}
\end{aligned}$$

which is the first part of (11).

Let $T = \left\{k \in N : j \in H_k^{tf}\right\}$ and $t = |T|$. We still need to prove that $\frac{1}{|\Pi|} \sum_{j \in H_i^{tf}} \sum_{\pi \in \Pi_{ij}} d_j = \sum_{j \in H_i^{tf}} \frac{d_j}{t}$. Clearly, it is enough to prove that $\frac{|\Pi_{ij}|}{|\Pi|} = \frac{1}{t}$ for all $j \in H_i^{tf}$. Notice that Π_{ij} is the set of orderings in which the predecessors of i are not in T. In particular, $\Pi_{ij} = \bigcup_{s=1,\ldots,n-t+1} \Pi_{ij}^s$ where $\Pi_{ij}^s = \{\pi \in \Pi_{ij} : \pi(s) = i\}$. Hence, $\frac{|\Pi_{ij}|}{|\Pi|} = \sum_{s=1}^{n-t+1} \frac{\left|\Pi_{ij}^s\right|}{|\Pi|}$. Moreover, $\frac{\left|\Pi_{ij}^s\right|}{|\Pi|}$ is the probability of randomly picking up an order in Π satisfying that node i is in position s and it is preceded by $s-1$ nodes in $N \setminus T$. Let $|N \setminus T| = n-t$. Then,

$$\frac{\left|\Pi_{ij}^{s}\right|}{|\Pi|} = \frac{n-t}{n} \cdot \frac{n-t-1}{n-1} \cdot \dots \cdot \frac{n-t-s+2}{n-s+2} \cdot \frac{1}{n-s+1} = \frac{(n-s)!(n-t)!}{n!(n-t-s+1)!}$$

So

$$\begin{aligned}
\frac{|\Pi_{ij}|}{|\Pi|} &= \frac{(n-t)!}{n!} \sum_{s=1}^{n-t+1} \frac{(n-s)!}{(n-s-t+1)!} \\
&= \frac{(n-t)!t!}{n!} \sum_{s=1}^{n-t+1} \frac{(n-s)!}{(n-s-t+1)!(t-1)!} \cdot \frac{1}{t} \\
&= \frac{1}{\binom{n}{t}} \sum_{s=1}^{n-t+1} \binom{n-s}{t-1} \frac{1}{t}.
\end{aligned}$$

Then, it is enough to prove that $\binom{n}{t} = \sum_{s=1}^{n-t+1} \binom{n-s}{t-1}$. This is trivially true when $n=1$. By induction hypothesis on n, and using Stidel formula:

$$\begin{aligned}
\binom{n}{t} &= \binom{n-1}{t-1} + \binom{n-1}{t} = \binom{n-1}{t-1} + \sum_{s=1}^{n-t} \binom{n-1-s}{t-1} \\
&= \binom{n-1}{t-1} + \sum_{s=2}^{n-t+1} \binom{n-s}{t-1} = \sum_{s=1}^{n-t+1} \binom{n-s}{t-1}.
\end{aligned}$$

■

We now define properties in the two-way flow case. All but one are analogous to the one-way flow case. Positivity and cost monotonicity are the same. The other properties are defined by adapting the ideas behind the properties defined in the one-way case to the two-way case.

Positivity (*Pos*) For any hub network problem P and each $i \in N$, we have $R_i(P) \geq 0$.

Given a hub network problem P we say that nodes i and j are *equal* when several conditions hold: First, $f_{ik} = f_{jk}$ and $f_{ki} = f_{ki}$ for all $k \in N \setminus \{i,j\}$. Second, $f_{ij} = f_{ji}$. Third, $i \in H$ if and only if $j \in H$. Forth, $\{i,k\} \in \overline{h}$ if and only if $\{j,k\} \in \overline{h}$. Fifth, for each $\{i,k\}, \{j,k\} \in \overline{h}$, $c_{ik} = c_{jk}$. These conditions are the same as in the one-way flow case, except for the first one. Equal nodes should not only send the same flow, but also receive it.

Equal Treatment of Equals (ETE) For any hub network problem P and each pair of equal nodes $i, j \in N$, we have that $R_i(P) = R_j(P)$.

Core Selection (CS) For any hub network problem P, we have that

$$R(P) \in Core\left(c_P^{tf}\right).$$

Null Flow (NF) For any hub network problem P and each $i \in N$ such that $f_{ij} = f_{ji} = 0$ for all $j \in N \setminus \{i\}$, we have that $R_i(P) = 0$.

Flow Monotonicity (FM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exist $i, j \in N$ satisfying $f_{ij} \geq f'_{ij}$ and $f_{kl} = f'_{kl}$ otherwise, then $R_i(P) \geq R_i(P')$ and $R_j(P) \geq R_j(P')$.

Hub Monotonicity (HM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then for each agent i such that $k \in H_i^{tf}$, we have that $R_i(P) \geq R_i(P')$.

Cost Monotonicity (CM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C', F, d, \alpha, h)$ such that there exists $i, j \in N$ satisfying $c_{ij} \geq c'_{ij}$ and $c_{kl} = c'_{kl}$ otherwise, then we have that $R_i(P) \geq R_i(P')$ and $R_j(P) \geq R_j(P')$.

Equal Treatment on Hubs (ETH) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then for all pair of agents i, j such that $k \in H_i^{tf} \cap H_j^{tf}$ or $k \notin H_i^{tf} \cup H_j^{tf}$, we have that

$$R_i(P) - R_i(P') = R_j(P) - R_j(P').$$

Independence of Irrelevant Hubs (IIH) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then we have that $R_i(P) = R_i(P')$ for each agent i such that $k \notin H_i^{tf}$.

Independence of Irrelevant Flows (IIF) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exist $j, k \in N$ satisfying $0 < f'_{jk} \leq f_{jk}$ and $f'_{j'k'} = f_{j'k'}$ otherwise, then we have that $R_i(P) = R_i(P')$ for each agent $i \in N \setminus \{j, k\}$.

The analogous results for Proposition 3.1 also holds in the two-flow case.

Proposition 4.1 *(a) CS implies Pos.*
(b) Pos, IIH and IIF imply CS.

Proof. It is analogous to the proof of Proposition 3.1 and we omit it. ■

CS does not imply neither IIH nor IIF. The rule in which each agent pays half the cost of sending and receiving her flow and the cost of each hub is paid equally by the agents that use the most expensive hubs among those that use that hub satisfies CS but not IIH. The rule in which each agent pays half the cost of sending and receiving her flow and the cost of each hub is paid equally by the agents sending more flow through this hub satisfies CS but not IIF.

The next property is new. It says that a variation of flow affects the sender and the receiver in the same way. Note that this requirement is not reasonable in the one-flow case but it is in the two-way case.

Equal Treatment on Flows (ETF) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exists $k, l \in N$ satisfying $0 < f'_{kl} \leq f_{kl}$ and $f'_{ij} = f_{ij}$ otherwise, we have that

$$R_i(P) - R_i(P') = R_j(P) - R_j(P')$$

for all pair of agents i, j such that $\{i, j\} = \{k, l\}$ or $\{i, j\} \cap \{k, l\} = \emptyset$.

In the next proposition we prove that the Shapley rule satisfy all these properties.

Proposition 4.2 *The Shapley rule satisfies Pos, ETE, CS, NF, FM, HM, CM, ETH, IIH, IIF and ETF.*

Proof. The proof for Pos, ETE, CS, NF, FM, HM, CM, ETH, IIH and IIF is analogous to that of Proposition 3.2 (using Theorem 4.2 instead of Theorem 3.2 and Proposition 4.1 instead of Proposition 3.1) and we omit it.

Let P, P' be given as in the definition of ETF. We consider two cases:

1. $\{i, j\} = \{k, l\}$. Let $\lambda^{h'}$ the λ^h associated with P'. By Theorem 4.2,

$$Sh_i\left(c_P^{tf}\right) - Sh_i\left(c_{P'}^{tf}\right) = \frac{\lambda_{ij}^h - \lambda_{ij}^{h'}}{2} = Sh_j\left(c_P^{tf}\right) - Sh_j\left(c_{P'}^{tf}\right),$$

and hence $Sh\left(c^{tf}\right)$ satisfies ETF.

2. $\{i, j\} \cap \{k, l\} = \emptyset$. By Theorem 4.2,

$$Sh_i\left(c_P^{tf}\right) - Sh_i\left(c_{P'}^{tf}\right) = 0 = Sh_j\left(c_P^{tf}\right) - Sh_j\left(c_{P'}^{tf}\right),$$

■

Similarly to Theorem 3.3, we give two characterizations of the Shapley rule.

Theorem 4.3 (a) *The Shapley rule is the unique rule satisfying CS, ETH and ETF.*

(b) *The Shapley rule is the unique rule satisfying Pos, IIH, IIF, ETH, and ETF.*

Proof. (a) By Proposition 4.2 the Shapley rule satisfies these properties. We now prove the uniqueness. Let R be a rule satisfying CS, ETH and ETF.

Let $P = (N, C, F, d, \alpha, h)$ be a hub network problem. We assume $d_i = 0$ for all $i \in H$; the extension to positive hub costs is analogous to the proof of Theorem 3.3 and we omit it.

Let $E = \{(i, j) : f_{ij} > 0\}$ and, for each $i \in N$ and $e \in E$, let $a^i(e) = 1$ when node i is adjacent to e, and $a^i(e) = 0$ otherwise. Denote $E = \{e_1, \ldots, e_\gamma\}$. We assume, w.l.o.g., $e_1 = (1, 2)$. We also assume, w.l.o.g., $e_2 = (2, 1)$ in case $(2, 1) \in E$.

For each $\varepsilon > 0$, let $P^\varepsilon = (N, C, F^\varepsilon, d, \alpha, h)$ defined by $f_{ij}^\varepsilon = \varepsilon$ for all (i,j) with $f_{ij} > 0$, and $f_{ij}^\varepsilon = f_{ij} = 0$ otherwise.

Let $a(P)$ de defined as in the proof of Proposition 3.1. Suppose that, for ε small enough, there exists $x^P \in \mathbb{R}^N$ with $-7^{|E|}a(P)\varepsilon \leq x_i^P \leq 7^{|E|}a(P)\varepsilon$ for all $i \in N$ such that

$$R_i(P) = \sum_{e \in E} \frac{\lambda_e^h}{2} a^i(e) + x_i^P \text{ for all } i \in N. \tag{12}$$

Since $R_i(P)$ does not depend on ε, we deduce that for all $i \in N$

$$R_i(P) = \sum_{e \in E} \frac{\lambda_e^h}{2} a^i(e) = \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2} = Sh_i(P).$$

Hence, we just need to prove that (12) holds.

For each $e_k \in E$, we define $P^{-k} = \left(N, C, F^{-k}, d, \alpha, h\right)$ with $f_{e_k}^{-k} = \varepsilon$ and $f_{ij}^{-k} = f_{ij}$ otherwise. For notational convenience, we write λ_k^h, f_{ij}^{-k} and $a^i(k)$ instead of $\lambda_{e_k}^h$, $f_{ij}^{-e_k}$ and $a^i(e_k)$, respectively.

We proceed by induction on $|E|$. Case $E = \emptyset$ is not possible because H is nonempty and for each $k \in H$ we assume that there exist $i, j \in N$ with $f_{ij} > 0$ and $k \in \{h(i), h(j)\}$.

Assume then $E = \{e_1\}$. In this case, $P^{-1} = P^\varepsilon$. Let $x_i^P = 0$ if $i \notin \{1, 2\}$ and $x_i^P = R_i(P) - \frac{\lambda_1^h}{2}$ if $i \in \{1, 2\}$. We prove that for all $i \in N$, x_i^P lies on the interval $[-7a(P)\varepsilon, 7a(P)\varepsilon]$.

Let $i \notin \{1, 2\}$. By CS, $R_i(P) \leq c_P^{tf}(\{i\}) = 0$. Since $a^i(e_1) = 0$ and $\lambda_e^h = 0$ when $e \neq e_1$, (12) holds trivially.

We now prove it for $i = 1$ (the case $i = 2$ is analogous). By ETF, there exists $y^{1,1} \in \mathbb{R}$ such that $R_1(P) - R_1\left(P^{-1}\right) = R_2(P) - R_2\left(P^{-1}\right) = y^{1,1}$. Hence,

$$y^{1,1} = \frac{R_1(P) + R_2(P) - R_1(P^{-1}) - R_2(P^{-1})}{2}.$$

By CS, $0 \leq R_1(P) + R_2(P) = \lambda_1^h$ and $0 \leq R_1(P^{-1}) + R_2(P^{-1}) \leq a(P)\varepsilon$. Hence,

$$y^{1,1} \in \left[\frac{\lambda_1^h}{2} - a(P)\varepsilon, \frac{\lambda_1^h}{2} + a(P)\varepsilon\right].$$

By induction hypothesis, $x_1^{P^{-1}} \in [-a(P)\varepsilon, a(P)\varepsilon]$. Thus,

$$R_1(P) = R_1\left(P^{-1}\right) + y^{1,1} = x_1^{P^{-1}} + y^{1,1} \in \left[\frac{\lambda_1^h}{2} - 2a(P)\varepsilon, \frac{\lambda_1^h}{2} + 2a(P)\varepsilon\right].$$

and so (12) holds with $x_1^P = R_1(P) - \frac{\lambda_1^h}{2}$.

Assume now (12) holds when $|E| < \gamma$ and suppose $|E| = \gamma$. We consider several cases:

Case 1. $\gamma = 2$ and $e_2 = (2,1)$, so that $E = \{e_1, e_2\}$. Then, we proceed as above defining $y^{1,1}$ in the same way.

Case 2. Either $\gamma > 2$ or $e_2 \neq (2,1)$. Notice that this implies $n > 2$. Fix $i \in N$. We consider two cases.

Case 2.1. $a^i(e) = 0$ for all $e \in E$. We take $x_i^P = 0$. Then, by CS, (12) holds because $R_i(P) = 0$.

Case 2.2. There exists $k \in E$ such that $a^i(k) = 1$. Fix also $e_l \in E \setminus \{e_k\}$ with different adjacent nodes than e_k. We can find such e_l because either $\gamma > 2$ or $e_2 \neq (2,1)$.

ETF implies that there exist $y^{k,0}, y^{k,1}, y^{l,0}$ and $y^{l,1}$ such that $R_j(P) - R_j\left(P^{-k}\right) = y^{k,a^j(k)}$ and $R_j(P) - R_j\left(P^{-l}\right) = y^{l,a^j(l)}$ for all $j \in N$.Since

$$\sum_{j\in N} R_j(P) - \sum_{j\in N} R_j\left(P^{-k}\right) = \lambda_k^h - \frac{\lambda_k^h}{f_k}\varepsilon,$$

we have

$$2y^{k,1} + (n-2)y^{k,0} = \lambda_k^h + z^{k,1} \tag{13}$$

where $z^{k,1} = -\frac{\lambda_k^h}{f_k}\varepsilon \in [-a(P)\varepsilon, 0]$. Analogously,

$$2y^{l,1} + (n-2)y^{l,0} = \lambda_l^h + z^{l,1} \tag{14}$$

with $z^{l,1} \in [-a(P)\varepsilon, 0]$.

On the other hand, $R_i(P) = R_i(P^{-k}) + y^{k,1} = R_i(P^{-l}) + y^{l,a^i(l)}$. Hence,

$$y^{k,1} - y^{l,a^i(l)} = R_i(P^{-l}) - R_i(P^{-k})$$

by induction hypothesis,

$$= \frac{\lambda_k^h}{2} - \frac{\lambda_l^h}{2}a^i(l) + x_i^{P^{-l}} - x_i^{P^{-k}}$$

with $x_i^{P^{-l}}, x_i^{P^{-k}} \in [-7^{\gamma-1}a(P)\varepsilon, 7^{\gamma-1}a(P)\varepsilon]$.

We define $z^{k,0} = x_i^{P^{-l}} - x_i^{P^{-k}} \in [-2 \cdot 7^{\gamma-1}a(P)\varepsilon, 2 \cdot 7^{\gamma-1}a(P)\varepsilon]$ so that

$$y^{k,1} - y^{l,a^i(l)} = \frac{\lambda_k^h}{2} - \frac{\lambda_l^h}{2}a^i(l) + z^{k,0}. \tag{15}$$

We repeat the reasoning for some $j \in N$ adjacent to e_l (i.e. $a^j(l) = 1$) but not to e_k (i.e. $a^j(k) = 0$). We can find such j because e_l has different adjacent nodes than e_k. Then, we get

$$y^{l,1} - y^{k,0} = \frac{\lambda_l^h}{2} + z^{l,0} \tag{16}$$

with $z^{l,0} \in [-2 \cdot 7^{\gamma-1}a(P)\varepsilon, 2 \cdot 7^{\gamma-1}a(P)\varepsilon]$.

Equations (13)-(14)-(15)-(16) form a system of linear equations given by

$$\begin{bmatrix} 2 & 0 & n-2 & 0 \\ 0 & 2 & 0 & n-2 \\ 1 & -a^i(l) & 0 & a^i(l)-1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y^{k,1} \\ y^{l,1} \\ y^{k,0} \\ y^{l,0} \end{bmatrix} = \begin{bmatrix} \lambda_k^h + z^{k,1} \\ \lambda_l^h + z^{l,1} \\ \frac{\lambda_k^h}{2} - \frac{\lambda_l^h}{2}a^i(l) + z^{k,0} \\ \frac{\lambda_k^h}{2} + z^{l,0} \end{bmatrix}.$$

The determinant of the first matrix is $(n + 2a^i(l) - 4)n$. We consider several cases.

Case 2.2.1. $a^i(l) = 1$. Then, $(n + 2a^i(l) - 4)n \neq 0$. Thus, the previous system of linear equations have a unique solution which is given for $y^{k,1}$ by

$$y^{k,1} = \frac{\lambda_k^h}{2} + \frac{z}{(n-2)n}$$

where $z = (n-2)z^{k,1} - nz^{l,1} + (n^2 - 4n + 4)z^{k,0} + (n^2 - 4n + 4)z^{l,0}$.

Since $z^{k,1}, z^{l,1}, z^{k,0}, z^{l,0} \in [-2 \cdot 7^{\gamma-1}a(P)\varepsilon, 2 \cdot 7^{\gamma-1}a(P)\varepsilon]$, we deduce that $z \in [-2(2n^2 - 6n + 6) \cdot 7^{\gamma-1}a(P)\varepsilon, 2(2n^2 - 6n + 6) \cdot 7^{\gamma-1}a(P)\varepsilon]$.

For $n > 2$, we have $\frac{2(2n^2-6n+6)}{(n-2)n} \leq 6$ and hence

$$\frac{z}{(n-2)n} \in [-6 \cdot 7^{\gamma-1}a(P)\varepsilon, 6 \cdot 7^{\gamma-1}a(P)\varepsilon]. \tag{17}$$

By induction hypothesis,

$$R_i(P) = R_i(P^{-k}) + y^{k,1} = \sum_{e \in E} \frac{\lambda_e^h}{2}a^i(e) + x_i^{P^{-l}} + \frac{z}{(n-2)n}.$$

Let us define $x_i^P = x_i^{P^{-l}} + \frac{z}{(n-2)n}$. By (17) and $x_i^{P^{-l}} \in [7^{\gamma-1}a(P)\varepsilon, 7^{\gamma-1}a(P)\varepsilon]$, we deduce that $x_i^P \in [-7^{\gamma}a(P)\varepsilon, 7^{\gamma}a(P)\varepsilon]$.

Case 2.2.2. $a^i(l) = 0$ and $n \neq 4$. Then, $(n + 2a^i(l) - 4)n \neq 0$. Thus, the previous system of linear equations have a unique solution which is given for $y^{l,0}$ by

$$y^{l,0} = \frac{z}{(n-4)n}$$

where $z = -2z^{k,1} + (n-2)z^{l,1} + 4z^{k,0} + (-2n+4)z^{l,0}$.

Since $z^{k,1}, z^{l,1}, z^{k,0}, z^{l,0} \in [-2 \cdot 7^{\gamma-1}a(P)\varepsilon, 2 \cdot 7^{\gamma-1}a(P)\varepsilon]$, we deduce that

$$z \in \left[-3n \cdot 7^{\gamma-1}a(P)\varepsilon, 3n \cdot 7^{\gamma-1}a(P)\varepsilon\right].$$

For $n \geq 3$, $n \neq 4$ we have $\frac{6n}{(n-4)n} \leq 6$ and hence

$$y^{l,0} \in [-6 \cdot 7^{\gamma-1}a(P)\varepsilon, 6 \cdot 7^{\gamma-1}a(P)\varepsilon]. \tag{18}$$

By induction hypothesis,

$$R_i(P) = R_i(P^{-l}) + y^{l,0} = \sum_{e \in E} \frac{\lambda_e^h}{2} a^i(e) + x_i^{P^{-l}} + y^{l,0}.$$

Let us define $x_i^P = x_i^{P^{-l}} + y^{l,0}$. By (18) and $x_i^{P^{-l}} \in [-7^{\gamma-1}a(P)\varepsilon, 7^{\gamma-1}a(P)\varepsilon]$, we deduce that $x_i^P \in [-7^{\gamma}a(P)\varepsilon, 7^{\gamma}a(P)\varepsilon]$.

Case 2.2.3. $a^i(l) = 0$ and $n = 4$. Then, $(n + 2a^i(l) - 4)n = 0$. In this case we replace equation (16) by either $y^{k,0} - y^{l,0} = z^{lk,0}$ or $y^{l,1} - y^{k,1} = \frac{\lambda_l^h}{2} - \frac{\lambda_k^h}{2} + z^{lk,1}$, with $z^{lk,\cdot} \in [-2 \cdot 7^{\gamma-1}\varepsilon, 2 \cdot 7^{\gamma-1}\varepsilon]$. Now the resulting determinant is non zero. The rest of the proof is similar and we omit further details.

(b) It follows from (a), Proposition 4.2, and Proposition 4.1. ■

Remark 4.1 *We now prove that the properties used in Theorem 4.3 are independent.*

- *Let R^0 be defined as $R^0(P) = x + Sh\left(c^{tf}\right)$ for some $x \in \mathbb{R}^N$ with $\sum_{i \in N} x_i = 0$ and $x_i \neq 0$ for some $i \in N$. R^0 satisfies IIH, IIF, ETH and ETF, but fails CS and Pos.*

- *Let $\omega \in \mathbb{R}^N$ be such that $\omega_i > 0$ for all $i \in N$ and $\omega_i \neq \omega_j$ for some $i \neq j$. Let R^1 be defined for each P and each $i \in N$ as follows:*

$$R_i^1(P) = \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2} + \sum_{j \in H_i^{tf}} \frac{\omega_i}{\sum_{k \in N : j \in H_k^{tf}} \omega_k} d_j.$$

R^1 satisfies CS, Pos, IIH, IIF and ETF, but fails ETH.

- *Let R^2 be defined for each P and $i \in N$ as follows:*

$$R_i^2(P) = \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2} + \sum_{j \in H} \frac{d_j}{n}.$$

R^2 satisfies Pos, IIF, ETH and ETF, but fails CS and IIH.

- *Let R^3 be defined for each P and $i \in N$ as follows:*

$$R_i^3(P) = \frac{\sum\limits_{k \in N} \sum\limits_{j \in N} \lambda_{kj}^h}{n} + \sum_{j \in H_i^{tf}} \frac{d_j}{\left|\left\{k \in N : j \in H_k^{tf}\right\}\right|}.$$

R^3 satisfies Pos, ETH, IIH and ETF, but fails CS and IIF.

- *Let R^4 be defined for each P and $i \in N$ as follows:*

$$R_i^4(P) = \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{tf}} \frac{d_j}{\left|\left\{k \in N : j \in H_k^{tf}\right\}\right|}.$$

R^4 satisfies CS, Pos, IIH, IIF and ETH but fails ETF.

References

Alcalde-Unzu, J., Gómez-Rúa, M., and Molis, E. (2015). Sharing the costs of cleaning a river: the upstream responsibility rule. *Games and Economic Behavior*, 90:134–150.

Alumur, S. and Kara, B. Y. (2008). Network hub allocation problems: The state of the art. *European Journal of Operational Research*, 190:1–21.

Aykin, T. (1994). Lagrangean relaxation based approaches to capacitated hub-and-spoke network design problem. *European Journal of Operational Research*, 79(3):501–523.

Bailey, J. (1997). The economics of Internet interconnection agreements. In McKnight, L. and Bailey, J., editors, *Internet Economics*, pages 115–168. MIT Press, Cambridge, MA.

Bergantiños, G. and Kar, A. (2010). On obligation rules for minimum cost spanning tree problems. *Games and Economic Behavior*, 69:224–237.

Bogomolnaia, A. and Moulin, H. (2010). Sharing the cost of a minimal cost spanning tree: beyond the folk solution. *Games and Economic Behavior*, 69(2):238–248.

Bryan, D. and O'Kelly, M. (1999). Hub-and-spoke networks in air transportation: An analytical review. *Journal of Regional Science*, 39(2):275–295.

Dutta, B. and Mishra, D. (2012). Minimum cost arborescences. *Games and Economic Behavior*, 74(1):120–143.

Ernst, A. and Krishnamoorthy, M. (1999). Solution algorithms for the capacitated single allocation hub location problem. *Annals of Operations Research*, 86:141–159.

Farahani, R. Z., Hekmatfar, M., Arabani, A. B., and Nikbakhsh, E. (2013). Hub location problems: A review of models, classification, solution techniques, and applications. *Computers & Industrial Engineering*, 64(4):1096–1109.

Greenfield, D. (2000). Europe's virtual conundrum. *Network Magazine*, 15:116–123.

Guardiola, L. A., Meca, A., and Puerto, J. (2009). Production-inventory games: A new class of totally balanced combinatorial optimization games.

Games and Economic Behavior, 65(1):205–219. Special Issue in Honor of Martin Shubik.

Helme, M. and Magnanti, T. (1989). Designing satellite communication networks by zero-one quadratic programming. *Networks*, 19:427–450.

Labbé, M., Yaman, H., and Gourdin, E. (2005). A branch and cut algorithm for the hub location problems with single assignment. *Mathematical Programming*, 102:371–405.

Matsubayashi, N., Umezawa, M., Masuda, Y., and Nishino, H. (2005). A cost allocation problem arising in hub-spoke network systems. *European Journal of Operational Research*, 160:821–838.

Moulin, H. (2014). Pricing traffic in a spanning network. *Games and Economic Behavior*, 86:475–490.

Sasaki, M. and Fukushima, M. (2003). On the hub-and-spoke model with arc capacity constraints. *Journal of the Operations Research Society of Japan*, 46(4):409–428.

Sim, T., Lowe, T. J., and Thomas, B. W. (2009). The stochastic-hub center problem with service-level constraints. *Computers & Operations Research*, 36(12):3166–3177. New developments on hub location.

Skorin-Kapov, D. (1998). Hub network games. *Networks*, 31:293–302.

Skorin-Kapov, D. (2001). On cost allocation in hub-like networks. *Annals of Operations Research*, 106:63–78.

Trudeau, C. (2012). A new stable and more responsible cost sharing solution for mcst problems. *Games and Economic Behavior*, 75(1):402–412.

Trudeau, C. (2014). Minimum cost spanning tree problems with indifferent agents. *Games and Economic Behavior*, 84:137–151.

Yang, T.-H. (2009). Stochastic air freight hub location and flight routes planning. *Applied Mathematical Modelling*, 33(12):4424–4430.