The axiomatic approach to three values in games with coalition structure*

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Abstract

We study three values for transferable utility games with coalition structure, including the Owen coalitional value and two weighted versions with weights given by the size of the coalitions. We provide three axiomatic characterizations using the properties of Efficiency, Linearity, Independence of Null Coalitions, and Coordination, with two versions of Balanced Contributions inside a Coalition and Weighted Sharing in Unanimity Games, respectively.

Keywords: coalition structure, coalitional value.

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1 Introduction

Coalition structures are important in many real-world contexts, such as the formation of cartels or bidding rings, alliances or trading blocs among nation states, research joint ventures, and political parties.

These situations can be modelled through transferable utility (TU, for short) games, in which the players partition themselves into coalitions for the purpose of bargaining. All players in the same coalition agree before the play that any cooperation with other players will only by carried out collectively. That is, either all the members of the coalition take part of it or none of them (Malawski, 2004).

Given a coalition structure, bargaining occurs between coalitions and between players in the same coalition. The main idea is that the coalitions play among themselves as individual players in a game among coalitions, and then, the profit obtained by each coalition is distributed among its members. Owen (1977) studied the allocation that arises from applying the Shapley value (Shapley, 1953b) twice: first in the game among coalitions, and then in a reduced game inside each coalition.

The same two-step approach has been applied by Pulido and Sánchez-Soriano (2009) to generalize the core and Weber, and by Casas-Méndez et al. (2003) to generate the τ-value (Tijs, 1981).

In general, this approach assumes a symmetric treatment for each coalition. As Harsanyi (1977) points out, in unanimity games this procedure implies that players would be better off bargaining by themselves than joining forces. This is known as the join-bargaining paradox, or the Harsanyi paradox.

An alternative approach is to give a different treatment, or weight, to each coalition. Following this idea, Levy and McLean (1989) apply the weighted Shapley value (Shapley, 1953a; Kalai and Samet, 1987, 1988) in the game among coalitions, as well as in the reduced games. Other weighted version of the Shapley value is provided by Haeringen (2006), whereas a weighted version of the Banzhaf value is provided by Radzik et al. (1997) and Nowak and Radzik (2000).

In all these works, the weigth system is exogenously given. Hoewever, a natural weight for each coalition can also be endogenously provived by its own cardinality. In fact, a motivation for the weighted Shapley value is precisely the difference in size[1]. Moreover, Kalai and Samet (1987, Corollary 2 in Section 7) show that the

[1]Kalai and Samet (1987) present the example of large constituencies with many individuals,

cardinality of coalitions are appropriate weights for the players. The reason is that if we force the players in a coalition to work together (by destroying their resources when they are not all together), then the aggregated Shapley value of each coalition in the new game coincides with the weighted Shapley value of the game among coalitions, with weights given by the cardinality of the coalition[2].

It is then reasonable to apply the Levy and McLean value with intracoalitional symmetry and weights given by the cardinality of the coalition. However, in Levy and McLean's model, the weight of the subcoalitions in the reduced game remains constant, even though these subcoalitions may have different cardinality. An alternative approach is to vary the weight of the coalitions in the reduced game. Vidal-Puga (2006) follows this approach to define a new coalitional value. This new coalitional value does not present the Harsanyi paradox.

We have then, three reasonable generalizations of the Shapley value for games with coalition structure: the coalitional Owen value (Owen, 1977), the coalitional Levy-McLean weighted value (Levy and McLean, 1989) with the weights given by the size of the coalition, and the new value presented by Vidal-Puga (2006). In order to compare the three coalitional values, we can study which properties are satisfied by each of them, so that we can decide from these properties which is the most reasonable one for each particular situation. Moreover, when these properties completely characterize the values, we can be sure that the properties catch their essence.

In this paper, we characterize the above coalitional values. These three values have in common the following feature: First, the worth of the grand coalition is divided among the coalitions following either the Shapley value (Owen), or the weighted Shapley value with weights given by the size of the coalitions (Levy and McLean, Vidal-Puga), and then the profit obtained by each coalition is distributed among its members following the Shapley value of an appropriately defined "reduced" game.

Some of the axioms used in the characterizations (*efficiency, intracoalitional symmetry,* and *linearity*) are standard in the literature, others (*independence of*

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in contrast with constituencies composed by a small number of individuals.

[2]Another possibility is to give the worth of any coalition to any of its nonempty subcoalitions. In this case, the aggregated Shapley value of each coalition coincides with the weighted Shapley value of the *dual* game among coalitions (see Kalai and Samet, 1987, Section 7, for further details).

null coalitions and two intracoalitional versions of *balanced contributions*) are used in many different frameworks. The property of independence of null coalitions is related to the standard property of *null player*. However, the role of null players is important as their only presence affects the size of the coalition. In fact, two of the three values (Owen, Levy and McLean) satisfy the null player property, whereas the third one (Vidal-Puga) does not.

Additionally, we introduce new properties in this kind of problems: *coordination* (which asserts that internal changes in a coalition which do no affect the game among coalitions, do not influence the final payment of the rest of the players) and two properties of *sharing in unanimity games* (which establish how should the payment be under the grand coalition in unanimity game).

The properties of efficiency, linearity, intracoalitional symmetry and independence of null coalitions are natural extensions of the classical properties that characterize the Shapley value (efficiency, linearity, symmetry and null player, respectively) to the game among coalitions. On the other hand, the properties of balanced contributions are applied to the game inside a coalition, and each of them is a natural extension of the property of balanced contribution that also characterizes, with efficiency, the Shapley value (Myerson, 1980). Moreover, the property of *balanced intracoalitional contributions* is used to characterize the value proposed by Vidal-Puga (2006) in Gómez-Rúa and Vidal-Puga (2008). Hence, the three values proposed here can be seen as natural extensions of the Shapley value for games with coalition structure. Additionally, the property of coordination formalizes the idea presented by Owen that the players inside a coalition negotiate among them, but always assuming that the rest of the coalitions remain together (see for example the game v_1 defined by Kalai and Samet, 1987, Section 7).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we define a family that includes the three coalitional values. In Section 4 we present the properties used in the characterization and we study which properties the coalitional values satisfy. In Section 5 we present the characterization results. In Section 6 we prove that the properties are independent. In Section 7 we present some concluding remarks.

2 Notation

Let $U = \{1, 2, ...\}$ be the (infinite) set of potential players.

Given a finite subset $N \subset U$, let $\Pi(N)$ denote the set of all orders in N. Given $\pi \in \Pi(N)$, let $Pre(i, \pi)$ denote the set of the elements in N which come before i in the order given by π, *i.e.* $Pre(i, \pi) = \{j \in N : \pi(j) < \pi(i)\}$. For any $S \subset N$, π_S denotes the order induced in S by π (for all $i, j \in S$, $\pi_S(i) < \pi_S(j)$ if and only if $\pi(i) < \pi(j)$).

Let $|C|$ denote the cardinality of a set C.

A *transfer utility game*, *TU game*, or simply a *game*, is a pair (N, v) where $N \subset U$ is finite and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. When N is clear, we can also denote (N, v) as v. Given a *TU* game (N, v) and $S \subset N$, $v(S)$ is called the *worth* of S. Given $S \subset N$, we denote the restriction of (N, v) to S as (S, v).

For simplicity, we usually write $S \cup i$ instead of $S \cup \{i\}$, $N \backslash i$ instead of $N \backslash \{i\}$, and $v(i)$ instead of $v(\{i\})$.

Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subset N \backslash \{i, j\}$. A player $i \in N$ is *null* in (N, v) if $v(T \cup i) = v(T)$ for all $T \subset N \backslash i$. The set of non-null players in (N, v) is the *minimal carrier* of (N, v), and we denote it as $MC(N, v)$. Given two games (N, v), (N, w), the game $(N, v + w)$ is defined as $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$. Given a game (N, v) and a real number α, the game $(N, \alpha v)$ is defined as $(\alpha v)(S) = \alpha v(S)$ for all $S \subset N$.

Given $N \subset U$ finite, we call *coalition structure* over N a partition of the player set N, *i.e.* $\mathcal{C} = \{C_1, C_2, ..., C_m\} \subset 2^N$ is a coalition structure if it satisfies $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. We also assume $C_q \neq \emptyset$ for all q.

We say that $C_q \in \mathcal{C}$ is a *null coalition* if all its members are null players.

For any $S \subset N$, we denote the restriction of $\mathcal{C}$ to the players in S as $\mathcal{C}_S$, *i.e.* $\mathcal{C}_S = \{C_q \cap S : C_q \in \mathcal{C} \text{ and } C_q \cap S \neq \emptyset\}$.

For any $S \subset C_q \in \mathcal{C}$, we will frequently study the case in which the players in $C_q \backslash S$ leave the game. In this case, we write $\mathcal{C}^S$ instead of the more cumbersome $\mathcal{C}_{N \backslash (C_q \backslash S)}$.

Given a game (N, v) and a coalition structure $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N, the *game among coalitions* is the TU game $(M, v/\mathcal{C})$ where $M = \{1, 2, ..., m\}$ and $(v/\mathcal{C})(Q) = v\left(\bigcup_{q \in Q} C_q\right)$ for all $Q \subset M$.

We denote the *game* (N, v) *with coalition structure* $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N as $(N, v, \mathcal{C})$ or $(v, \mathcal{C})$. When no confusion can arise, we write v instead of $(N, v, \mathcal{C})$.

Given $S \subset N$, $S \neq \emptyset$, the *unanimity game with minimal carrier* S, (N, u_N^S) is defined as $u_N^S(T) = 1$ if $S \subset T$ and $u_N^S(T) = 0$ otherwise, for all $T \subset N$. We call

(N, u_N^N) the *unanimity game.*

A *value* is a function that assigns to each game (N, v) a vector in $\mathbb{R}^N$ representing the amount that each player in N expects to get in the game. One of the most important values in TU games is the *Shapley value* (Shapley, 1953b). We denote the Shapley value of the TU game (N, v) as $Sh(N, v) \in \mathbb{R}^N$.

Similarly, a *coalitional value* is a function that assigns to each game with coalition structure $(N, v, \mathcal{C})$ a vector in $\mathbb{R}^N$. Each value can also be considered as a coalitional value. Hence, we define the *coalitional Shapley value* of the game $(N, v, \mathcal{C})$ as $Sh\,(N, v, \mathcal{C}) = Sh\,(N, v)$. One of the most important coalitional values is the *Owen value* (Owen, 1977).

Another generalization for a value is the following: a *weighted value* ϕ^ω is a function that assigns to each TU game (N, v) and each $x \in \mathbb{R}_+^N$ a vector $\phi^x\,(N, v)$ in $\mathbb{R}^N$. For each $i \in N$, x_i is the *weight* of player i. We will say that a weighted value ϕ^ω *extends* or *generalizes* a value ϕ if $\phi^x\,(N, v) = \phi\,(N, v)$ for any weight vector x with $x_i = x_j$ for all $i, j \in N$. The most prominent weighted generalization of the Shapley value is the *weighted Shapley value* Sh^ω (Shapley (1953a), Kalai and Samet (1987, 1988)), which is defined as

$$Sh_i^\omega\,(N, v) = \sum_{S \ni i} \frac{\omega_i}{\sum_{j \in S} \omega_j} \sum_{T \subset S} (-1)^{|S|-|T|}\, v\,(T)$$

for all $i \in N$ and each $\omega \in \mathbb{R}_+^N$.

3 Games with coalition structure

We now focus on games with coalition structure. Fix $\mathcal{C} = \{C_1, ..., C_m\}$ and let $M = \{1, ..., m\}$. For each pair (γ, ϕ^ω), where γ is a value and ϕ^ω is a weighted value, we define two coalitional values $\gamma\,[\phi^\omega]$ and $\gamma\,\langle\phi^\omega\rangle$. In both cases, the idea is to divide the worth of the grand coalition in two steps: In the first step, ϕ^ω is used in the game among coalitions, with weights given by the size of each coalition. In the second step, γ is used inside each coalition.

For each coalition structure $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N, let $\sigma\,(\mathcal{C}) \in \mathbb{R}_+^M$ be defined as $\sigma_q\,(\mathcal{C}) = |C_q|$ for all $q \in M$. Given $C_q \in \mathcal{C}$, the *reduced TU game with fixed weights* $\left(C_q, v_{C_q}^{[\phi^\omega]N}\right)$ is defined as $v_{C_q}^{[\phi^\omega]N}(S) := \phi_q^{\sigma(\mathcal{C})}\left(M, v/\mathcal{C}^S\right)$ for all

$S \subset C_q$. The *reduced TU game with relaxed weights* $\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$ is defined as $v_{C_q}^{\langle\phi^\omega\rangle N}(S) := \phi_q^{\sigma(\mathcal{C}^S)}\left(M, v/\mathcal{C}^S\right)$ for all $S \subset C_q$.

Thus, both $v_{C_q}^{[\phi^\omega]N}(S)$ and $v_{C_q}^{\langle\phi^\omega\rangle N}(S)$ are interpreted as the value that ϕ^ω assigns to coalition S in the game among coalitions assuming that the members of $C_q \backslash S$ are out. In the first case, coalition S maintains the weight of the original coalition C_q. In the second case, coalition S plays with a weight proportional to its own (reduced) size.

In the particular case $\phi^x = \phi$ for all x, both reduced TU games coincide and we write $\left(C_q, v_{C_q}^{(\phi)N}\right)$ instead of $\left(C_q, v_{C_q}^{[\phi^\omega]N}\right)$ or $\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$.

Definition 1 *Given a value γ and a weighted value ϕ^ω, we define respectively the coalitional values $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ as $\gamma[\phi^\omega]_i(N, v, \mathcal{C}) := \gamma_i\left(C_q, v_{C_q}^{[\phi^\omega]N}\right)$ and $\gamma\langle\phi^\omega\rangle_i(N, v, \mathcal{C}) := \gamma_i\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$ for all $i \in C_q \in \mathcal{C}$. In the particular case $\phi^x = \phi$ for all x, both expressions coincide and hence we write $\gamma(\phi) := \gamma[\phi^\omega] = \gamma\langle\phi^\omega\rangle$.*

We concentrate on three particular members of this family, that have been previously studied in the literature:

Example 2 *$Sh(Sh)$ is the Owen value (Owen, 1977).*

$Sh[Sh^\omega]$ is the weighted coalitional value with intracoalitional symmetry, and weights given by the size of the coalitions (Levy and McLean, 1989).

$Sh\langle Sh^\omega\rangle$ has been studied by Vidal-Puga (2006).

We provide an example to compute these three values, in order to highlight their differences: Let $(N, v, \mathcal{C})$ given by $N = \{1, 2, 3\}$, $v(N) = v(\{1, 2\}) = 1$, $v(\{1, 3\}) = \alpha$ with $\alpha \in [0, 1)$, and $v(S) = 0$ otherwise. This game arises when a seller has an item that is valued diferenty by two potential buyers; or when an employer should decide to hire an employee from two possible candidates with different capabilities; or when two complementary items (produced by players 1 and 2, respectively) face the presence of a third item that may imperfectly substitute the item produced[3] by player 2.

[3]Hence, items $\{1, 2\}$ are worth more than items $\{1, 3\}$ but no other pair of items is valuable.

When players 1 and 3 act together, we have the coalition structure $\mathcal{C} = \{C_1, C_2\}$ with $C_1 = \{1,3\}$ and $C_2 = \{2\}$. The game among coalitions is then $(\{1,2\}, v/\mathcal{C})$ with $(v/\mathcal{C})(\{1\}) = \alpha$, $(v/\mathcal{C})(\{2\}) = 0$, and $(v/\mathcal{C})(\{1,2\}) = 1$. The Shapley value is $\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right)$, whilst the weighted Shapley value (with weights $\omega_1 = 2$ and $\omega_2 = 1$) is $\left(\frac{2+\alpha}{3}, \frac{1-\alpha}{3}\right)$.

In order to compute $Sh\,(Sh)$, $Sh\,[Sh^\omega]$ and $Sh\,\langle Sh^\omega \rangle$, we need to know how these values change when either player 1 or 3 leaves coalition C_1. These values are summarized in the following table.

	$Sh\,(Sh)$	$Sh\,[Sh^\omega]$	$Sh\,\langle Sh^\omega \rangle$
None leaves	$\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right)$	$\left(\frac{2+\alpha}{3}, \frac{1-\alpha}{3}\right)$	$\left(\frac{2+\alpha}{3}, \frac{1-\alpha}{3}\right)$
Player 1 leaves	$(0,0)$	$(0,0)$	$(0,0)$
Player 3 leaves	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$
Gains of cooperation	$\frac{\alpha}{2}$	$\frac{\alpha}{3}$	$\frac{1+2\alpha}{6}$

In the first column $(Sh\,(Sh))$, the Shapley value is computed. In the second column and third colum, the weighted Shapley value is computed. For $Sh\,[Sh^\omega]$, coalition C_1 always has weight 2. For $Sh\,\langle Sh^\omega \rangle$, coalition C_1 has weight 1 when eiher player 1 or 3 leaves.

We now focus on the second step for coalition C_1: For $Sh\,(Sh)$, the reduced game is given by $v_{C_1}^{(Sh)N}(\{1\}) = \frac{1}{2}$, $v_{C_1}^{(Sh)N}(\{3\}) = 0$, and $v_{C_1}^{(Sh)N}(\{1,3\}) = \frac{1+\alpha}{2}$. The Shapley value is $\left(\frac{2+\alpha}{4}, \frac{\alpha}{4}\right)$. Hence, the final pyaoff allocation is $Sh\,(Sh)\,(N, v, \mathcal{C}) = \left(\frac{2+\alpha}{4}, \frac{1-\alpha}{2}, \frac{\alpha}{4}\right)$.

Analogously, we have $Sh\,[Sh^\omega]\,(N, v, \mathcal{C}) = \left(\frac{4+\alpha}{6}, \frac{1-\alpha}{3}, \frac{\alpha}{6}\right)$ and $Sh\,\langle Sh^\omega \rangle\,(N, v, \mathcal{C}) = \left(\frac{7+2\alpha}{12}, \frac{1-\alpha}{3}, \frac{1+2\alpha}{12}\right)$. Notice that $Sh\,\langle Sh^\omega \rangle$ always assings player 3 a positive payoff, even when $\alpha = 0$ (and therefore, player 3 is null).

There exist other relevant coalitional values that belong to this family. Let Ba be the *Banzhaf value* (Banzhaf 1965, Owen 1975). Let In be the *individual value* (Owen[4], 1978) defined as $In_i(N, v) = v(\{i\})$ for all $i \in N$. Given $p \in [0,1]$, let B^p be the p-binomial value (Puente, 2000). Let DP be the *Deegan-Packel value* (Deegan and Packel, 1978). Let LSP be the *least square prenucleolus* (Ruiz, Valenciano and Zarzuelo, 1996).

Example 3 *$Sh\,(In)$ is the Aumann-Drèze value (Aumann and Drèze, 1974).*

[4]Owen uses the term *dictatorial* instead of *individual*.

$Ba\,(Ba)$ is the Banzhaf-Owen value (Owen 1975).

$Sh\,(Ba)$ is the symmetric coalitional Banzhaf value (Alonso-Meijide and Fiestras-Janeiro, 2002).

$Ba\,(Sh)$ is defined and studied by Amer, Carreras and Giménez (2002).

$\{Sh\,(B^p)\}_{p\in[0,1]}$ is the family of symmetric coalitional binomial values (Carreras and Puente, 2006).

$DP\,(DP)$ and $LSP\,(LSP)$ are defined and studied by Młodak (2003).

4 Properties

In this section we present some properties of the values. Moreover, we provide several results.

4.1 Classical properties

Efficiency (Eff) For any game (N, v), $\sum_{i\in N} f_i\,(N, v) = v\,(N)$.

That is, the worth of the grand coalition is distributed.

Linearity (Lin) Given (N, v), (N, w) and real numbers α and β,

$$f\,(N, \alpha v + \beta w) = \alpha f\,(N, v) + \beta f\,(N, w).$$

That is, if a game is a linear combination of two games, the value assigns the linear combination of the values of the games.

Symmetry (Sym) Given two symmetric players $i, j \in N$ in a game (N, v), $f_i\,(N, v) = f_j\,(N, v)$.

That is, two symmetric players in (N, v) receive the same.

Null Player (NP) Given a null player $i \in N$ in (N, v), $f_i\,(N, v) = 0$.

That is, any null player receives zero.

Independence of Null Players (INP) Given a null player $i \in N$ in a game (N, v), for all $j \in N\backslash i$, $f_j\,(N, v) = f_j\,(N\backslash i, v)$

That is, no player gets a different value if a null player is removed from the game.

We say that a weighted value ϕ^ω satisfies some property if ϕ^x satisfies this property for each x.

Proposition 4 *a) The Shapley value Sh is the only value that satisfies Eff, Lin, Sym and INP.*

b) The weighted Shapley value Sh^ω satisfies INP.

Proof. a) It is well-known that Sh satisfies Eff, Lin and Sym. It is also clear that Sh satisfies INP. On the other hand, it is straightforward to check that Eff and INP imply NP. Since Sh is the only value that satisfies Eff, Lin, Sym and NP (Shapley, 1953b), we deduce the result.

b) From Kalai and Samet (1987, Theorem 1) and a classical induction hypothesis on the number of players, it is straightforward to check that Sh^ω satisfies INP. ■

These properties can be adapted to games with coalition structure without changes. For Sym and INP, we will also apply them inside the coalitions and to null coalitions, respectively:

Intracoalitional Symmetry (IS) Given two symmetric players in the same coalition $i, j \in C_q \in \mathcal{C}$, $f_i(N, v, \mathcal{C}) = f_j(N, v, \mathcal{C})$.

Independence of Null Coalitions (INC) Given a game $(N, v, \mathcal{C})$ and a null coalition $C_q \in \mathcal{C}$, $f_i(N, v, \mathcal{C}) = f_i\left(N \backslash C_q, v, \mathcal{C}_{N \backslash C_q}\right)$ for all $i \in N \backslash C_q$.

INC asserts that if a coalition is *null*, it does not influence the allocation within the rest of the players. It is a weaker property than INP. Notice that INC and Eff imply that the aggregated payment of the players in a null coalition is zero.

Proposition 5 *a) If both γ and ϕ^ω satisfy Eff, then both $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ satisfy Eff.*

b) If both γ and ϕ^ω satisfy Lin, then both $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ satisfy Lin.

c) If γ satisfies Sym, then both $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ satisfy IS.

d) If ϕ^ω satisfies INP, then both $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ satisfy INC.

Proof. *Parts a), b) and c) are straightforward from the definition.*

d) We prove the result for $\gamma[\phi^\omega]$. *The result for* $\gamma\langle\phi^\omega\rangle$ *is analogous. Let* $\mathcal{C}=\{C_1,...,C_m\}$ *and let* $C_q\in\mathcal{C}$ *be a null coalition. Denote* $M=\{1,2,...,m\}$. *To prove that* $\gamma[\phi^\omega]_i(N,v,\mathcal{C})=\gamma[\phi^\omega]_i(N\backslash C_q,v,\mathcal{C}_{N\backslash C_q})$ *for all* $i\in N\backslash C_q$ *it is enough to prove that* $v_{C_r}^{[\phi^\omega]N}(S)=v_{C_r}^{[\phi^\omega]N\backslash C_q}(S)$ *for all* $S\subset C_r\in\mathcal{C}\backslash\{C_q\}$.

Take $S\subset C_r\in\mathcal{C}\backslash\{C_q\}$. *By definition,* $v_{C_r}^{[\phi^\omega]N}(S)=\phi_r^{\sigma(\mathcal{C})}(M,v/\mathcal{C}^S)$.

Since $\phi^{\sigma(\mathcal{C})}$ *satisfies* INP, *we have* $\phi_r^{\sigma(\mathcal{C})}\left(M,v/\mathcal{C}^S\right)=\phi_r^{\sigma(\mathcal{C})}\left(M\backslash q,v/\mathcal{C}_{N\backslash C_q}^S\right)$.

Notice that there is no ambiguity in the notation $v/C_{N\backslash C_q}^S$ *because* $\left(\mathcal{C}^S\right)_{N\backslash C_q}=\left(\mathcal{C}_{N\backslash C_q}\right)^S$. *By definition,* $\phi_r^{\sigma(\mathcal{C})}\left(M\backslash q,v/\mathcal{C}_{N\backslash C_q}^S\right)=v_{C_r}^{[\phi^\omega]N\backslash C_q}(S)$.

Combining the three last expressions we obtain the result. ■

Corollary 6 $Sh(Sh)$, $Sh[Sh^\omega]$ *and* $Sh\langle Sh^\omega\rangle$ *satisfy* Eff, *Lin*, IS *and* INC.

4.2 Properties of Balanced Contributions

The principle of Balanced Contributions is used in different contexts. Myerson (1977) was the first to use it for *games with graphs*. He called it *Fairness*. Later, Myerson (1980) characterized the Shapley value with balanced contributions and efficiency. The principle of balanced contributions has also been used in other contexts: Amer and Carreras (1995) and Calvo, Lasaga and Winter (1996) characterized the Owen value; Calvo and Santos (2000) characterized a value for multi-choice games; Bergantiños and Vidal-Puga (2005) characterized an extension of the Owen value for non-transferable utility games; Calvo and Santos (2006) characterized the subsidy-free serial cost sharing method (Moulin, 1995) in discrete cost allocation problems; Alonso-Meijide, Carreras and Puente (2007) characterized a parametric family of coalitional values; and Lorenzo-Freire et al. (2007) defined a property of balanced contributions in the context of cooperative games with transferable utility and awards.

Balanced Contributions (BC) Given a game (N,v), for all $i,j\in N$,

$$f_i(N,v)-f_i(N\backslash j,v)=f_j(N,v)-f_j(N\backslash i,v).$$

This property states that for any two players, the amount that each player would gain or lose by the other's withdrawal from the game should be equal.

A remarkable property of this principle is that it completely characterizes the Shapley value with the only help of efficiency.

Proposition 7 *(Myerson, 1980) Sh is the only value that satisfies Eff and BC.*

A similar, yet different version of BC arises when we make the players become null, instead of leaving the game: Given (N, v) and $i \in N$, we define (N, v^{-i}) as $v^{-i}(S) = v(S \cap (N \backslash i))$ for all $S \subset N$. Hence, in (N, v^{-i}) player i becomes a null player.

Null Balanced Contributions (NBC) Given a game (N, v), for all $i, j \in N$,
$f_i(N, v) - f_i(N, v^{-j}) = f_j(N, v) - f_j(N, v^{-i})$.

This property sates that for any two players, the amount that each player would gain or lose by the other's becaming null should be equal. In this context, the fact that an agent becomes null can be interpreted as this player losing his resources.

Under Eff and Sym, NBC and BC are equivalent:

Proposition 8 *Sh is the only value that satisfies Eff, NBC and Sym.*

Proof. It is well-known that Sh satisfies Eff, Sym and INP. For any null player $j \in N$, $(N \backslash j, v^{-j}) = (N \backslash j, v)$. Since Sh satisfies INP, we have $Sh_i(N, v^{-j}) = Sh_i(N \backslash j, v^{-j}) = Sh_i(N \backslash j, v)$ for any $i \in N \backslash j$. Hence, BC and NBC are equivalent for Sh. Since Sh satisfies BC (Proposition 7), Sh also satisfies NBC.

To see the uniqueness, let f be a value satisfying these properties. Fix (N, v). We proceed by induction on $|Carr(N, v)|$. If $|Carr(N, v)| = 0$, the result holds from Eff and Sym. Assume the result holds for less than $|Carr(N, v)|$ non-null players, with $|Carr(N, v)| > 0$. Let $i \in N$.

Assume first that player i is a null player. Obviously, $(N, v) = (N, v^{-i})$. For any $j \in Carr(N, v)$, under NBC, $f_i(N, v) - f_i(N, v^{-j}) = f_j(N, v) - f_j(N, v^{-i}) = 0$
and hence $f_i(N, v) = f_i(N, v^{-j})$. By induction hypothesis, $f_i(N, v) = Sh_i(N, v^{-j}) = 0$ because i is also a null player in (N, v^{-j}).

Assume now $i \in Carr(N,v)$. Under NBC, $f_i(N,v) - f_i(N,v^{-j}) = f_j(N,v) - f_j(N,v^{-i})$ for all $j \in N \backslash i$, and hence

$$\begin{aligned} & (n-1) f_i(N,v) - \sum\nolimits_{j \in N \backslash Carr(N,v)} f_i\left(N,v^{-j}\right) - \sum\nolimits_{j \in Carr(N,v) \backslash i} f_i\left(N,v^{-j}\right) \\ = & \sum\nolimits_{j \in N \backslash i} f_j(N,v) - \sum\nolimits_{j \in N \backslash i} f_j\left(N,v^{-i}\right). \end{aligned}$$

Obviously, $f_i(N,v) = f_i(N,v^{-j})$ for all $j \in N \backslash Carr(N,v)$. Hence,

$$\begin{aligned} & (|Carr(N,v)| - 1) f_i(N,v) - \sum\nolimits_{j \in Carr(N,v) \backslash i} f_i(N,v^{-j}) \\ & = \sum\nolimits_{j \in N \backslash i} f_j(N,v) - \sum\nolimits_{j \in N \backslash i} f_j(N,v^{-i}). \end{aligned}$$

Under Eff, $\sum_{j \in N \backslash i} f_j(N,v) = v(N) - f_i(N,v)$ and hence,

$$f_i(N,v) = \frac{1}{|Carr(N,v)|}\left[v(N) + \sum\nolimits_{j \in Carr(N,v) \backslash i} f_i\left(N,v^{-j}\right) - \sum\nolimits_{j \in N \backslash i} f_j\left(N,v^{-i}\right)\right].$$

Under the induction hypothesis, $f(N,v^{-j}) = Sh(N,v^{-j})$ for all $j \in Carr(N,v)$ and hence

$$f_i(N,v) = \frac{1}{|Carr(N,v)|}\left[v(N) + \sum\nolimits_{j \in Carr(N,v) \backslash i} Sh_i\left(N,v^{-j}\right) - \sum\nolimits_{j \in N \backslash i} Sh_j\left(N,v^{-i}\right)\right]$$

from where we deduce that $f_i(N,v)$ is unique for all $i \in Carr(N,v)$. ■

Remark 9 *Sym is needed in the previous characterization. Let $f^{\{1,2\}}$ be defined as follows: If $\{1,2\} \subseteq N$, then $f_1^{\{1,2\}}(N,v) = Sh_1(N,v) + 1$, $f_2^{\{1,2\}}(N,v) = Sh_2(N,v) - 1$, and $f_i^{\{1,2\}}(N,v) = Sh_i(N,v)$ otherwise. If $\{1,2\} \nsubseteq N$, then $f^{\{1,2\}}(N,v) = Sh(N,v)$. This value satisfies Eff and NBC, but $f^{\{1,2\}} \neq Sh$.*

Remark 10 *Young (1985) characterized Sh as the only value that satisfies Eff, Sym and Strong Monotonicity (SM). This last property says that $f_i(N,v) \geq f_i(N,v')$ whereas $v(S \cup i) - v(S) \geq v'(S \cup i) - v'(S)$ for all $S \subset N \backslash i$. Hence, Proposition 8 implies that NBC and SM are equivalent under Eff and Sym.*

In order to keep the essence of the Shapley value at the intracoalitional level, we force (null) balanced contributions inside a coalition:

Balanced Intracoalitional Contributions (BIC) Given a game $(N, v, \mathcal{C})$, for all $i, j \in C_q \in \mathcal{C}$, $f_i(N, v, \mathcal{C}) - f_i\left(N \backslash j, v, \mathcal{C}_{N \backslash j}\right) = f_j(N, v, \mathcal{C}) - f_j\left(N \backslash i, v, \mathcal{C}_{N \backslash i}\right).$

This property states that for any two players that belong to the same coalition in $\mathcal{C}$, the amount that each player would gain or lose by the other's withdrawal from the game should be equal.

Null Balanced Intracoalitional Contributions $(NBIC)$ Given a game $(N, v, \mathcal{C})$, for all $i, j \in C_q \in \mathcal{C}$, $f_i(N, v, \mathcal{C}) - f_i(N, v^{-j}, \mathcal{C}) = f_j(N, v, \mathcal{C}) - f_j(N, v^{-i}, \mathcal{C}).$

This property states that for any two players that belong to the same coalition in $\mathcal{C}$, the amount that each player would gain or lose if the other becomes null should be equal.

Proposition 11 *a) If γ satisfies NBC, then $\gamma[\phi^\omega]$ satisfies $NBIC$.*

b) If γ satisfies BC, then $\gamma\langle\phi^\omega\rangle$ satisfies BIC.

Proof. *Fix $C_q \in \mathcal{C}$ and $i, j \in C_q$.*

a) By definition of $\gamma[\phi^\omega]$, $v_{C_q}^{[\phi^\omega]N}$ and (N, v^{-j}), it is straightforward to check that $\gamma[\phi^\omega]_i(N, v, \mathcal{C}) - \gamma[\phi^\omega]_i(N, v^{-j}, \mathcal{C}) = \gamma_i\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_i\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-j}\right)$. Since γ satisfies NBC, we have $\gamma[\phi^\omega]_i(N, v, \mathcal{C}) - \gamma[\phi^\omega]_i(N, v^{-j}, \mathcal{C}) = \gamma_j\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_j\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-i}\right)$.

Reasoning as before, $\gamma[\phi^\omega]_j(N, v, \mathcal{C}) - \gamma[\phi^\omega]_j(N, v^{-i}, \mathcal{C}) = \gamma_j\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_j\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-i}\right)$ and hence the result.

b) By definition of $\gamma\langle\phi^\omega\rangle$ and $v_{C_q}^{\langle\phi^\omega\rangle N}$, it is straightforward to check that $\gamma\langle\phi^\omega\rangle_i(N, v, \mathcal{C}) - \gamma\langle\phi^\omega\rangle_i(N \backslash j, v, \mathcal{C}_{N \backslash j}) = \gamma_i\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right) - \gamma_i\left(C_q \backslash j, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$. Since γ satisfies BC, we have $\gamma\langle\phi^\omega\rangle_i(N, v, \mathcal{C}) - \gamma\langle\phi^\omega\rangle_i(N \backslash j, v, \mathcal{C}_{N \backslash j}) = \gamma_j\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right) - \gamma_j\left(C_q \backslash i, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$.

Reasoning as before, $\gamma\langle\phi^\omega\rangle_j(N, v, \mathcal{C}) - \gamma\langle\phi^\omega\rangle_j(N \backslash i, v, \mathcal{C}_{N \backslash j}) = \gamma_j\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right) - \gamma_j\left(C_q \backslash i, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$ and hence the result. ■

Corollary 12 *The Owen value $Sh\,(Sh)$ satisfies both BIC and $NBIC$; $Sh\,[Sh^{\omega}]$ satisfies $NBIC$; $Sh\,\langle Sh^{\omega}\rangle$ satisfies BIC.*

Even though Proposition 7 and Proposition 8 show that BC and NBC are equivalent under Eff and Sym, this is not the case for their intracoalitional versions:

Remark 13 *a) $Sh\,[Sh^{\omega}]$ does not satisfy BIC. Let $N = \{1,2,3\}$ and v defined as $v\,(S) = 1$ if $\{1,2\} \subset S$ or $\{1,3\} \subset S$, and $v\,(S) = 0$ otherwise. Let $\mathcal{C} = \{\{1,2\},\{3\}\}$. Then,*

$$\begin{aligned} Sh\,[Sh^{\omega}]_1\,(N,v,\mathcal{C}) - Sh\,[Sh^{\omega}]_1\left(N\backslash 2, v, \mathcal{C}_{N\backslash 2}\right) &= \frac{5}{6} - \frac{1}{2} = \frac{1}{3} \\ Sh\,[Sh^{\omega}]_2\,(N,v,\mathcal{C}) - Sh\,[Sh^{\omega}]_2\left(N\backslash 1, v, \mathcal{C}_{N\backslash 1}\right) &= \frac{1}{6} - 0 = \frac{1}{6}. \end{aligned}$$

b) $Sh\,\langle Sh^{\omega}\rangle$ does not satisfy $NBIC$. Let $(N,v,\mathcal{C})$ be defined as in a). Then,

$$\begin{aligned} Sh\,\langle Sh^{\omega}\rangle_1\,(N,v,\mathcal{C}) - Sh\,\langle Sh^{\omega}\rangle_1\left(N, v^{-2}, \mathcal{C}\right) &= \frac{3}{4} - \frac{7}{12} = \frac{1}{6} \\ Sh\,\langle Sh^{\omega}\rangle_2\,(N,v,\mathcal{C}) - Sh\,\langle Sh^{\omega}\rangle_2\left(N, v^{-1}, \mathcal{C}\right) &= \frac{1}{4} - 0 = \frac{1}{4}. \end{aligned}$$

Notice that these values apply Sh in the game inside the coalitions, and, moreover, the Shapley value satisfies BC. However, BIC does not hold for $Sh\,[Sh^{\omega}]$ because the weight of coalition C_q is affected by the withdrawn of one of its players. On the other hand, $NBIC$ does not hold in $Sh\,\langle Sh^{\omega}\rangle$ because of a subtler different reason. The player that becomes null does so in v, but not in the intracoalitional game, where it still maintains the weight of the coalition.

4.3 Other properties

Coordination (Co) Fix $\mathcal{C}$. For all v, v' and $C_q \in \mathcal{C}$, if $v\left(T \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = v'\left(T \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$for all $T \subset C_q$ and all $\mathcal{R} \subset \mathcal{C}\backslash\{C_q\}$, then, $f_i\,(N,v,\mathcal{C}) = f_i\,(N,v',\mathcal{C})$ for all $i \in C_q$.

This property says that, given a coalition C_q, if there are changes inside other coalitions, but these changes do not affect the worth of any subset of C_q with the rest of coalitions, then these internal changes in the other coalitions do not affect the final payment of each player in C_q.

Proposition 14 *$\gamma(\phi)$, $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ satisfy Co for every ϕ, ϕ^ω and γ.*

Proof. *Let $\mathcal{C}$, v and v' be such that $v\left(T\cup\bigcup_{C_r\in\mathcal{R}}C_r\right)=v'\left(T\cup\bigcup_{C_r\in\mathcal{R}}C_r\right)$ for all $T\subset C_q$ and all $\mathcal{R}\subset\mathcal{C}\setminus\{C_q\}$. It is enough to prove that $v_{C_q}^{[\phi^\omega]N}(S)=v_{C_q}^{\prime[\phi^\omega]N}(S)$ and $v_{C_q}^{\langle\phi^\omega\rangle N}(S)=v_{C_q}^{\prime\langle\phi^\omega\rangle N}(S)$ for all $S\subset C_q$. By the condition satisfied by v and v' we have that $\left(M,v/\mathcal{C}^S\right)=\left(M,v'/\mathcal{C}^S\right)$ for all $S\subset C_q$. Hence, $\phi_q^{\sigma(\mathcal{C})}\left(M,v/\mathcal{C}^S\right)=\phi_q^{\sigma(\mathcal{C})}\left(M,v'/\mathcal{C}^S\right)$ and $\phi_q^{\sigma\left(\mathcal{C}^S\right)}\left(M,v/\mathcal{C}^S\right)=\phi_q^{\sigma\left(\mathcal{C}^S\right)}\left(M,v'/\mathcal{C}^S\right)$ for all $S\subset C_q$. By the definition of the reduced games, we have the result.* ■

Frequently, it is interpreted that players form coalitions in order to improve their bargaining strength (Hart and Kurz, 1983). However, as Harsanyi (1977) points out, the bargaining strength does not improve in general. An individual can be worse off bargaining as a member of a coalition than bargaining alone. This is what is known as the "Harsanyi paradox".

The following property prevents the "Harsanyi paradox" in the case where all the players are symmetric. In the unanimity game (with minimal carrier N) all the players are necessary to obtain a positive payment. Hence it seems reasonable that their assignment should be independent of the coalitional structure:

Equal Sharing in Unanimity Games (ESUG) For any $\mathcal{C}$, and for all $i,j\in N$, $f_i\left(N,u_N^N,\mathcal{C}\right)=f_j\left(N,u_N^N,\mathcal{C}\right)$

This property asserts that in the unanimity game, each player should receive the same payment, regardless of $\mathcal{C}$.

The Owen value does not satisfy $ESUG$ but a weighted version:

Inverse Proportional Sharing in Unanimity Games ($IPSUG$) For any game $\left(N,u_N^N,\mathcal{C}\right)$, any coalitions $C_q,C_r\in\mathcal{C}$ and for all $i\in C_q$ and $j\in C_r$, $|C_q|\,f_i\left(N,u_N^N,\mathcal{C}\right)=|C_r|\,f_j\left(N,u_N^N,\mathcal{C}\right)$.

This property asserts that under the unanimity game, each player should receive a payment inversely proportional to the size of the coalition he belongs to. A similar property is the following:

Coalitional Symmetry in Unanimity Games ($CSUG$) For any game $\left(N, u_N^N, \mathcal{C}\right)$, and any coalitions $C_q, C_r \in \mathcal{C}$, $\sum_{i \in C_q} f_i\left(N, u_N^N, \mathcal{C}\right) = \sum_{i \in C_r} f_i\left(N, u_N^N, \mathcal{C}\right)$.

The latter three properties are defined following the philosophy of other axioms in the literature, which determine how the payoff allocation should be in unanimity games. See, for instance, Aumann (1985, Axiom 3) and Winter (1991, Axiom A6-Unanimity Games).

It is straightforward to check that, under IS, $CSUG$ is equivalent to $IPSUG$. We use $IPSUG$ because it follows the same formulation as $ESUG$.

In addition to Eff, either $ESUG$ or $IPSUG$ would determine the coalitional value for $\left(N, u_N^N, \mathcal{C}\right)$:

Proposition 15 *a) If a coalitional value f satisfies Eff and $ESUG$, then $f_i\left(N, u_N^N, C\right) = \frac{1}{|N|}$ for all $i \in N$.*

b) If a coalitional value f satisfies Eff and $IPSUG$, then $f_i\left(N, u_N^N, \mathcal{C}\right) = \frac{1}{|C_q||\mathcal{C}|}$ for all $i \in C_q \in \mathcal{C}$.

Proof. Part a) is trivial. As for part b), notice that $IPSUG$ implies that all the coalitions should receive the same aggregate value, and hence, under Eff, this value is $\frac{1}{|\mathcal{C}|}$. Moreover, $IPSUG$ also implies that all the players in the same coalition should receive the same value. Hence the result. ■

However, these properties are still very weak, since they only apply to a very specific unanimity game u_N^N. The following result gives us sufficient conditions to have these properties for the family of coalitional values defined before:

Proposition 16 *a) If both γ and ϕ^ω satisfy Eff and Sym, then $\gamma\left[\phi^\omega\right]$ and $\gamma\left\langle\phi^\omega\right\rangle$ satisfy $IPSUG$.*

b) If γ satisfies Eff and Sym, ϕ^ω satisfies Eff, and $\phi_i^x\left(N, u_N^N\right)/x_i = \phi_j^x\left(N, u_N^N\right)/x_j$ for all $i, j \in N$ and all $x \in \mathbb{R}_+^N$, then $\gamma\left[\phi^\omega\right]$ and $\gamma\left\langle\phi^\omega\right\rangle$ satisfy $ESUG$.

Proof. *Clearly, $\left(M, u_N^N/\mathcal{C}\right) = \left(M, u_M^M\right)$ and $\left(M, u_N^N/\mathcal{C}^S\right) = (M, null)$ for all $S \subsetneq C_q \in \mathcal{C}$, where $null\,(Q) = 0$ for all $Q \subset M$.*

a) Under Eff and Sym of ϕ^ω, we have $\left(u_N^N\right)_{C_q}^{[\phi^\omega]N} = \left(u_N^N\right)_{C_q}^{\langle\phi^\omega\rangle N} = \frac{1}{|\mathcal{C}|} u_{C_q}^{C_q}$ for all $C_q \in \mathcal{C}$. Under Eff and Sym of γ, we conclude that $\gamma\left[\phi^\omega\right]_i(N, v, \mathcal{C}) = \gamma\left\langle\phi^\omega\right\rangle_i(N, v, \mathcal{C}) = \frac{1}{|C_q||\mathcal{C}|}$ for all $i \in C_q \in \mathcal{C}$ and hence the result.

b) Under our hypothesis over ϕ^{ω}, we have $\left(u_N^N\right)_{C_q}^{[\phi^{\omega}]N} = \left(u_N^N\right)_{C_q}^{\langle\phi^{\omega}\rangle N} = \frac{|C_q|}{|N|} u_{C_q}^{C_q}$ for all $C_q \in \mathcal{C}$. Under Eff and Sym of γ, we conclude that $\gamma\left[\phi^{\omega}\right]_i(N, v, \mathcal{C}) = \gamma\left\langle\phi^{\omega}\right\rangle_i(N, v, \mathcal{C}) = \frac{1}{|C_q|}\frac{|C_q|}{|N|} = \frac{1}{|N|}$ for all $i \in C_q \in \mathcal{C}$ and hence the result. ■

Corollary 17 *a) The Owen value $Sh(Sh)$ satisfies $IPSUG$.*
b) $Sh[Sh^{\omega}]$ and $Sh\langle Sh^{\omega}\rangle$ satisfy $ESUG$.

5 Characterization

In this section, we present our main result:

Theorem 18 *Among all the coalitional values that satisfy Eff, Lin, INC and Co,*
a) the Owen value $Sh(Sh)$ is the only one that satisfies $NBIC$, $IPSUG$ and IS;
b) the Owen value $Sh(Sh)$ is the only one that satisfies BIC and $IPSUG$;
c) $Sh[Sh^{\omega}]$ is the only one that satisfies $NBIC$, $ESUG$ and IS; and
d) $Sh\langle Sh^{\omega}\rangle$ is the only one that satisfies BIC and $ESUG$.

Proof. We know by Corollary 6, Corollary 12, Proposition 14 and Corollary 17 that these rules satisfy the corresponding properties. Let $\mathcal{C} = \{C_1, ..., C_m\}$ be a coalition structure. Let $M = \{1, ..., m\}$.

Let f^1 and f^2 be two coalitional values satisfying Eff, Lin, INC, Co, and the properties stated in one of the four sections. We prove $f^1 = f^2$ by induction over the number of players n. If $n = 1$, under Eff, $f^1(N, v, \mathcal{C}) = f^2(N, v, \mathcal{C})$ and hence the result holds.

Assume the result holds for less than n players. Now we prove that the result holds for n players.

It is well-know that every TU game can be expressed as a linear combination of unanimity games. Since f^1 and f^2 satisfy Lin, we can restrict our proof to unanimity games.

Let $S \subset N$, $S \neq \emptyset$. Consider the game u_N^S. First, we will show that it is enough to restrict the proof to the case where all the coalitions intersect the carrier S. To prove that, suppose that there exists some coalition, say $C_m \in \mathcal{C}$, that does not intersect the carrier; that is, $S \cap C_m = \emptyset$.

Clearly, C_m is a null coalition. Under INC, $f_i^x(N,u_N^S,\mathcal{C}) = f_i^x(N\backslash C_m, u_{N\backslash C_m}^S, \mathcal{C}_{N\backslash C_m})$ for all $i \in N\backslash C_m$ and $x = 1,2$. By induction hypothesis, $f_i^1\left(N\backslash C_m, u_{N\backslash C_m}^S, \mathcal{C}_{N\backslash C_m}\right) = f_i^2\left(N\backslash C_m, u_{N\backslash C_m}^S, \mathcal{C}_{N\backslash C_m}\right)$ for all $i \in N\backslash C_m$. Moreover, as an implication of INC and Eff, $\sum_{i\in C_m} f_i^x(N,u_N^S,\mathcal{C}) = 0$ for $x = 1,2$. We still need to prove that every player in C_m receives the same under both coalitional values. In particular, we will prove that each of them receives zero. We have two possibilities:

Cases a and c (the coalitional values satisfy IS): Under IS, it is clear that $f_i^x(N,u_N^S,\mathcal{C}) = 0$ for all $i \in C_m$, $x = 1,2$, because all the players in C_m are symmetric and their values sum up zero.

Cases b and d (the coalitional values satisfy BIC): If $|C_m| = 1$, it is clear that $f_i^x\left(N,u_N^S,\mathcal{C}\right) = 0$, $i \in C_m$, $x = 1,2$. Assume $f_i^x\left(N,u_N^S,\mathcal{C}\right) = 0$ for all null coalitions with less than l players. If $|C_m| = l$, $l > 1$, from BIC, $f_i^x(N,u_N^S,\mathcal{C}) - f_i^x(N\backslash j, u_{N\backslash j}^S, \mathcal{C}_{N\backslash j}) = f_j^x(N,u_N^S,\mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^S, \mathcal{C}_{N\backslash i})$ for all $i,j \in C_m$, $x = 1,2$. By induction hypothesis on $|C_m|$, $f_i^x(N\backslash j, u_{N\backslash j}^S, \mathcal{C}_{N\backslash j}) = f_j^x(N\backslash i, u_{N\backslash i}^S, \mathcal{C}_{N\backslash i}) = 0$, for all $i,j \in C_m$, $x = 1,2$. Hence, we have that $f_i^x(N,u_N^S,\mathcal{C}) = f_j^x(N,u_N^S,\mathcal{C})$ for all $i,j \in C_m$ and $x = 1,2$. Moreover, since $\sum\limits_{i\in C_m} f_i^x(N,u_N^S,\mathcal{C}) = 0$, we obtain that $f_i^x(N,u_N^S,\mathcal{C}) = 0$ for all $i \in C_m$ and $x = 1,2$.

From now on, we assume that $S \cap C_q \neq \emptyset$ for all $C_q \in \mathcal{C}$.

Fix $i \in C_q \in \mathcal{C}$. We should prove that $f_i^1(N,u_N^S,\mathcal{C}) = f_i^2(N,u_N^S,\mathcal{C})$.

Let $S_q := C_q \cap S$ and $T := S_q \cup (N\backslash C_q)$. It is straightforward to check that $u_N^S\left(T' \cup \bigcup_{C_r\in\mathcal{R}} C_r\right) = u_N^T\left(T' \cup \bigcup_{C_r\in\mathcal{R}} C_r\right)$ for all $T' \subset C_q$ and all $\mathcal{R} \subset \mathcal{C}\backslash\{C_q\}$. Since we are under the assumptions of Co (Claim **??**), we have $f_i^x(N,u_N^S,\mathcal{C}) = f_i^x(N,u_N^T,\mathcal{C})$ for $x = 1,2$. Hence, it is enough to prove that $f_i^1(N,u_N^T,\mathcal{C}) = f_i^2(N,u_N^T,\mathcal{C})$. As a previous step, consider the unanimity game $\left(N,u_N^N,\mathcal{C}\right)$.

By an analogous argument as before, we have $u_N^T\left(T' \cup \bigcup_{C_l\in\mathcal{Q}} C_l\right) = u_N^N\left(T' \cup \bigcup_{C_l\in\mathcal{Q}} C_l\right)$ for all $T' \subset C_r \in \mathcal{C}\backslash\{C_q\}$ and all $\mathcal{Q} \subset \mathcal{C}\backslash\{C_r\}$. Under Co, for all $j \in N\backslash C_q$,

$$f_j^x(N,u_N^N,\mathcal{C}) = f_j^x(N,u_N^T,\mathcal{C}) \tag{1}$$

We have two possibilities:

Cases a and c (the coalitional values satisfy $NBIC$ and IS): Under Eff and $ESUG/IPSUG$, by Proposition 15, we have $\sum_{i\in C_q} f_i^x(N,u_N^T,\mathcal{C}) = \beta_q$ where $\beta_q = \frac{1}{|\mathcal{C}|}$ (when f^x satisfies $IPSUG$) or $\beta_q = \frac{|C_q|}{|N|}$ (when f^x satisfies $ESUG$).

Under IS, we have $f_i^x(N, u_N^T, \mathcal{C}) = f_j^x(N, u_N^T, \mathcal{C})$ for all $i, j \in S_q$ (respectively, $i, j \in C_q \backslash S_q$) and $x = 1, 2$. Hence it is enough to prove that $f_i^x(N, u_N^T, \mathcal{C}) = 0$ for all $i \in C_q \backslash S_q$, $x = 1, 2$. This is clear for $S_q = C_q$. Let $i \in S_q$ and $j \in C_q \backslash S_q$. Player j is a null player in (N, u_N^T) and hence $(N, u_N^T) = (N, \left(u_N^T\right)^{-j})$. Under $NBIC$, $0 = f_i^x(N, u_N^T, \mathcal{C}) - f_i^x\left(N, \left(u_N^T\right)^{-j}, \mathcal{C}\right) = f_j^x(N, u_N^T, \mathcal{C}) - f_j^x\left(N, \left(u_N^T\right)^{-i}, \mathcal{C}\right)$.

Obviously, $(N, \left(u_N^T\right)^{-i}, \mathcal{C})$ is the null game $\left(u_N^T\right)^{-i}(S) = 0$ for all $S \subset N$ and thus Eff and IS imply $f_j^x(N, \left(u_N^T\right)^{-i}, \mathcal{C}) = 0$. Thus, $f_j^x(N, u_N^T, \mathcal{C}) = 0$ for $x = 1, 2$.

Cases b and d (the coalitional values satisfy BIC): Fix $x \in \{1, 2\}$. Under BIC, $f_i^x(N, u_N^T, \mathcal{C}) - f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}) = f_j^x(N, u_N^T, \mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i})$ for all $j \in C_q\backslash i$. Hence, $\sum_{j \in C_q \backslash i}\left(f_i^x(N, u_N^T, \mathcal{C}) - f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})\right)$ equals $\sum_{j \in C_q \backslash i}\left(f_j^x(N, u_N^T, \mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i})\right)$. Rearranging terms,

$$\begin{aligned}&(|C_q| - 1)\, f_i^x\left(N, u_N^T, \mathcal{C}\right) = \\ &= \textstyle\sum_{j \in C_q \backslash i}\left(f_j^x(N, u_N^T, \mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})\right).\end{aligned} \quad (2)$$

On the other hand, by Proposition 15,

$$f_j^x(N, u_N^N, \mathcal{C}) = \alpha_q \text{ for all } j \in C_q \in \mathcal{C} \quad (3)$$

where $\alpha_q = \frac{1}{|N|}$ (if f^x satisfies $ESUG$) and $\alpha_q = \frac{1}{|C_q||\mathcal{C}|}$ (if f^x satisfies $IPSUG$).

Hence, $\sum_{j \in N \backslash C_q} f_j^x(N, u_N^T, \mathcal{C}) \overset{(1)}{=} \sum_{j \in N \backslash C_q} f_j^x(N, u_N^N, \mathcal{C}) \overset{(3)}{=} \sum_{C_r \in \mathcal{C} \backslash \{C_q\}} |C_r|\, \alpha_r$.

Moreover, by Eff, $\sum_{j \in C_q \backslash i} f_j^x(N, u_N^T, \mathcal{C}) = u_N^T(N) - f_i^x(N, u_N^T, \mathcal{C}) - \sum_{C_r \in \mathcal{C} \backslash \{C_q\}} |C_r|\, \alpha_r$.
Since $u_N^T(N) = 1$, $\sum_{j \in C_q \backslash i} f_j^x(N, u_N^T, \mathcal{C}) = 1 - f_i^x(N, u_N^T, \mathcal{C}) - \sum_{C_r \in \mathcal{C} \backslash \{C_q\}} |C_r|\, \alpha_r$.

It is not difficult to check that $1 - \sum_{C_r \in \mathcal{C} \backslash \{C_q\}} |C_r|\, \alpha_r = \beta_q$ (defined in the previous case). Hence $\sum\limits_{j \in C_q \backslash i} f_j^x(N, u_N^T, \mathcal{C}) = \beta_q - f_i^x(N, u_N^T, \mathcal{C})$.

Replacing this expression in (2) and rearranging terms, $|C_q|\, f_i^x(N, u_N^T, \mathcal{C}) = \beta_q - \sum_{j \in C_q \backslash i} f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + \sum_{j \in C_q \backslash i} f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})$. And so, $f_i^x(N, u_N^T, \mathcal{C}) = \frac{1}{|C_q|}\left[\beta_q - \sum_{j \in C_q \backslash i} f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + \sum_{j \in C_q \backslash i} f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})\right]$

But by induction hypothesis: $f_j^1(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) = f_j^2(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i})$ and $f_i^1(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}) = f_i^2(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})$ for all $j \neq i$. Hence we conclude that $f_i^1(N, u_N^T, \mathcal{C}) = f_i^2(N, u_N^T, \mathcal{C})$. ■

Remark 19 *In parts a and c we need to add IS. The reason is that, when $NBIC$ is involved, some transfers can be forced inside a coalition depending on the inner structure of other coalitions. These transfers are ruled out in parts b and d because BIC only applies when the inner structure of the coalition structure is affected.*

Take for example the coalitional value F given by $F(N,v,\mathcal{C}) = Sh[Sh^{\omega}](N,v,\mathcal{C})$ if $\{1,2\} \notin \mathcal{C}$ or $\{3\} \notin \mathcal{C}$. When $\{1,2\},\{3\} \in \mathcal{C}$, take $F_i(N,v,\mathcal{C}) = Sh[Sh^{\omega}]_i(N,v,\mathcal{C})$ for all $i \in N\backslash\{1,2\}$ and moreover $F_1(N,v,\mathcal{C}) = Sh[Sh^{\omega}]_1(N,v,\mathcal{C}) + v(\{3\})$ and $F_2(N,v,\mathcal{C}) = Sh[Sh^{\omega}]_2(N,v,\mathcal{C}) - v(\{3\})$.

This coalitional value satisfies Eff, Lin, INC, BIC, Co and $ESUG$, but fails IS. For example, in the game (N,v) given by $N = \{1,2,3\}$ and $v(S) = 1$ for all $S \neq \emptyset$, we have $F(N,v,\{\{1,2\},\{3\}\}) = \left(\frac{4}{3}, \frac{-2}{3}, \frac{1}{3}\right)$.

Analogously, define the coalitional value F' as before, but taking $Sh(Sh)$ instead of $Sh[Sh^{\omega}]$. Then, F' satisfies Eff, Lin, INC, $NBIC$, Co and $IPSUG$, but fails IS. For example, in the previous game, we have $F(N,v,\{\{1,2\},\{3\}\}) = \left(\frac{5}{4}, \frac{-3}{4}, \frac{1}{2}\right)$.

6 Independence of the axioms

In this section we show that the axioms used in Theorem 18 are independent.

The Aumann-Drèze value $Sh(In)$ satisfies Lin, INC, Co, BIC, $NBIC$, $IPSUG$, $ESUG$, IS and fails Eff.

Define the *bounded egalitarian value* BE as $BE_i(N,v) = v(N)/|Carr(N,v)|$ if $i \in Carr(N,v)$ and $BE_i(N,v) = 0$ otherwise.

$Sh(BE)$ satisfies Eff, INC, Co, BIC, $NBIC$, $IPSUG$, IS and fails Lin.

Define the *egalitarian value* E as $E_i(N,v) = v(N)/|N|$ for all $i \in N$.

$Sh(E)$ satisfies Eff, Lin, Co, BIC, $NBIC$, $IPSUG$, IS and fails INC.

Take the coalitional value G given by $G(N,v,\mathcal{C}) = Sh(Sh)(N,v,\mathcal{C})$ if $\{3,4\} \notin \mathcal{C}$, $1,2 \notin N$ or $1,2 \in N$ and they belong to the same coalition in $\mathcal{C}$. When $\{3,4\} \in \mathcal{C}$, $1,2 \in N$ and $1,2$ do not belong to the same coalition in $\mathcal{C}$, take $G_i(N,v,\mathcal{C}) = Sh(Sh)_i(N,v,\mathcal{C})$ for all $i \in N\backslash\{1,2\}$ and moreover $G_1(N,v,\mathcal{C}) = Sh(Sh)_1(N,v,\mathcal{C}) + \delta^v_{\{1,2,3\}}$ and $G_2(N,v,\mathcal{C}) = Sh(Sh)_2(N,v,\mathcal{C}) - \delta^v_{\{1,2,3\}}$, where $\delta^v_{\{1,2,3\}} := v(\{1,2,3\}) - v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(1) + v(2) + v(3)$ is the *Harsanyi dividend* for $u_N^{\{1,2,3\}}$. This coalitional value G satisfies Eff, Lin, INC, BIC, $IPSUG$, IS and fails Co.

$E\,(Sh)$ satisfies Eff, Lin, INC, Co, $IPSUG$, IS and fails both BIC and $NBIC$.

$Sh\,\langle Sh^{\omega}\rangle$ satisfies Eff, Lin, INC, Co, BIC, IS and fails $IPSUG$.

$Sh\,[Sh^{\omega}]$ satisfies Eff, Lin, INC, Co, $NBIC$, IS and fails $IPSUG$.

The second coalitional value presented in Remark 19 satisfies Eff, Lin, INC, Co, $NBIC$, $IPSUG$ and fails IS.

Define the *weighted bounded egalitarian value* BE^{ω} as $BE_i^x\,(N,v) = x_i v\,(N) \,/ \sum_{j\in Carr(N,v)} x_j$ if $i \in Carr\,(N,v)$ and $BE_i^x\,(N,v) = 0$ otherwise, for all $x \in \mathbb{R}_{++}^N$.

$Sh\,[BE^{\omega}]$ satisfies Eff, INC, Co, $NBIC$, $ESUG$, IS and fails Lin.

$Sh\,\langle BE^{\omega}\rangle$ satisfies Eff, INC, Co, BIC, $ESUG$ and fails Lin.

$Sh\,[E^{\omega}]$ satisfies Eff, Lin, Co, $NBIC$, $ESUG$, IS and fails INC.

$Sh\,\langle E^{\omega}\rangle$ satisfies Eff, Lin, Co, BIC, $ESUG$ and fails INC.

The coalitional Shapley value Sh satisfies Eff, Lin, INC, $NBIC$, BIC, $ESUG$, IS and fails Co.

$E\,[Sh^{\omega}]$ satisfies Eff, Lin, INC, Co, $ESUG$, IS and fails both $NBIC$ and BIC.

The Owen value $Sh\,(Sh)$ satisfies Eff, Lin, INC, Co, $NBIC$, BIC, IS and fails $ESUG$.

The coalitional value F presented in Remark 19 satisfies Eff, Lin, INC, Co, $NBIC$, $ESUG$ and fails IS.

In the following table we summarize the results presented in this Section:

	Eff	Lin	INC	BIC	$NBIC$	Co	$ESUG/IPSUG$	IS
$Sh\,(Sh)$	OK^{*+}	OK^{*+}	OK^{*+}	OK*	OK^{+}	OK^{*+}	$IPSUG^{*+}$	OK^{+}
$Sh\,[Sh^{\omega}]$	OK*	OK*	OK*	no	OK*	OK*	$ESUG^{*}$	OK*
$Sh\,\langle Sh^{\omega}\rangle$	OK*	OK*	OK*	OK*	no	OK*	$ESUG^{*}$	OK
$Sh\,(In)$	no	OK	OK	OK	OK	OK	BOTH	OK
$Sh\,(BE)$	OK	no	OK	OK	OK	OK	$IPSUG$	OK
$Sh\,(E)$	OK	OK	no	OK	OK	OK	$IPSUG$	OK
G	OK	OK	OK	OK	OK	no	$IPSUG$	OK
$E\,(Sh)$	OK	OK	OK	no	no	OK	$IPSUG$	OK
F'	OK	OK	OK	no	OK	OK	$IPSUG$	no
$Sh\,[BE^{\omega}]$	OK	no	OK	no	OK	OK	$ESUG$	OK
$Sh\,\langle BE^{\omega}\rangle$	OK	no	OK	OK	no	OK	$ESUG$	OK
$Sh\,[E^{\omega}]$	OK	OK	no	no	OK	OK	$ESUG$	OK
$Sh\,\langle E^{\omega}\rangle$	OK	OK	no	OK	no	OK	$ESUG$	OK
Sh	OK	OK	OK	OK	OK	no	$ESUG$	OK
$E\,[Sh^{\omega}]$	OK	OK	OK	no	no	OK	$ESUG$	OK
F	OK	OK	OK	no	OK	OK	$ESUG$	no

Table 1: Properties satisfied by the coalitional values. "*" (resp. "+") means that this property together with the others with "*" (resp. "+") in the line, characterizes the coalitional value.

7 Concluding remarks

In this paper we characterize three generalizations of the Shapley value. As for the Owen value, one of its most controversial properties is that of symmetry in the game among coalitions. In our characterization, this symmetry is in fact implied by $IPSUG$. Other characterizations of the Owen value also include some property that leads to this symmetry. This is the case of property A3 in the original characterization by Owen (1977); the *coalitional symmetry* in Winter (1989) and Albizuri (2008); the *intermediate game property* in Peleg (1989), called *game between coalitions property* in Winter (1992) and *quotient game property* in Vázquez-Brage et al. (1997); the property of *symmetry among coalitions* in Zhang (1995); the property of *block strong symmetry* in Amer and Carreras (1995), called *balanced contributions in the coalitions* in Calvo et al. (1996); the property of *symmetry* in Chae and Heidhues (2004); and the properties of *unanimity coalitional game*, *symmetry between exchangeable coalitions* and *coalitional symmetry* in the various characterizations presented in Bergantiños et al. (2007).

As opposed to the Owen value, the other two coalitional values do not satisfy $IPSUG$, but the more intuitive $ESUG$. In both cases, however, the proposed property does not suffice to characterize the payoff allocation in a unanimity game with minimal carrier $S \neq N$. The reason is that, as opposed to other characterizations presented in the literature, we do not have the property of null player. Instead, we have to apply the properties of balanced contributions and the independence of null coalitions.

Hart and Kurz (1983) presented an alternative characterization of the Owen value without the property of symmetry in the game among coalitions. Instead, they used a property of *Carrier*, which implies that the value should not be affected by the presence of null players. Various axiomatic characterizations of the Owen value also use this property: Hamiache (1999 and 2001), Albizuri and Zarzuelo (2004), and Albizuri (2008).

One may wonder whether the Carrier axiom is a reasonable requirement in games with coalition structure. Since null players affect the size of the coalition, their role could be relevant (as far as we accept that size is important). Take for example the unanimity game $\left(N, u_N^S\right)$ with $N = \{1, 2, 3\}$ and $S = \{1, 2\}$. Take $\mathcal{C} = \{\{1\}, \{2, 3\}\}$. This game models the following situation, as described in Hart and Kurz (1983):

> As an everyday example of such a situation, "I will have to check this with my wife/husband" may (but not necessarily) lead to a better bargaining position, due to the fact that the other party has to convince *both* the player and the spouse.

The Owen value would simply ignore the presence of player 3: $Sh\,(Sh)\left(N, u_N^S, \mathcal{C}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

In this example, the role of the symmetry in the game among coalitions is clear: since both $\{1\}$ and $\{2, 3\}$ are equally necessary to get a positive payoff, this payoff should be shared equally among them, irrespectively of their respective size. This idea is appropriate to describe situations where the negotiations take place among representatives with the same power of negotiation.

As opposed, $Sh\left[Sh^{\omega}\right]$ would assign twice as much to coalition $\{2,3\}$ than to coalition $\{1\}$, but still maintaining the null player property: $Sh\left[Sh^{\omega}\right]\left(N,u_N^S,\mathcal{C}\right)=\left(\frac{1}{3},\frac{2}{3},0\right)$.

This idea is appropriate to describe situations where the power of negotiation among coalitions depend on their size. One may think for example on political parties that join forces in a Parliament, maintaining however their respective proposal prerogatives. In fact, Kalandrakis (2006) shows that proposal making has a very significant impact on outcomes.

Notice that player 2 would only expect to get $\frac{1}{2}$ in case player 3 be not present. Hence, the benefit of cooperation between players 2 and 3 is $\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$. $Sh\left\langle Sh^{\omega}\right\rangle$ proposes to share this benefit equally between players 2 and 3: $Sh\left\langle Sh^{\omega}\right\rangle\left(N,u_N^S,\mathcal{C}\right)=\left(\frac{1}{3},\frac{7}{12},\frac{1}{12}\right)$.

In this case, the null player property is not satisfied. However, one may find examples of real situations where this null player property also fails. Consider the Basque Country[5] Parliament that arose in 2001 election. Five parties got representation: Coalition EAJ-PNV / EA, Partido Popular (PP), Partido Socialista de Euskadi - Euskadiko Ezquerra (PSE-EE / PSOE), Euskal Herritarrok (EH) and Ezker Batua-Izquierda Unida (EB-IU). The number of representatives is given in Table 2. The number of seats needed to win a vote is 38.

Party	Number of Seats
EAJ-PNV / EA	33
PP	19
PSE-EE / PSOE	13
EH	7
EB-IU	3

Table 2: Number of seats in the Basque Country Parliament.

Even though EB-IU is a null player in the associated voting game[6], a minority

[5]Autonomous community of Spain.

[6]This game is defined as $v(S)=1$ if the members of S sum up at least 38 seats, and $v(S)=0$ otherwise.

government was formed with the coalition of EAJ-PNV / EA and EB-IU. Whatever the reason for this decision could be, it suggests that null players can also play a significant role.

The three values characterized in this paper can be applied to many other contexts. Apart from computing the power of differents parties in the Parliament, we can also use them in other situations such as a the problem of the airport or a bankrupty problem.

In a bankruptcy problem there exist a person, firm, or institution that does not have sufficient funds to meet the claims of all its creditors. In such a situation, the goal is how to divide the available funds among the creditors. Formally, a bankruptcy problem consists on a pair (E, d), where $E \in \mathbb{R}$ is the available funds of the debtor, "the estate", and $d = (d_1, ..., d_n) \in \mathbb{R}^N_+$ is the vector of claims (or demands) of the n creditors, satistying that $0 \leq E \leq \sum_{i \in N} d_i$. Given a bankrupty problem, a corresponding game $(N, v_{E;d})$ is defined, where $v_{E;d}(S) = \max \left\{ E - \sum_{i \in N \setminus S} d_i, 0 \right\}$ for all $S \subset N$. Let consider an example that appears in Casas-Mendez et al. (2003). In this case, we have $N = \{1, 2, 3\}$, $\mathcal{C} = \{\{1, 2\}, 3\}$, $E = 400$ and $d = (100, 200, 300)$. In this case, we obtain that

$$\begin{aligned} Sh(Sh)(N, v_{E;d}, \mathcal{C}) &= (75, 125, 200) \\ Sh[Sh^{\omega}](N, v_{E;d}, \mathcal{C}) &\simeq (83, 150, 167) \\ Sh \langle Sh^{\omega} \rangle (N, v_{E;d}, \mathcal{C}) &\simeq (92, 142, 167) . \end{aligned}$$

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