A non-cooperative approach to the folk rule in minimum cost spanning tree problems*

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September 27, 2019

Abstract

This paper deals with the problem of finding a way to distribute the cost of a minimum cost spanning tree problem between the users. A rule that assigns a payoff to each agent provides this distribution. An optimistic point of view is considered to devise a cooperative game. Following this optimistic approach, a sequential game exerts this construction to define the action sets of the agents. The main result states the existence of a unique cost allocation in subgame perfect equilibria. This cost allocation matches the one suggested by the folk rule.

Keywords: Minimum cost spanning tree, cost allocation, subgame perfect equilibrium.

*The authors thank the support of the Spanish Ministry of Science, Innovation and Universities, the Spanish Ministry of Economy and Competitiveness, the Spanish Agency of Research, co-funded with FEDER funds, under the projects ECO2016-77200-P, ECO2017-82241-R, ECO2017-87245-R, and Xunta de Galicia (ED431B 2019/34).

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1 Introduction

In this paper, we study the implementation of the folk solution associated with a minimum cost spanning tree problem. An optimistic point of view is considered to devise a cooperative game. Following this optimistic approach, a sequential game exerts this construction to define the action sets of the agents. The main result states the existence of a unique cost allocation in subgame perfect equilibria. This cost allocation matches the one suggested by the folk rule.

The situation of constructing a tree with minimal cost known as *minimal cost spanning tree problems* is quite familiar in the literature of operation research, economics, management, among others. Assume a group of agents requires a service that can be only provided by a source. A network, whose arcs entail some cost to build or to use, provides access to this source. Agents can connect to the source either directly through an existing network that already provides the service to other users. No congestion nor depreciation of the service is assumed, which implies that the optimal network is always a tree. Stream video, voice-conference or software distribution applications, or an irrigation system that supplies water to irrigated land from a water dam, are some examples of such situations.

Assuming that agents agree to build a network and decide on how to share its cost, there are two possible approaches to tackle this situation.

The first approach arises when the agents let the decision to a neutral referee. This referee may be either a regulator whose decision is mandatory for the agents, or an adviser whose proposal is not compulsory, but all the agents have incentives to follow. In this sense, a fundamental property is *core selection*, which assures that no coalition of agents can connect to the source by themselves at a lower cost than the one suggested by the adviser.[1] A rele-

[1] Non-emptiness of the core in minimum cost spanning tree problems has been first noted by (Bird, 1976) and deeply studied by Granot and Huberman (1981, 1984). More recently, Dutta and Mishra (2012), Sziklai et al. (2016) proved the non-emptiness of the core in two more general classes of games, respectively, and Kobayashi and Okamoto (2014)

vant core-selection rule is the folk solution (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007a) which, moreover, also satisfies many other relevant properties (Bergantiños and Vidal-Puga, 2008). The second approach arises when the agents achieve agreements directly among themselves, following the rules of a non-cooperative game. In this second case, the final network in equilibrium is not guaranteed to be optimal nor the final payoff allocation to be efficient. Joint the two approaches, it could be suitable to find a mechanism leading to an optimal network together to a fair allocation of its cost.

In this paper, we focus on the second approach. We define a non-cooperative game in which utility-maximizers players agree on how to share the cost of an efficient graph. The non-cooperative game is as follows: first, we fix a random order of choices of the agents. Then agents act sequentially according to the above order: the first agent selects to whom she connects looking for the cheapest connection; then, the second agent decides with whom she wants to connect taking into account that, in case the first agent had previously connected to her, then she can choose an arc adjacent to the first agent, and so on.[2] The only restriction is that no cycles are allowed. At the end of the last round, an optimal tree arises. The cost allocation that arises by charging each player with her chosen arc provides a stable share of the total cost such that the final share is fair. Consequently, players accept both the optimal tree and a cost-share given by the folk solution.

Bergantiños and Vidal-Puga (2010) propose a non-cooperative game in which players always agree on an optimal tree and a cost-share given by the folk solution. In the first stage, the agents offer prices to each other. These prices represent the amount that the agents are willing to pay to other agents if they connect. Then, the agent with a maximum net offer is asked to connect to the source or to propose a different network.

Moulin and Velez (2013) and Hougaard and Tvede (2013) consider two

focus on concave problems, where the core has a well-known structure.

[2]Such a mechanism resembles the well-known Kruskal algorithm.

mixed approaches, respectively. In Moulin and Velez (2013), nodes are sellers who bid to supply individual arcs, so that a single buyer purchases a minimum cost spanning tree. They show that an optimal tree arises in equilibrium. In Hougaard and Tvede (2013), a planner asks for the costs of the arcs to the adjacent agents, who have a priori private information about their actual costs. With this information, the planner builds the optimal network (under the assumption of truth-telling), so that costs become common knowledge for the arcs that belong to this optimal network. They show that the folk rule is, among all the known rules in the literature, the only consistent with truthful announcements.

In Hernández et al. (2016), a different strategic game is defined associated to a minimum cost spanning tree problem in which the set of actions of some agent i is the set of nodes (including the source) with whom agent i may connect in a spanning tree. Under this approach, subgame perfect equilibria may appear such that the provided spanning tree is not efficient. This inefficiency cannot occur under our approach.

The paper organizes as follows. In Section 2, we present the model. In Section 3, we introduce the non-cooperative game. In Section 4, we discuss the results.

2 The model

Let $N_0 = N \cup \{0\}$ be a set of *nodes* where $N = \{1, 2, \ldots, n\}$ is a finite set of *agents* and 0 is the *source* they need to connect.

Let $C = (c_{ij})_{i,j \in N_0}$ be the *cost matrix*, where $c_{ij} \in \mathbb{R}_+$ represents the connection cost between nodes i and j. We assume, as usual, that $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N_0$. We denote the set of all cost matrices on N as $\mathcal{C}^N$. A *minimum cost spanning tree problem*, briefly *mcstp*, is a pair (N_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of g are called *arcs*. We assume that the arcs are undirected, i.e. (i, j) and (j, i) represent the same arc.

Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of distinct nodes $\{i_0, \ldots, i_l\}$ satisfying $i = i_0$, $j = i_l$ and $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, \ldots, l\}$.

A *spanning tree* over N_0 is a network t such that for all $i, j \in N$ there exists a unique path from i to j. In that case, we denote as $\tau_{ij}^t = \{(i_{h-1}, i_h)\}_{h=1}^l$ the network formed by the nodes in the unique path between i and j. Let $\mathcal{T}_0^N$ denote the set of all spanning trees over N_0.

Given $t \in \mathcal{T}_0^N$, we define the *cost* associated with t in (N_0, C) as

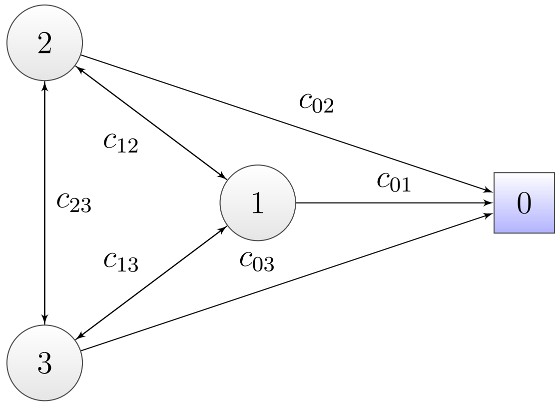
$$c(N_0, C, t) = \sum_{(i,j)\in t} c_{ij}.$$

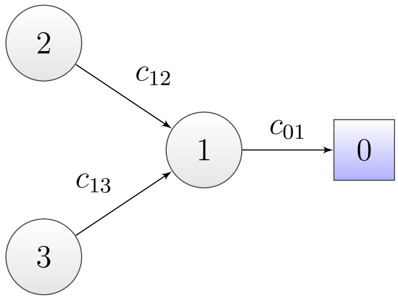
When there is no ambiguity, we write $c(t)$ or $c(C, t)$ instead of $c(N_0, C, t)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a spanning tree $t^* \in \mathcal{T}_0^N$ such that $c(t^*) = \min_{t\in\mathcal{T}_0^N}\{c(t)\}$. Given a *mcstp* (N_0, C), an *mt* always exists, but it may not be unique. We denote the cost associated with any *mt* on (N_0, C) as $m(N_0, C)$.

There are several algorithms in the literature to construct an *mt*. Prim (1957) provides one. Sequentially, the agents connect, either directly or indirectly to the source. At each stage, we add one of the cheapest arcs between the connected and the unconnected nodes.

Example 2.1 *Consider the* mcstp (N_0, C) *with* $N = \{1, 2, 3\}$ *and a cost matrix* $C \in \mathcal{C}^N$ *satisfying* $c_{12} < c_{13} < c_{23} < c_{01} < c_{02} < c_{03}$. *The Prim algorithm proceeds as follows: At stage 1, the arc formed is* $(0, 1)$*, because it is the cheapest one between a connected node (the source), and a non-connected one (players in* N*). At stage 2, the arc formed is* $(1, 2)$ *because it is the cheapest one between a connected node (the source and agent* 1*) and a non-connected one (agents* 2 *and* 3*). At stage 3, the arc formed is* $(1, 3)$ *because it is the cheapest one between a connected node (the source and agents* 1 *and* 2*) and a non-connected one (agent* 3*). The* mt *formed is then* $\{(0, 1), (1, 2), (1, 3)\}$*, which in this example is unique.*





Given $S \subset N$, we denote the restriction to S of the *mcstp* (N_0, C) as (S_0, C), and the cost associated with any *mt* on (S_0, C) as $m(S_0, C)$; that is, $m(S_0, C)$ is the cost of connection of the agents in S to the source.

For each minimum cost spanning tree problem (N_0, C), we construct an associated cooperative cost game (N, v_C) given by $v_C(S) = m(S_0, C)$ where the worth of a coalition S depends on nodes only in S, *i.e.*, those nodes outside S are unavailable. This approach is pessimistic because each coalition S should build their network without counting with agents in $N \setminus S$.

Example 2.2 *With the data in Example 2.1, the cost game* (N, v_C) *is given by* $v_C(\{i\}) = c_{0i}$ *for all* $i \in N$, $v_C(\{1,2\}) = c_{01} + c_{12}$, $v_C(\{1,3\}) = c_{01} + c_{13}$, $v_C(\{2,3\}) = c_{02} + c_{23}$, *and* $v_C(N) = c_{01} + c_{12} + c_{13}$.

Nevertheless, we may consider an optimistic approach if for each S, let $C^S \in \mathcal{C}^S$ be the cost matrix given by $c^S_{ij} = c_{ij}$ for all $i, j \in S$ and $c^S_{i0} = \min\{c_{ij} : j \in N_0 \setminus S\}$ for all $i \in S$. This formulation means that each coalition S can build a network assuming that agents in $N \setminus S$ are already connected. The cost problem $\left(N, v_C^+\right)$ is then defined as $v_C^+(S) = m\left(S_0, C^S\right)$ for all $S \subseteq N$. Bergantiños and Vidal-Puga (2007b) are the first to propose this alternative associated cooperative cost game (N, v_C^+).

Example 2.3 *With the data in Example 2.1, the optimistic cost game* (N, v_C^+) *is given by* $v_C^+(\{1\}) = v_C^+(\{2\}) = c_{12}$*,* $v_C^+(\{3\}) = c_{13}$*,* $v_C^+(\{1,2\}) = c_{12} + c_{13}$*,* $v_C^+(\{1,3\}) = c_{13} + c_{12}$*,* $v_C^+(\{2,3\}) = c_{12} + c_{13}$*,* $v_C^+(N) = c_{01} + c_{12} + c_{13}$*.*

Let Π^N be the set of orders $\pi : \{1, \ldots, n\} \to N$. Then, given some $\pi \in \Pi^N$, the *marginal contributions payoff vector of the optimistic game* (N, v_C^+) *with order* π is $m_\pi^1 = v_C^+\left(\{\pi(1)\}\right)$ and, for $k = 2, \ldots, n$

$$m_\pi^k = v_C^+\left(\{\pi(1), \pi(2), \ldots, \pi(k)\}\right) - v_C^+\left(\{\pi(1), \pi(2), \ldots, \pi(k-1)\}\right)$$

The *folk rule*, as defined by Bergantiños and Vidal-Puga (2007a), provides a criterion for sharing the cost of an *mt* between the agents. The definition of the folk rule is made by applying the Prim algorithm to an irreducible cost matrix C^*. Remarkably, the folk rule can also be defined as the Shapley value of the optimistic game (N, v_C^+) or as the Shapley value of the pessimistic cost game (N, v_C^*) obtained from the irreducible cost matrix.[3]

Example 2.4 *Since the Shapley value is the average of marginal contributions payoff vectors, we can obtain the folk rule by computing these payoff vectors in the optimistic game* (N, v_C^+) *for each possible order. The following table represents these vectors with the data in Example 2.1 and the average of these contributions that corresponds with the folk rule:*

[3]See Bergantiños and Vidal-Puga (2007a,b) for details and additional properties.

order	*agent* 1	*agent* 2	*agent* 3
[123]	$v_C^+(\{1\}) = c_{12}$	$v_C^+(\{1,2\}) - v_C^+(\{1\})$ $= c_{13}$	$v_C^+(N) - v_C^+(\{1,2\})$ $= c_{01}$
[132]	$v_C^+(\{1\}) = c_{12}$	$v_C^+(N) - v_C^+(\{1,3\})$ $= c_{01}$	$v_C^+(\{1,3\}) - v_C^+(\{1\})$ $= c_{13}$
[213]	$v_C^+(\{1,2\}) - v_C^+(\{2\})$ $= c_{13}$	$v_C^+(\{2\}) = c_{12}$	$v_C^+(N) - v_C^+(\{1,2\})$ $= c_{01}$
[231]	$v_C^+(N) - v_C^+(\{2,3\})$ $= c_{01}$	$v_C^+(\{2\}) = c_{12}$	$v_C^+(\{2,3\}) - v_C^+(\{2\})$ $= c_{13}$
[312]	$v_C^+(\{1,3\}) - v_C^+(\{3\})$ $= c_{12}$	$v_C^+(N) - v_C^+(\{1,3\})$ $= c_{01}$	$v_C^+(\{3\}) = c_{13}$
[321]	$v_C^+(N) - v_C^+(\{2,3\})$ $= c_{01}$	$v_C^+(\{2,3\}) - v_C^+(\{3\})$ $= c_{12}$	$v_C^+(\{3\}) = c_{13}$
Average	$\dfrac{2c_{01} + 3c_{12} + c_{13}}{6}$	$\dfrac{2c_{01} + 3c_{12} + c_{13}}{6}$	$\dfrac{2c_{01} + 4c_{13}}{6}$

3 The non-cooperative extensive game

We define the non-cooperative game inductively as follows:

- At the first stage ($k = 0$), nature chooses some $\pi \in \Pi^N$, being each π chosen with the same probability $\frac{1}{n!}$. We define $\Sigma_i^0 = \{i\}$ for all $i \in N_0$.

- At stage $k = 1$, player $\pi(1)$ chooses an action from the following set:

$$S_{\pi(1)} = \left\{(i,j) : i \in \Sigma_{\pi(1)}^0, j \in N_0 \setminus \Sigma_{\pi(1)}^0\right\}.$$

  That is, player $\pi(1)$ selects arc $s_{\pi(1)} = (i^1 = \pi(1), j^1) \in S_{\pi(1)}$ to be built. Once done, nodes i^1 and j^1 become connected, and we set $\Sigma_{i^1}^1 = \Sigma_{j^1}^1 = \{i^1, j^1\}$. We also define $\Sigma_i^1 = \Sigma_i^0$ for any other $i \in N_0 \setminus \{i^1, j^1\}$.

- In general, at stage $k \geq 1$, player $\pi(k)$ chooses an action from the set:

$$S_{\pi(k)} = \left\{(i,j) : i \in \Sigma^{k-1}_{\pi(k)}, j \in N_0 \setminus \Sigma^{k-1}_{\pi(k)}\right\}.$$

  That is, player $\pi(k)$ selects some arc $s_{\pi(k)} = (i^k, j^k) \in S_{\pi(k)}$ to be built. Once this action is done, nodes i^k and j^k become connected and we set $\Sigma^k_{i^k} = \Sigma^k_{j^k} = \Sigma^{k-1}_{i^k} \cup \Sigma^{k-1}_{j^k}$. We also define $\Sigma^k_l = \Sigma^k_{i^k}$ for all $l \in \Sigma^{k-1}_{i^k} \cup \Sigma^{k-1}_{j^k}$, and $\Sigma^k_l = \Sigma^{k-1}_l$ in another case.

- At stage $k = n+1$, the game finishes and the payoff for each player $i \in N$ is given by

$$u_i(s_i) = c_{s_i}.$$

  That is, player i pays the cost of the arc she selected.

Following Maschler et al. (2013), we define the non-cooperative game in extensive form with perfect information and chance moves as:

$$\Gamma = (N, V, E, x^0, (V_i)_{i \in N_0}, (p_x)_{x \in V_0}, u)$$

where

- $N = \{1, 2, \ldots, n\}$ is the set of players.

- V is the set of vertices in the game tree.[4] Each $v \in V$ is determined by some stage $k \in \{0, 1, \ldots, n+1\}$, some $\pi \in \Pi^N$ that determines the order (only for $k > 0$), and some function $f^\pi_k : \{1, \ldots, k-1\} \to N \times N_0$ such that $f^\pi_k(l) \in S_{\pi(l)}$ for all $l = 1, \ldots, k-1$. Pair (π, f^π_k) determines the history, i.e. the (feasible) choice of each predecessor of $\pi(k)$ in π. Hence, for $k = 1, \ldots, n$, the set of arcs already paid, before $\pi(k)$ chooses, is

$$\left\{f^\pi_k(\pi(1)), f^\pi_k(\pi(2)), \ldots, f^\pi_k(\pi(k-1))\right\}.$$

  Notice that this set is empty for $k = 1$. For $k = n+1$, the node is a terminal vertex. If $k = 0$, the agent at such a vertex is the nature, and $\pi(k)$ otherwise.

[4]To avoid ambiguities, we use the terms *vertices* and *edges* in the game tree, as opposed to *nodes* and *arcs* defined for the spanning tree.

- $E \subset V \times V$ is the set of edges. For a vertex v determined by (k, π, f_k^π), the edge (v, v') belongs to E when v' is determined by $\left(k+1, \pi, f_{k+1}^\pi\right)$ such that $f_{k+1}^\pi(l) = f_k^\pi(l)$ for all $l < k$.

- x^0 is the vertex determined by $k = 0$.

- $(V_i)_{i \in N_0}$ is a partition of the set of non-terminal vertices, and it determines the decision-maker at that vertex (or nature, when $i = 0$). In particular, $V_0 = \{x^0\}$ and, given $i \in N$, we have $v \in V_i$ when v is determined by (k, π, f_k^π) with $k \in \{1, \ldots, n\}$ and $\pi(k) = i$.

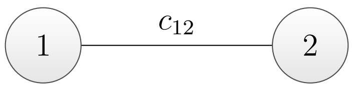
- p_0 is a probability distribution over the edges emanating from x^0. In particular, $p_0(e) = \frac{1}{n!}$ for each such an edge e.

- u is the function that associates each terminal node with a game outcome. In particular, if the terminal node is given by $\left(n+1, \pi, f_{n+1}^\pi\right)$, the game outcome is the payoff vector $\left(c_{f_{n+1}^\pi(k)}\right)_{k \in \{1,\ldots,n\}}$ provided by the spanning tree $t = \left\{f_{n+1}^\pi\left(k\right)\right\}_{k \in \{1,\ldots,n\}}$.

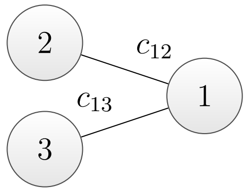
Given $\pi \in \Pi$, we denote as Γ_π the subgame that begins after nature chooses π.

Example 3.1 *With the data in Example 2.1, let us now construct Γ_π with $\pi(i) = i$ for all i:*

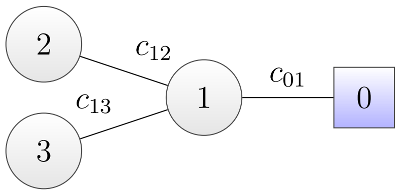
- *At the first stage, agent* 1 *decides the arc she wants to pay,* $s_1 \in \{(1,0), (1,2), (1,3)\}$. *Say, for example* $s_1 = (1, 2)$.



- *Now, agent* 2 *decides which arc* s_2 *to pay by taking into account* s_1. *Assuming that* $s_1 = (1, 2)$, *we have* $s_2 \in \{(2,0), (1,3), (2,3)\}$, *i.e. agent* 2 *cannot choose* $(1, 2)$ *(already taken) but she can choose* $(1, 3)$ *(because she is already connected to agent* 1*). Say, for example,* $s_2 = (1, 3)$.



- *Finally, agent* 3 *decides which arc* s_3 *to pay by taking into account* s_1 *and* s_2. *Assuming* $s_1 = (1,2)$ *and* $s_2 = (1,3)$, *we have* $s_3 \in \{(0,1),(0,2),(0,3)\}$. *In either case, the three agents get connected to the source simultaneously through a spanning tree.*



The formed spanning tree determines the payoffs. For instance, if the players select their cheapest available options, the spanning tree is $\{(1,2),(1,3),(0,1)\}$ *and the cost of each node is distributed in the following way: Player* 1 *pays* $c_{s_1} = c_{12}$; *player* 2 *pays* $c_{s_2} = c_{13}$; *and player* 3 *pays* $c_{s_3} = c_{01}$.

The following table represents the payoff allocation for each π, *assuming each player selects her cheapest available option:*

order	*mt in C*	*agent* 1	*agent* 2	*agent* 3
[123]	$\{(1,2),(1,3),(0,1)\}$	c_{12}	c_{13}	c_{01}
[132]	$\{(1,2),(1,3),(0,1)\}$	c_{12}	c_{01}	c_{13}
[213]	$\{(1,2),(1,3),(0,1)\}$	c_{13}	c_{12}	c_{01}
[231]	$\{(1,2),(1,3),(0,1)\}$	c_{01}	c_{12}	c_{13}
[312]	$\{(1,3),(1,2),(0,1)\}$	c_{12}	c_{01}	c_{13}
[321]	$\{(1,3),(1,2),(0,1)\}$	c_{01}	c_{12}	c_{13}
Average		$\frac{2c_{01}+3c_{12}+c_{13}}{6}$	$\frac{2c_{01}+3c_{12}+c_{13}}{6}$	$\frac{2c_{01}+4c_{13}}{6}$

Given the sequential structure of Γ_π, we will study the subgame perfect equilibria. The equilibrium strategies should specify optimal behavior from any information node up to the end of the game. That is, any agent's strategy should prescribe what is optimal from that node onwards given the other agents' strategies.

As Example 3.1 shows, the only equilibrium payoff in Γ_π is $f^{\pi^{-1}}$, where $\pi^{-1} \in \Pi^N$ is the order defined as $\pi^{-1}(k) = \pi(n-k+1)$, that corresponds with the marginal contributions payoff vector of the optimistic game (N, v_C^+) with order π. Hence, the expected equilibrium payoff in Γ is the one provided by the folk rule.

Our main result establishes that this happens in general:

Theorem 3.1 *Given $\pi \in \Pi^N$, there exists a unique subgame perfect equilibrium payoff allocation for Γ_π, given by the marginal contributions payoff vector of the optimistic game $\left(N, v_C^+\right)$ with order π. Moreover, this equilibrium is strong and uses undominated strategies.*

Proof. We will prove that for all Γ_π, each player $\pi(k)$ has a strategy that assigns her a cost so that she pays at most

$$m_\pi^{\pi(k)} = v_C^+\left(\{\pi(1), \ldots, \pi(k)\}\right) - v_C^+\left(\{\pi(1), \ldots, \pi(k-1)\}\right),$$

independently of the strategies of the other players. Thus, this strategy profile constitutes a strong subgame perfect equilibrium and the strategies are undominated.

By a standard backwards argument, it is clear that there exists a subgame perfect equilibrium for each Γ_π and, moreover, each player will select one of her cheapest available arcs. Hence, even though the subgame perfect equilibrium may not be unique, the subgame perfect equilibrium payoff is. Assume w.l.o.g. $\pi(i) = i$ for all $i \in N$. Hence, at the first stage, player 1 would choose one of her cheapest adjacent arcs $f_1^\pi(1) = (1, i)$ for some $i \in N_0 \setminus \{1\}$, whose cost is precisely $v_C^+(\{1\}) = c_{01}^{\{1\}}$.

For clarification purposes, we analyse stage 2. At this stage, player 1 has selected some arc $(1, j_1)$ and player 2 would choose her cheapest adjacent arc $(2, j_2)$, whose cost is $c_{02}^{\{2\}}$, unless $2 = f_1^\pi(1)$ and $j_2 = 1$. In this latter case, player 2 cannot choose arc $(2,1)$, but other arcs (those adjacent to player 1) would be available, and in particular the chosen arc would cost $\min\left\{c_{01}^{\{1,2\}}, c_{02}^{\{1,2\}}\right\}$. We show that, in either case, player 2 pays at most $v_C^+(\{1,2\}) - v_C^+(\{1\})$. We distinguish two cases:

a) If $f_1^\pi(1) \neq 2$, or $f_1^\pi(1) = 2$ and $j_2 \neq 1$, then player 2 chooses her cheapest adjacent arc $(2, j_2)$ and pays $c_{2j_2} = c_{02}^{\{2\}}$. In this case,

$$v_C^+(\{1,2\}) = \min\left\{c_{12} + c_{01}^{\{1\}}, c_{12} + c_{02}^{\{2\}}, c_{01}^{\{1\}} + c_{02}^{\{2\}}\right\} \qquad v_C^+(\{1\}) = c_{01}^{\{1\}}$$

and

$$v_C^+(\{1,2\}) - v_C^+(\{1\}) = \min\left\{c_{12}, c_{12} + c_{02}^{\{2\}} - c_{01}^{\{1\}}, c_{02}^{\{2\}}\right\} = c_{02}^{\{2\}}$$

so player 2 pays $c_{2j_2} = v_C^+(\{1,2\}) - v_C^+(\{1\})$.

b) If $f_1^\pi(1) = 2$, and $j_2 = 1$, then player 2 selects the arc that minimizes c_{ij}, $i \in \{1,2\}$, $j \in N_0 \setminus \{1,2\}$. We have two subcases:

- $c_{01}^{\{1\}} = c_{12}$, then player 2 pays $\min\left\{c_{01}^{\{1,2\}}, c_{02}^{\{1,2\}}\right\} = v_C^+(\{1,2\}) - v_C^+(\{1\})$.
- $c_{01}^{\{1\}} < c_{12}$, then player 2 pays $c_{01}^{\{1\}} < v_C^+(\{1,2\}) - v_C^+(\{1\})$.

We now prove the result in general. Assume we are in stage k, so that player $\pi(k) = k$ chooses an arc to be built. Notice that we do not assume that the previous players, denoted as $S = \{1, \ldots, k-1\}$, have followed any particular strategy profile. Player k would choose one of her cheapest adjacent arcs, that may connect her to a *previous* player (some $j \in S$) or not (some $j \notin S \cup \{k\}$). The cost of this arc is $c_{kj_k} = \min_{i \in N_0, i \neq k}\{c_{ki}\}$. However, as in stage 2 case, this arc might be available or not. We distinguish the following possibilities:

a) If $k \notin \bigcup_{i \in S} \Sigma_i^{k-1}$, then we have three subcases:

- If $j_k \in S$ and (k, j_k) is not one of the cheapest arcs that connect a node in S with a node in $N_0 \setminus S$, then $v_C^+(S \cup \{k\}) = v_C^+(S) + c_{kj_k}$, so player k would pay $c_{kj_k} = v_C^+(S \cup \{k\}) - v_C^+(S)$.

- If $j_k \in S$ and (k, j_k) is one of the cheapest arcs that connect a node in $N_0 \setminus S$ with a node in S; that is, then there is some mt t_S in S_0 such that k is connected with players $S^k \subseteq S$ throughout t_S. In this case,

$$v_C^+(S \cup \{k\}) = v_C^+(S) + \min_{i \in S^k \cup \{k\}, l \notin S^k \cup \{k\}} \{c_{il}\}$$

  and $\min_{i \in S^k \cup \{k\}, l \notin S^k \cup \{k\}} \{c_{il}\} \geq c_{kj}$. So player k would pay

$$c_{kj_k} \leq \min_{i \in S^k \cup \{k\}, l \notin S^k \cup \{k\}} \{c_{il}\} = v_C^+(S \cup \{k\}) - v_C^+(S).$$

- If $j_k \notin S$, then $v_C^+(S \cup \{k\}) = v_C^+(S) + c_{kj_k}$, so player k would pay $c_{kj_k} = v_C^+(S \cup \{k\}) - v_C^+(S)$.

b) If $k \in \bigcup_{i \in S} \Sigma_i^{k-1}$, this means that arc (r, k) has been built for some $r \in S$, so $k \in \Sigma_r^{k-1}$. If there is $j_k \in N_0 \setminus \Sigma_r^{k-1}$ such that $c_{kj_k} = \min_{i \in N_0, i \neq k} \{c_{ki}\}$, arc (k, j_k) is available for player k, and the same reasoning as in the previous case applies.

c) Finally, it remains the case in which $k \in \bigcup_{i \in S} \Sigma_i^{k-1}$ and for each j_k such that $c_{kj_k} = \min_{i \in N_0, i \neq k} \{c_{ki}\}$, the arc (k, j_k) is not available for player k; that is, $k, j_k \in \Sigma_r^{k-1}$, for some $r \in S$. Then, player k would choose one of the cheapest available arcs (j, l) with $j \in \Sigma_r^{k-1}$ and $l \notin \Sigma_r^{k-1}$, so that

$$c_{jl} = \min_{i \in \Sigma_r^{k-1}, i^* \notin \Sigma_r^{k-1}} \{c_{ii^*}\}. \tag{1}$$

The cost of this arc and the final payoff for player k is c_{jl}.

Let t^* be an mt, and let $t_S^* = \{(i, i^*) \in t^* : i, i^* \in S\}$ be the restriction of t^* to nodes in S. Clearly, t_S^* induces a partition $\{S_1, \ldots, S_\lambda\}$ of S into $\lambda \geq 1$ connected components. For each $\alpha = 1, \ldots, \lambda$, let $(i_\alpha, i_\alpha^*) \in t^*$ such that $i_\alpha \in S_\alpha$, $i_\alpha^* \notin S_\alpha$, and

$$c_{i_\alpha i_\alpha^*} = \min_{i \in S_\alpha, i^* \notin S_\alpha} \{c_{ii^*}\}.$$

Clearly, $i^*_\alpha \notin S$ for all α (however, $i^*_\alpha = i^*_{\alpha'}$ is possible for some $\alpha \neq \alpha'$). Let $t = t^*_S \cup \{(i_\alpha, i^*_\alpha)\}_{\alpha=1}^{\lambda}$. It is not difficult to check that

$$v^+(S) = \sum_{(i,i^*)\in t} c_{ii^*}. \tag{2}$$

We have two subcases:

- If $k = i^*_\alpha$ for some α, let $\hat{S} = \bigcup_{\alpha:k=i^*_\alpha} S_\alpha$. Then,

  $$v^+(S \cup \{k\}) = v^+(S) + c_{hh^*}$$

  where $(h, h^*) \in t^*$, $h \in \hat{S} \cup \{k\}$, $h^* \notin \hat{S} \cup \{k\}$, and

  $$c_{hh^*} = \min_{i\in\hat{S}\cup\{k\}, i^*\notin\hat{S}\cup\{k\}} \{c_{ii^*}\}.$$

  So, $m^k_\pi = c_{hh^*}$.

  Let (i, i^*) be the first arc in the (unique) path in t from k to l such that $i \in \Sigma^{k-1}_r$ and $i^* \notin \Sigma^{k-1}_r$. Under (1), $c_{jl} \leq c_{ii^*}$. Under (2), $c_{ii^*} \leq c_{hh^*}$. Hence, $c_{jl} \leq c_{hh^*} = v^+_C(S \cup \{k\}) - v^+_C(S)$.

- If $k \neq i^*_\alpha$ for all α,

  $$v^+ (S \cup \{k\}) = v^+(S) + c_{kk^*}$$

  where $(k, k^*) \in t^*$ and

  $$c_{kk^*} = \min_{i\neq k} \{c_{ki}\} = v^+(\{k\}).$$

  So, $m^k_\pi = c_{kk^*}$. In case $k^* \notin \Sigma^{k-1}_r$, under (1) we deduce $c_{jl} \leq c_{kk^*} = v^+_C(S \cup \{k\}) - v^+_C(S)$. In case $k^* \in \Sigma^{k-1}_r$, let (i, i^*) be the first arc in the (unique) path in t^* from k to l such that $i \in \Sigma^{k-1}_r$ and $i^* \notin \Sigma^{k-1}_r$. Under (1), $c_{jl} \leq c_{ii^*}$. Under (2), $c_{ii^*} \leq c_{kk^*}$. Hence, $c_{jl} \leq c_{kk^*} = v^+_C(S \cup \{k\}) - v^+_C(S)$.

Finally, observe that given an *mt* t^* in N_0, with cost $c(t^*)$, the following relationships are fulfilled in equilibrium, where $f(k) = f^\pi_{k+1}(k)$ denotes the arc selected by player k

$$c(t^*) \leq \sum_{k=1}^{n} c_{f(k)} \leq \sum_{k=1}^{n} v_C^+(\{1,\ldots,k\}) - v_C^+(\{1,\ldots,k-1\}) = c(t^*)$$

and the equality in the above relationships is obtained, $c_{f(k)} = v_C^+(\{1,\ldots,k\}) - v_C^+(\{1,\ldots,k-1\})$, for all $k \in N$. ■

Corollary 3.1 *The folk rule arises as a unique expected subgame perfect equilibrium payoff allocation for* Γ.

4 Concluding remarks

The operations research literature has explored the design the efficient algorithms to build optimal trees, as well as their computational complexity. More recently, the cost-sharing aspect has received increasing attention, from both the operational research and the economics literature. The idea is that the agents involved are responsible for paying the total cost of the implementation of an optimal tree. This idea leads to taking into account the agents' incentives to guarantee the construction of such an optimal network. Within this context, the problem of finding an optimal network structure does not rely only on its total cost but also on the amount that should be charged to each agent.

Our non-cooperative game gets the folk rule in expected terms. Following Bag and Winter (1999) and Mutuswami and Winter (2002), we can achieve a complete implementation by adding a previous stage in which one of the agents, chosen at random, proposes a spanning tree and a cost-sharing allocation. If all the other agents accept this proposal (they vote sequentially in any order), both the tree and the cost-sharing allocation are imposed, and the game finishes. In case any of them rejects the proposal, they play game Γ in the known terms. Assuming either that: a) agents are risk-averse, or b) they are risk-neutral but prefer to finish as soon as possible, then the only final cost allocation is the one given by the folk rule.

Another relevant characteristic of our approach is that the equilibrium strategy profiles do not need to anticipate the moves of the following players in the order. Hence, we can define the non-cooperative game by choosing only the first agent at random; after this agent chooses her available arc, another agent is chosen at random, and so on. Moreover, the optimal strategy is to choose the cheapest available arc. Hence, the subgame perfect equilibrium is also a strong perfect equilibrium and an equilibrium with dominant strategies.

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