On the set of extreme core allocations for minimal cost spanning tree problems

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Abstract

Minimal cost spanning tree problems connect agents efficiently to a source when agents are located at different points and the cost of using an edge is fixed. We propose a method, based on the concept of marginal games, to generate all extreme points of the corresponding core. We show that three of the most famous solutions to share the cost of mcst problems, the Bird, folk and cycle-complete solutions, are closely related to our method.

Keywords: Minimal cost spanning tree problems, extreme core allocations, reduced game, Bird solution, folk solution, cycle-complete solution.

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1 Introduction

Minimal cost spanning tree (mcst) problems model a situation in which agents are located at different points and need to be connected to a source in order to obtain a good or information. Agents do not care if they are

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connected directly to the source or indirectly through other agents. The cost to build a link between two agents or an agent and the source is a fixed number, meaning that the cost is the same whether one or ten agents use that particular link. Mcst problems can be used to model various real-life problems, from telephone and cable TV to water supply networks.

The core of mcst problems has been an early focus of attention, with Bird (1976) and Granot and Huberman (1981) showing that it is always non-empty and Granot and Huberman (1984) providing an algorithm to generate multiple core allocations. For the special case for which all edges have a cost of 0 or 1 (called elementary cost matrices in this paper), Kuipers (1993) shows that all extreme points of the core are marginal vectors (a property labeled as the CoMa property by Hamers et al. (2002)). One can thus find all extreme points of the core by enumerating all marginal vectors and verifying which ones belong to the core. We present an improvement over these results by providing a method that allows to obtain the full set of extreme core allocations for all cost matrices.

The method is based on the concept of marginal games (Núñez and Rafels, 1998), in which we assign an agent her marginal cost to join the grand coalition, remove her from the problem and update the stand-alone costs of the remaining coalitions: they can either keep their original stand-alone cost or the stand-alone cost of them with the departing player, net of her cost share. This reduction is itself a special case of the Davis-Maschler reduction (Davis and Maschler, 1965). Given an ordering of the agents, repeating the process until all players are removed allows to find an extreme core allocation.

This method or very similar ones have been implemented for the assignment problem (Núñez and Rafels, 2003) and shortest path problems (Bahel and Trudeau, 2014), among others[1]. In the non-cooperative setting, a similar approach consists in ordering buyers according to a given permutation and letting them buy goods in that order (Pérez-Castrillo and Sotomayor, 2002; Vidal-Puga, 2004).

The method does not work as well on all problems. Even though there exist sufficient conditions for the method to always generate the full set of extreme core allocations (Potters et al., 1989; Driessen, 1988; Núñez and Rafels, 1998), none of them are satisfied, in general, by mcst problems. We are still

[1]A related approach is that of Tijs (2005); Funaki et al. (2007); Kongo et al. (2010); Tijs et al. (2011), who also look for extreme core allocations given some lexicographic order. However, their approach is explicitly based on the core constraints and not on marginal games.

able to prove that the method generates the full set of extreme core allocations, using a representation of marginal games as minimal cost spanning tree problems with priced nodes. This new problem is a generalization of both mcst problems and Steiner tree problems (Hwang and Richards, 1992; Skorin-Kapov, 1995).

By taking the average of these extreme core allocations for all permutations, we obtain a very natural cost sharing solution, identified as the selective value in Vidal-Puga (2004) and that coincides with the so-called Average lexicographic value, or Alexia (Tijs, 2005). If the game is concave, it also corresponds to the Shapley value. We show that our procedure is very close to three well-known cost-sharing solutions for mcst problems.

Firstly, if we only consider permutations that correspond to the order in which we connect agents in an optimal network configuration, we obtain directly the Bird solution (Bird, 1976). The Bird solution was the first solution to be shown to always be in the core and it is known for its simplicity, as we may assign cost at the same time as we construct an optimal tree.

Secondly, we show that for elementary problems (where all costs are either 0 or 1), our solution corresponds to the cycle-complete solution (Trudeau, 2012). The cycle-complete solution is obtained by modifying the cost of some links before taking the Shapley value of the corresponding cost game: we reduce the cost of edge (i, j) if there exists a cycle that goes through nodes i and j and such that its most expensive edge is cheaper than the direct edge (i, j). The modification is enough to make the corresponding cost game concave, and thus the Shapley value stable.

Thirdly, we show that for elementary problems, our solution applied to the public version of the mcst problem (where a coalition can use the nodes of its neighbors to connect to the source) corresponds to the folk solution (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007). The folk solution is also the Shapley value of a modified cost game, with the modifications similar to those made for cycle-complete solution, with the distinction that we look at paths instead of cycles.

An interpretation of our result is that for elementary problems, the folk solution is the permutation-weighted average of extreme core allocations of the public game, while the cycle-complete solution is the permutation-weighted average of extreme core allocations of the private game (where a coalition can only build a network among the nodes of its members).

The paper is divided as follows: In Section 2 we define the minimal cost spanning tree problems. In Section 3 we describe our method and show that it generates extreme core allocations. In Section 4 we show that it generates the full set of extreme core allocations. In Section 5 we discuss the potential generalization of our results to general nonconcave balanced games. In Section 6 we explore some links with popular cost sharing solutions. Section 7 contains some discussions. Counter-examples are in appendix.

2 The model

A (cost sharing) game is a pair (N, C) where $N = \{1, \ldots, n\}$ is a nonempty, finite set of agents, and C is a characteristic function that assigns to each nonempty coalition $S \subseteq N$ a nonnegative cost $C(S) \in \mathbb{R}_+$ that represents the price agents in S should pay in order to receive a service. In particular, we assume that the agents in N need to be connected to a source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between nodes i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with $N_0^p = (N_0)^p$ and c_e representing the cost of edge e. Given $E \subset N_0^p$, its associated cost is $c(E) = \sum_{e \in E} c_e$.

Let Γ be the set of all cost vectors. Since c assigns cost to all edges e, we often abuse language and call c a cost matrix. A minimal cost spanning tree problem is a triple $(0, N, c)$. Since 0 and N do not change, we omit them in the following and simply identify a mcst problem $(0, N, c)$ by its cost matrix c.

A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges.

We call mcst a spanning tree that has a minimal cost. It can be obtained using Prim's algorithm, which has n steps. First, pick an edge $(0, i)$ such that $c_{0i} \leq c_{0j}$ for all $j \in N$. We then say that i is connected. In the second step, we choose an edge with the smallest cost connecting an agent in $N \setminus \{i\}$ either directly to the source or to i, which is connected. We continue until all agents are connected, at each step connecting an agent not already connected to an agent already connected or to the source. Note that the mcst might not be unique. Let $\mathcal{T}^*(c)$ be the set of all mcst for the cost matrix c. Let $C(N, c)$ be the cost of a mcst. Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $C(S, c)$ be the cost of the mcst of the problem (S, c^S). Given these definitions, we say that C is the stand-alone cost function associated with c.

For any $S \subseteq N$, let $x(S) = \sum_{i \in S} x_i$. An *allocation* is a vector $x \in \mathbb{R}^N$ such that $x(N) = C(N)$. Given $S \subseteq N$ and $x \in \mathbb{R}^N$, we denote as $x_S \in \mathbb{R}^S$ the restriction of x to $\mathbb{R}^S$.

For any cost matrix c, the associated cost game is given by (N, C) with $C(S) = C(S, c)$ for all $S \subseteq N$. We then say that C is a mcst game. We define the set of stable allocations as $Core(C)$. Formally, an allocation $x \in Core(C)$ if $x(S) \leq C(S)$ for all $S \subseteq N$.

3 A method to find extreme core allocations

Our method to find extreme core allocations is based on the concept of marginal games of Núñez and Rafels (1998), but we define it in terms of a closely related source connection problem. A *mcst problem with priced nodes* is a tupla (N, P, y, c) where $P \subseteq N$ are nodes that do not need to be connected and $y \in \mathbb{R}^P$ is the vector whose coordinates are the prices that nodes in P pay to agents in $N \setminus P$ if they are actually connected. Nodes in P are called *priced nodes*. Hence, the cost of (N, P, y, c) is defined as

$$C(N, P, y, c) = \min_{T \subseteq P} \{C((N \setminus P) \cup T) - y(T)\}$$

and the cost of a subset $S \subseteq N \setminus P$ is given by

$$C(S, P, y, c) = \min_{T \subseteq P} \{C(S \cup T) - y(T)\}.$$

In particular, $C(N, P, y, c) = C(N \setminus P, P, y, c)$.

As usual, the *core* of a mcst problem with priced nodes (N, P, y, c) is the set of allocations $x \in \mathbb{R}^{N \setminus P}$ that satisfy $x(N \setminus P) = C(N, P, y, c)$ and $x(S) \leq C(S, P, y, c)$ for all $S \subseteq N \setminus P$.

Notice that mcst problems with priced nodes generalize both mcst problems (when $P = \emptyset$) and minimal cost Steiner tree problems (when $y_i = 0$ for all $i \in P$).

Let Π be the set of orders of N. Let $\pi = (\pi_1, \ldots, \pi_n) \in \Pi$. Given $\pi \in \Pi$ and $i \in N$, let $P^{\pi i} = \{\pi_i, \ldots, \pi_n\}$ be the set of nodes that come after π_i (including π_i) in the order π. For notational convenience, we denote $P^{\pi n+1} = \emptyset$. We define the reduced marginal cost vector of C related to permutation π, denoted as $y^{r\pi}(c)$, or simply $y^{r\pi}$, in a recursive manner, starting with $y^{r\pi}_{\pi_n}$ and making our way down to $y^{r\pi}_{\pi_1}$

$$y^{r\pi}_{\pi_i} = C(\{\pi_1 \ldots \pi_i\}, P^{\pi i+1}, y^{r\pi}, c) - C(\{\pi_1 \ldots \pi_{i-1}\}, P^{\pi i+1}, y^{r\pi}, c)$$

for $i = n, \ldots, 1$.[2]

It is not difficult to check that we are applying the marginal games idea of Núñez and Rafels (1998). After removing players, we allow coalitions of remaining players to either keep their original stand-alone cost or take the stand-alone cost they had with (some of) the departing players, net of their assigned cost shares. In our interpretation with priced nodes, this means that a coalition has to decide to build a network that includes or not some of these priced nodes, consisting of the set of departed players.

Example 1 *Let $N = \{1, 2, 3, 4\}$ and c be as described in the following (i horizontally, j vertically) and illustrated in Figure 1:*

c_{ij}	*1*	*2*	*3*	*4*
0	*1*	*6*	*5*	*5*
1		*6*	*4*	*2*
2			*5*	*5*
3				*5*

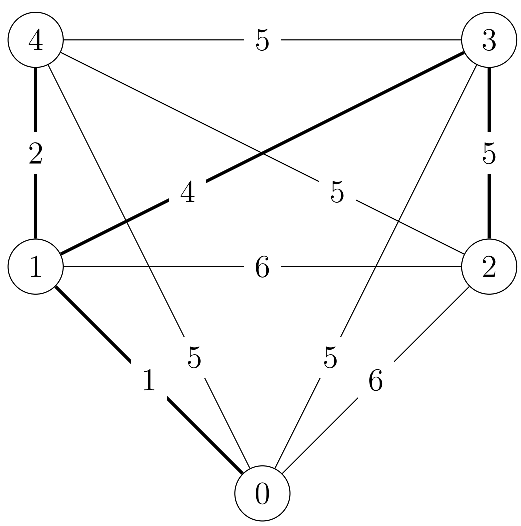

Figure 1: Example of a minimal cost spanning tree problem.

Let $\pi = [1234]$. We can see that $C(N, c) = 12$ and $C(\{1, 2, 3\}, c) = 10$, yielding $y_4^{r\pi} = 2$. Agent 4 then becomes a priced node. Now, we have $C(\{1, 2, 3\}, \{4\}, y^{r\pi}, c) = 10$ (as there's no advantage to connect agent 4 and

[2] For simplicity, we write $(N, P, y^{r\pi}, c)$ instead of $(N, P, (y^{r\pi})_P, c)$.

collect 2) and $C(\{1,2\},\{4\},y^{r\pi},c) = 6$ *(as connecting through agent 4 is advantageous), yielding* $y_3^{r\pi} = 4$. *We then have both agents 3 and 4 as priced nodes. We have* $C(\{1,2\},\{3,4\},y^{r\pi},c) = 6$ *and* $C(\{1\},\{3,4\},y^{r\pi},c) = 1$ *and thus* $y_2^{r\pi} = 5$. *Finally, we have* $C(\{1\},\{2,3,4\},y^{r\pi},c) = 1 = y_1^{r\pi}$. *We thus obtain* $y^{r\pi} = (1,5,4,2)$.

Núñez and Rafels (1998) provide a sufficient condition for $y^{r\pi}$ to be an extreme point of the core, and in fact for the set $\{y^{r\pi}\}_{\pi\in\Pi}$ to be the set of extreme points of the core. The sufficient condition is that of almost-concavity of the cost game: $C(S) + C(T) \geq C(S\cup T) + C(S\cap T)$ for all $S, T \subset N$ such that $S\cup T \neq N$. We thus have all concavity conditions except those involving the grand coalition. We show in appendix that the condition is not satisfied by games generated by mcst problems.

Nevertheless, we show that our method provides core allocations. To do so, we use four lemmas on the behavior of optimal trees with respect to the modification of the mcst problem when priced nodes are added.

The first lemma shows that as the set of priced nodes grows, with the prices given by $y^{r\pi}$, then the set of optimal trees can only grow. In particular, a mcst for the problem with no priced nodes is optimal in any problem with priced nodes as long as prices are given by $y^{r\pi}$.

Lemma 1 *Given* $\pi \in \Pi$ *and* $\pi_i \in N$, *any optimal tree in* $(N, P^{\pi i}, y^{r\pi}, c)$ *is also an optimal tree in* $(N, P^{\pi j}, y^{r\pi}, c)$ *for all* $j < i$.

Proof. Assume w.l.o.g. $\pi = [1\ldots n]$. Let t^i be an optimal tree in the mcst problem with priced nodes $(N,\{i,\ldots,n\},y^{r\pi},c)$ that uses the priced nodes set $T^i \subseteq \{i,\ldots,n\}$ and let $e \in t$. Let $j \leq i$. We prove that t^i is also an optimal tree in $(N,\{j,\ldots,n\},y^{r\pi},c)$ by backward induction on j. For $j = i$, the result holds trivially. Assume the result holds for $j+1$. We have to prove that for each tree t that connects all the nodes in $(\{1,\ldots,j-1\}\cup T$ to the source, with $T \subseteq \{j,\ldots,n\}$, it holds that

$$\sum_{e\in t^i} c_e - y^{r\pi}\left(\{j,\ldots,i-1\}\cup T^i\right) \leq \sum_{e\in t} c_e - y^{r\pi}(T). \qquad (1)$$

By induction hypothesis, t^i is an optimal tree in $(N,\{j+1,\ldots,n\},y^{r\pi},c)$. Assume first $j \in T$. Then, $\sum_{e\in t^i} c_e - y^{r\pi}\left(\{j+1,\ldots,i-1\}\cup T^i\right) \leq \sum_{e\in t} c_e -$

$y^{r\pi}(T \setminus j)$, which is equivalent to (1). Assume now $j \notin T$. Hence

$$\begin{aligned}
&\sum_{e\in t^i} c_e - y^{r\pi}\left(\{j,\ldots,i-1\}\cup T^i\right) \\
&= \sum_{e\in t^i} c_e - y^{r\pi}\left(\{j+1,\ldots,i-1\}\cup T^i\right) - y^{r\pi}_j \\
&= C\left(N,\{j+1,\ldots,n\},y^{r\pi},c\right) - y^{r\pi}_j.
\end{aligned}$$

By definition,

$$y^{r\pi}_j = C\left(N,\{j+1,\ldots,n\},y^{r\pi},c\right) - C\left(\{1,\ldots,j-1\},\{j+1,\ldots,n\},y^{r\pi},c\right).$$

Hence the previous expression equals $C\left(\{1,\ldots,j-1\},\{j+1,\ldots,n\},y^{r\pi},c\right)$ which, by definition, is less or equal than $\sum_{e\in t} c_e - y^{r\pi}(T)$. ■

Given an order π and $\pi_i \in N$, we say that $e \in N_0^p$ is a *critical edge* in π for π_i if $e \in t$ for some optimal tree t in the mcst problem with priced nodes $(N, P^{\pi i}, y^{r\pi}, c)$.

Lemma 2 *Given $\pi \in \Pi$ and $\pi_i \in N$, any critical edge in π for π_i is also a critical edge in π for π_j for all $j < i$.*

Proof. Assume w.l.o.g. $\pi = [1 \ldots n]$ and let $e \in N_0^p$ be a critical edge in π for i, i.e. $e \in t$ for some t optimal tree in $(N, \{i,\ldots,n\}, y^{r\pi}, c)$. Let $j < i$. Under Lemma 1, t is also optimal in $(N, \{j,\ldots,n\}, y^{r\pi}, c)$. Hence, e is a critical edge in π for j. ■

Using the previous two results on the behavior of the set of optimal trees, we show that for a given cost matrix, permutation, and rank i in that permutation, there exists a matrix with non-larger costs that generate the same allocation vector $y^{r\pi}$ and such that when we want to connect the first $i-1$, there is no gain to use any of the other agents as price nodes.

Lemma 3 *For all $\pi \in \Pi$ and all $\pi_i \in N$, there exists c^{π_i} satisfying:*

a) $c^{\pi_i} \leq c$,

b) $y^{r\pi}(c) = y^{r\pi}\left(c^{\pi_i}\right)$, *and*

c) $C\left(\{\pi_1,\ldots,\pi_{i-1}\},c^{\pi_i}\right) = \min_{T\subseteq P^{\pi i}}\left\{C\left(\{\pi_1,\ldots,\pi_{i-1}\}\cup T, c^{\pi_i}\right) - y^{r\pi}(T)\right\}$.

Proof. Assume w.l.o.g. $\pi = [1 \dots n]$. We proceed by backward induction on node i. For $i = n$, we take $c^n = c$, which trivially satisfies conditions a) and b). Moreover, it satisfies condition c) by definition of $y_n^{r\pi}$. Assume now the result holds for node $i+1$. We can then assume $c = c^{i+1}$. Fix an optimal tree $t^* \in \mathcal{T}^*(c)$. We denote $t^* = \{(j, j^*)\}_{j \in N}$, where j^* is the predecessor of node j in t^*, i.e. j^* is the adjacent node to node j in the (unique) path in t^* from node j to the source. For each $j \in N$, let

$$U^j = \{(k, k^*) \in t^* : k, k^* \in \{0, 1, \dots, j-1\}\}$$

be the set of edges in t^* that do not use priced nodes in $(N, \{j, \dots, n\}, y^{r\pi}, c)$. For each $j \in N$, let t^j be an optimal tree in $(N \setminus \{j\}, \{j+1, \dots, n\}, y^{r\pi}, c)$ which, by definition of $y_j^{r\pi}$, is also an optimal tree in $(N, \{j, \dots, n\}, y^{r\pi}, c)$. Given the optimality of t^*, we can choose each t^j such that $t^j = U^j \cup E^j$, where E^j is an optimal set of edges that connect to each other the connected components that arise from t^* when we remove the priced nodes $j, \dots, n$. We also denote the set of priced nodes used by E^j (and t^j) as $T^j \subseteq \{j+1, \dots, n\}$. Under the induction hypothesis, we can choose each t^j such that $T^j = \emptyset$ for all $j > i$. Analogously, let t be an optimal tree in $(\{1, \dots, i-1\}, c)$, i.e. $c(t) = C(\{1, \dots, i-1\})$. Again, given the optimality of t^*, we can choose t such that $t = U^i \cup E$, where E does not use priced nodes. In case of more than one minimizing E, we choose one with a maximum number of edges in E^i. If $T^i = \emptyset$, then $c^i = c$ satisfies the three conditions. Hence, we assume that $T^i \neq \emptyset$ and, moreover,

$$c(t) > c\left(t^i\right) - y^{r\pi}\left(T^i\right). \tag{2}$$

We show that we cannot have $E \subset E^i$. Assume, by contradiction, $E \subset E^i$ or, equivalently, $t \subset t^i$. This means that, for each node $k \in T^i$, there exists a unique path in $E^i \setminus E = t^i \setminus t$ that connects k with $\{0, 1, \dots, i-1\}$. Let $j \in T^i$ be the node with lowest index in T^i. Then, $t' = t^j \cup (E^i \setminus E)$ is a tree that connects all the nodes in $\{1, \dots, j-1\}$ to the source, so $c(t^j) - y^{r\pi}(T^j) \leq c(t') - y^{r\pi}(T^j \cup T^i)$. Since $T^j = \emptyset$ and $c(t') = c(t^j) + c(E^i \setminus E)$, we deduce $y^{r\pi}(T^i) \leq c(E^i \setminus E)$ and hence

$$c(t^i) - y^{r\pi}\left(T^i\right) \geq c(t^i) - c(E^i \setminus E) = c(t^i) - c(t^i \setminus t) = c(t)$$

which contradicts (2).

Hence, there exists some $e \in t$ such that $e \notin t^i$. Since E has a maximum number of edges in E^i, we can choose e so that it is not a critical edge in π

for i. Under Lemma 2, e is not a critical edge in π for any $j > i$. Moreover, we can choose e so that it is not a critical edge in π for any $j < i$. Otherwise, all the edges in t would be critical and hence t would be an optimal tree in $(N, \{i, \ldots, n\}, y^{r\pi}, c)$, which is a contradiction. We can then define c' as c except for edge e, whose cost we reduce (satisfying part a)) until it becomes a critical edge in c'. We then repeat the procedure until eventually we find $T^i = \emptyset$ and so we can define $c^i = c'$ satisfying the three conditions. ■

Our final lemma allows us to extend the previous result, as we show that for a given permutation of the agents, there exists a matrix with non-larger costs that generate the same allocation vector $y^{r\pi}$ and such that for any rank i, when we want to connect the first $i-1$, there is no gain to use any of the other agents as price nodes.

Lemma 4 *For all $\pi \in \Pi$, there exists c^π satisfying:*

a) $c^\pi \leq c$,

b) $y^{r\pi}(c) = y^{r\pi}(c^\pi)$, and

c) $C(\{\pi_1, \ldots, \pi_{i-1}\}, c^\pi) = \min_{T \subseteq P^{\pi i}} \{C(\{\pi_1, \ldots, \pi_{i-1}\} \cup T, c^\pi) - y(T)\}$ for all $\pi_i \in N$.

Proof. For each c and π_i, let c^{π_i} be the cost function given in Lemma 3. Define $c^n = c$ and assume we have defined c^{i+1}. We define $c^i = (c^{i+1})^{\pi_i}$. Under Lemma 3, it is clear that $c^\pi = c^1$ satisfies the three conditions. ■

We are now ready for the main result of this section.

Proposition 1 *For all $\pi \in \Pi$, $y^{r\pi} \in Core(C)$.*

Proof. Assume w.l.o.g. $\pi = [1 \ldots n]$. Let c^π be the cost matrix whose existence is guaranteed by Lemma 4 and C^π its associated cost game. Conditions a) and b) in Lemma 4 imply $Core(C^\pi) \subseteq Core(C)$. Hence, we can assume $c = c^\pi$ so that conditions a), b) and c) in Lemma 4 hold for c. Let $T = \{\alpha_1, \ldots, \alpha_k\} \subset N$ with $\alpha_1 < \cdots < \alpha_k$. Given a tree t, we denote as $t(N) \subseteq N$ the set of nodes that connect to the source through t. Let $t = \{(i, i^t)\}_{i \in T}$ with $t(N) = T$ and $C(T) = c(t)$. We will prove that $y^{r\pi}(T) \leq C(T)$. Let t^1 be a tree with $\{0, 1, \ldots, \alpha_1 - 1\} \subset t^1(N) \subset N \setminus \{\alpha_1\}$ and such that

$$C(\{1, \ldots, \alpha_1 - 1\}, \{\alpha_1 + 1, \ldots, n\}, y^{r\pi}, c) = c(t^1) - y^{r\pi}(T^1)$$

where $T^1 = t^1(N) \cap \{\alpha_1 + 1, \ldots, n\}$. Under condition c) in Lemma 4, we can choose t^1 such that $T^1 = \emptyset$. Hence, $t^1 \cap t = \emptyset$ and, moreover, $t^1 \cup t$ is a tree that connects $\{1, \ldots, \alpha_1\}$ to the source using the priced nodes in $T \setminus \{\alpha_1\}$. Hence,

$$\begin{aligned} y_{\alpha_1} &= C(N, \{\alpha_1 + 1, \ldots, n\}, y^{r\pi}) - C(\{1, \ldots, \alpha_1 - 1\}, \{\alpha_1 + 1, \ldots, n\}, y^{r\pi}) \\ &= C(N, \{\alpha_1 + 1, \ldots, n\}, y^{r\pi}) - c(t^1) \\ &\leq c(t^1 \cup t) - y^{r\pi}(T \setminus \{\alpha_1\}) - c\left(t^1\right) \\ &= c(t) - y^{r\pi}(T \setminus \{\alpha_1\}) \end{aligned}$$

and so $y^{r\pi}(T) \leq c(t) = C(T)$. ■

4 The set of extreme core allocations

We now have shown that our procedure generates allocations in the core. Consider the permutation $\pi = [1 \ldots n]$. Agent n pays her minimal allocation compatible with the core (as coalition $N \setminus i$ pays exactly $C(N \setminus i)$, their highest allocation compatible with the core). At the following stage, we assign to agent $n - 1$ her smallest allocation compatible with the core of the reduced game, given the allocation of agent n. We continue in this manner, at each step k assigning to agent $n - k$ her smallest allocation compatible with the core of the corresponding reduced game, given the allocations of agents coming after her in the permutation. If the allocation $y^{r\pi}$ is in the core, which, in our case, was proven in Proposition 1, then at each step k we assign to agent $n - k$ her smallest allocation compatible with the core and the allocations given to agents coming after her in the permutation. It is thus also an extreme core allocation, defined as follows:

Definition 1 *An allocation $y \in Core(C)$ is an extreme core allocation if there do not exist $y', y'' \in Core(C)$, $y' \neq y''$ and $\lambda \in (0, 1)$ such that $y = \lambda y' + (1 - \lambda) y''$.*

We denote the set of extreme core allocations as $ExtCore(C)$. Funaki et al. (2007) define a leximal as the lexicographic minimum of the core with respect to a given order.[3] It is not difficult from the description above to see

[3] Funaki et al. (2007) define the leximal as the lexicographic maximum of the core of a value game. The lexicographic minimum of the core is conceptually equivalent for cost games.

that our allocation $y^{r\pi}$ corresponds to the leximal with respect to the reverse order of π.

The convex hull of $\{y^{r\pi}\}_{\pi\in\Pi}$ is thus the convex hull of the leximals, defined by Funaki et al. (2007) as the lexicore. By definition, the lexicore is a subset of the core. A sufficient condition for the lexicore to be equal to the core is for the corresponding exact game (where all coalitions can attain their stand-alone cost in the core) to be concave. We show in the appendix that this condition does not hold generally for mcst problems.

Nevertheless, we next show that the core allocations defined in the previous section actually constitute the whole set of extreme core allocations. We proceed by first showing that for any coalition, we attain the maximal allocation compatible with the core. We need to prove two intermediary results before getting to the main result of this section.

First, we define the alternative stand-alone cost $\widehat{C}(S)$, where we let S pick a partner T and compute the sum of the costs of S with T and $N \setminus T$, to which we subtract the cost of the grand coalition.

Formally, for all $c \in \Gamma$, let $\left(N, \widehat{C}\right)$ be defined in the following way:

Step 0: $\widehat{C}(S) = C(S)$ for all S such that $|S| \geq n-1$.

Step k: $\widehat{C}(S) = \min\left\{C(S), \min_{\emptyset\neq T\subset N\setminus S} \widehat{C}(S\cup T) + \widehat{C}(N\setminus T) - \widehat{C}(N)\right\}$ for all S such that $|S| = n-1-k$, for $k = 1,\ldots,n-2$.

Step $n-1$: $\widehat{C}(\emptyset) = 0$.

Hence,

$$\widehat{C}(S) = \min_{T\subseteq N\setminus S}\left\{\widehat{C}(S\cup T) + \widehat{C}(N\setminus T) - C(N)\right\}$$

for all $S \subseteq N$.

The cost function $\widehat{C}$ has the same core as C and a coalition S cannot receive more than $\widehat{C}(S)$ in any core allocation.

Lemma 5 *For all $c \in \Gamma$, $Core\left(\widehat{C}\right) = Core(C)$.*

We refer the reader to Bondareva and Driessen (1994) and Branzei et al. (2005), who prove stronger results, in which the upper bound of the cost allocation of coalition S is obtained by minimizing the value of $C(R_1) + \cdots + C(R_k) - (k-1)C(N)$ over the sets $\{R_1,\ldots,R_k\}$ such that $N\setminus S = (N\setminus R_1)\cup\cdots\cup(N\setminus R_k)$. By contrast, $\widehat{C}$ only uses sets with $k=2$, which proves sufficient for our purposes.

Our next step is to show that for any S, there exists at least one permutation π such that $y^{r\pi}(S) = \widehat{C}(S)$. To do so, we once again use the mcst with priced nodes representation.

We know (Proposition 1) that $y^{r\pi}_{N \setminus P^{\pi i}}$ belongs to the core of the game generated by $(N, P^{\pi i}, y^{r\pi}, c)$ and, moreover, that *any* $t^* \in \mathcal{T}^*(c)$ is an optimal tree in *any* $(N, P^{\pi i}, y^{r\pi}, c)$.

The next lemma will be needed in our main result. See Figure 2 for a graphical interpretation.

Lemma 6 *Let $(N, P^a, y^{r\pi}, c)$ and $\left(N, P^b, y^{r\pi}, c\right)$ be two mcst problems with priced nodes and assume there exists $P_1 \subseteq P^a \cap P^b$ such that P_1 is a maximal subset of both P^a and P^b whose nodes are connected in t^* for some $t^* \in \mathcal{T}^*(c)$. For any $p_1 \in P_1$, there exists a tree t^1 with $N \setminus P_1 \subseteq t^1(N) \subseteq N \setminus \{p_1\}$ such that t^1 is optimal in both $(N \setminus \{p_1\}, P^a \setminus \{p_1\}, y^{r\pi}, c)$ and $\left(N \setminus \{p_1\}, P^b \setminus \{p_1\}, y^{r\pi}, c\right)$.*

Proof. Let t^1 be an optimal tree in $(N \setminus \{p_1\}, P_1 \setminus \{p_1\}, y^{r\pi}, c)$. We will prove that t^1 is also an optimal tree in both $(N \setminus \{p_1\}, P^a \setminus \{p_1\}, y^{r\pi}, c)$ and $\left(N \setminus \{p_1\}, P^b \setminus \{p_1\}, y^{r\pi}, c\right)$. Given a tree t, we denote as $t(N) \subseteq N$ the set of nodes that connect to the source through t. Notice first that t^1 connects to the source all the nodes in $N \setminus P_1$, which includes both $S^a = N \setminus P^a$ and $S^b = N \setminus P^b$. Let t^a be an optimal tree in $(N \setminus \{p_1\}, P^a \setminus \{p_1\}, y^{r\pi}, c)$ that connects all the nodes in S^a to the source (the case for $(N \setminus \{p_1\}, P^b \setminus \{p_1\}, y^{r\pi}, c)$ is analogous). In case there are more than one possible optimal tree, we choose t^a with maximal number of nodes $|t^a(N)|$. We need to prove that $c(t^1) - y^{r\pi}(t^1(N) \cap P^a) \leq c(t^a) - y^{r\pi}(t^a(N) \cap P^a)$. Assume first $P^a \setminus P_1 \subseteq t^a(N)$. Since t^1 is optimal in $(N \setminus \{p_1\}, P_1 \setminus \{p_1\}, y^{r\pi}, c)$, we have

$$
\begin{aligned}
c(t^1) - y^{r\pi}\left(t^1(N) \cap P^a\right) &= c(t^1) - y^{r\pi}\left(t^1(N) \cap P_1\right) - y^{r\pi}\left(P^a \setminus P_1\right) \\
&\leq c(t^a) - y^{r\pi}\left(t^a(N) \cap P_1\right) - y^{r\pi}\left(P^a \setminus P_1\right) \\
&= c(t^a) - y^{r\pi}\left(t^a(N) \cap P^a\right).
\end{aligned}
$$

Assume now there exists $p_2 \in P^a \setminus P_1$ such that $p_2 \notin t^a(N)$. Denote $t^* = \{(i, i^*)\}_{i \in N}$, where i^* is the predecessor of node i in t^*, i.e. i^* is the adjacent node to node i in the (unique) path in t^* from node i to the source. Assume w.l.o.g. that $p_2^* \in t^a(N)_0$, i.e. the immediate predecessor of p_2 in t^* is connected to the source (or it is the source itself) through t^a. Let P_2 be the maximal subset of P^a whose nodes are connected to p_2 in t^* (including

p_2 itself) without using any node in $t^a(N)$ (this implies $p_2 \in P_2$ and $p_2^* \notin P_2$). Let $R^* = \{i \in N_0 \setminus P_2 : i^* \in P_2\} \cup \{p_2^*\}$ be the set of nodes in N_0 (and also in $t^a(N)_0$) that are linked in t^* to some node in P_2. Assume w.l.o.g. $R^* = \{1, \ldots, k-1, p_2^*\}$. For each $i \in R^* \setminus \{p_2^*\}$, let F^{*i} be the set of nodes that follow node i in t^* (including node i itself). We define the set of edges $E = \{e_1, \ldots, e_{k-1}\} \subset N_0^p$ inductively as follows: Let $e_1 = (i_1, i_1') \in N_0^p$ be the first edge in the (unique) path in t^a from node 1 to node p_2^* such that $i_1 \in F^{*1}$ and $i_1' \notin F^{*1}$. Let $e_2 = (i_2, i_2') \in N_0^p$ be the first edge in the (unique) path in t^a from node 2 to node p_2^* such that $i_2 \in F^{*1} \cup F^{*2}$ and $i_2' \notin F^{*1} \cup F^{*2}$. Let $e_3 = (i_3, i_3') \in N_0^p$ be the first edge in the (unique) path in t^a from node 3 to node p_2^* such that $i_3 \in F^{*1} \cup F^{*2} \cup F^{*3}$ and $i_3' \notin F^{*1} \cup F^{*2} \cup F^{*3}$. And so on. Thus, E is the set of edges that connect R^* in t^a, and so that $t^a \setminus E$ leaves k connected components. Since nodes in R^* are connected in t^* using the arcs in $E^* = \{(i,j) \in t^* : i,j \in R^* \cup P_2\}$, we deduce that

$$t^2 = (t^a \setminus E) \cup E^* \tag{3}$$

is, like t^a, a tree that connects all the nodes in $N \setminus P^a$ to the source and, moreover,

$$t^a(N) = t^2(N) \setminus P_2. \tag{4}$$

Optimality of t^* in $(N, N, y^{r\pi}, c)$ implies that

$$c\,(E^*) - y^{r\pi}(P_2) \leq c\,(E) \tag{5}$$

because, otherwise, $(t^* \setminus E^*) \cup E$ would improve t^*. We now check that t^2 is also, like t^a, an optimal tree in $(N \setminus \{p_1\}, P^a \setminus \{p_1\}, y^{r\pi}, c)$:

$$\begin{aligned} c(t^2) - y^{r\pi}\left(t^2(N) \cap P^a\right) &\overset{(3)}{=} c(t^a) - c(E) + c\,(E^*) - y^{r\pi}\left(t^2(N) \cap P^a\right) \\ &\overset{(5)}{\leq} c(t^a) + y^{r\pi}(P_2) - y^{r\pi}\left(t^2(N) \cap P^a\right) \\ &\overset{(4)}{=} c(t^a) - y^{r\pi}\left(t^a(N) \cap P^a\right) \end{aligned}$$

which contradicts $|t^a(N)|$ be maximal. ■

Let $t^* \in \mathcal{T}^*(c)$. We then define $i \preceq_* j$ as the partial relation in N given by "i precedes j in t^*", that is, $i \preceq_* j$ iff $j \in F^{*i}$, where F^{*i} is the set of followers of node i in t^* (including node i). As usual, we denote $t^* = \{(i, i^*)\}_{i \in N}$, where i^* is the predecessor of node i in t^*, i.e. i^* is the adjacent node to node i in the (unique) path in t^* from node i to the source.

Given $P \subseteq N$ and $\pi \in \Pi$, we say that π is t^*-compatible with P if the following two conditions hold:

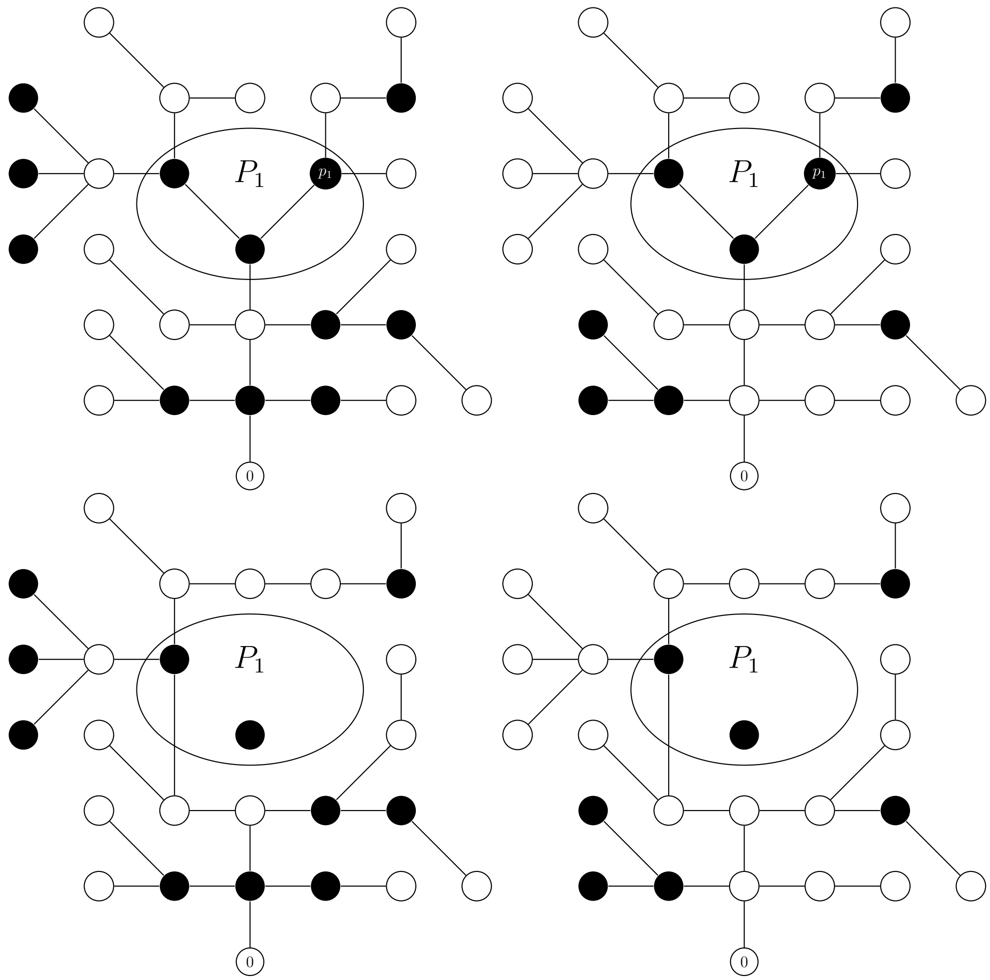

Figure 2: The two figures at the top represent respectively two mcst problems with priced nodes and the same cost matrix. Priced nodes are those in black. P_1 is a maximal set of priced nodes in both problems. Lemma 6 says that, if we remove some node $p_1 \in P_1$, then we can find a common optimal tree for both reduced problems, represented in the two figures at the bottom, respectively.

External t^*-compatibility Nodes in $S = N \setminus P$ that follow or immediately precede in t^* a node from P come first in π:

$$\left.\begin{array}{l} \pi_i \preceq_* \pi_j \text{ or } \pi_i^* = \pi_j \\ \pi_i \in P \\ \pi_j \in S \end{array}\right\} \Longrightarrow j < i.$$

Internal t^*-compatibility Nodes inside P follow in π the partial order $\preceq_*$:

$$\left.\begin{array}{l} \pi_i \preceq_* \pi_j \\ \pi_i, \pi_j \in P \end{array}\right\} \Longrightarrow i \leq j.$$

It is clear that there always exists an order t^*-compatible with P. For example, let $\pi \in \Pi$ be an order obtained by Prim's algorithm, where nodes that are closer to the source in t^* come first. Clearly, this order satisfies internal t^*-compatibility. We then move the nodes in P to the last positions, maintaining the internal order in P and $N \setminus P$ unaffected. The resulting order satisfies both internal and external t^*-compatibility.

Example 2 *We revisit Example 1, with* $P = \{1, 3, 4\}$.

Take $t^* = \{(0,1), (1,4), (1,3), (3,2)\}$. *External t^*-compatibility implies that nodes* 1 *and* 3 *should go after node* 2. *Internal t^*-compatibility implies that nodes* 3 *and* 4 *should go after node* 1. *These two conditions together give two unique t^*-compatible orders:* [2143] *and* [2134]. *In both cases,* $y^{r\pi} = (0, 6, 4, 2)$.

The same happens if we take $t^* = \{(0,1), (1,3), (1,4), (4,2)\}$ *instead.*

We say that π is c-compatible with P if there exists some $t^* \in \mathcal{T}^*(c)$ such that π is t-compatible with P.

In the previous example, $y^{r\pi}(\{2\}) = \widehat{C}(\{2\}) = 6$ for each π c-compatible with $\{1, 3, 4\}$. We now show that this holds in general.

Proposition 2 *Given* $S \subseteq N$, *we have* $y^{r\pi}(S) = \widehat{C}(S)$ *for all* $\pi \in \Pi$ *order c-compatible with* $N \setminus S$.

Proof. Fix $t^* = \{(i, i^*)\}_{i \in N} \in \mathcal{T}^*(c)$, where i^* is the predecessor of node i in t^*, i.e. i^* is the adjacent node to node i in the (unique) path in t^* from node i to the source. Given $S \subseteq N$ and $P = N \setminus S$, let $\pi \in \Pi$ be an order t^*-compatible with P. We prove that the following two statements hold:

(I) Either $y^{r\pi}(S) = C(S)$ or there exists $\emptyset \neq T' \subset P$ such that

> **(I*a*)** $y^{r\pi^a}(T') = y^{r\pi}(T')$ for all $\pi^a \in \Pi$ order t^*-compatible with T', and
>
> **(I*b*)** $y^{r\pi^b}(P \setminus T') = y^{r\pi}(P \setminus T')$ for all $\pi^b \in \Pi$ order t^*-compatible with $P \setminus T'$.

(II) $y^{r\pi}(S) = \widehat{C}(S)$.

We proceed by induction on $|P|$. For $P = \emptyset$, both statements hold trivially. Assume now both statements hold when $|P| < \alpha$ and suppose $|P| = \alpha$ for some $\alpha > 0$.

We first prove statement (I). Let $p_1 = \pi_s \in P$ be the first element in P according to π (that is, $i < s$ implies $\pi_i \notin P$). Hence, for all $\pi_i \in P \setminus \{p_1\}$, we have $s < i$. Under internal t^*-compatibility, we deduce $p_1 \notin F^{*i}$ for all $i \in P \setminus \{p_1\}$.

Let $S' = S \cup \{p_1\}$ and $P' = P \setminus \{p_1\}$. Let $P_1 = \{i \in N : \psi^*_{ip_1} \subseteq P\}$, where $\psi^*_{ip_1}$ is the (unique) path from node i to node p_1 in t^*, be the set of nodes in P that are adjacent to p_1.

Let t' be an optimal tree in $(N \setminus \{p_1\}, P', y^{r\pi}, c)$. That is:

$$
\begin{aligned}
S &\subseteq t'(N) \\
p_1 &\notin t'(N) \\
c(t') - y^{r\pi}(t'(N) \cap P') &= \min_{T \subseteq P'} \{C(S \cup T) - y^{r\pi}(T)\}.
\end{aligned}
$$

In case there are more than one tree satisfying the above conditions, we take t' such that $|t'(N)|$ is maximal among them.

Assume first $S = t'(N)$, which means $t'(N) \cap P' = \emptyset$ and $c(t') = C(S)$. Moreover, under Lemma 6, we deduce that $P = P_1$. We will prove that $y^{r\pi}(S) = C(S)$, so that statement (I) holds.

Let $\sigma \in \Pi$ defined from π by moving nodes in P to the last positions, leaving the rest of the order unaffected[4]. Hence, $P = P^{\sigma s'}$ for some $s' \geq s$. From the proof of Proposition 1, we know that $y^{r\sigma}_{N \setminus P^{\sigma s'}}$ belongs to the core of $\left(N, P^{\sigma s'}, y^{r\sigma}, c\right)$. Since $P = P^{\sigma s'}$, this can be rewritten as $y^{r\sigma}_S \in Core\,(N, P, y^{r\sigma}, c)$, which implies $y^{r\sigma}(S) = C\,(S, P, y^{r\sigma}, c)$. Moreover, we know that $C\,(S, P, y^{r\pi}, c) = C(S)$. Hence, it is enough to prove that $y^{r\sigma} = y^{r\pi}$. Let $\pi_m \in S$ such that $\{\pi_s, \ldots, \pi_{m-1}\} \subseteq P = P_1$. We will prove that

[4]For example, if $\pi = [123456789]$ and $P = \{3, 4, 6, 8\}$, then $\sigma^P = [125793468]$.

$y^{r\pi} = y^{r\pi'}$ where $\pi' \in \Pi$ coincides with π after changing nodes π_{m-1} and π_m, i.e.

$$\pi' = [\pi_1 \dots \pi_{m-2}\pi_m\pi_{m-1}\pi_{m+1} \dots \pi_n]$$

so that a repeated reasoning leaves to $y^{r\pi} = y^{r\sigma}$.

It is clear that $y^{r\pi}_{\pi_i} = y^{r\pi'}_{\pi_i}$ for all $i > m$. We now check that $y^{r\pi}_{\pi_m} = y^{r\pi'}_{\pi_m}$. Under t^*-compatibility, it is straightforward to check that P_1 is a maximal connected component of both $\{\pi_{m+1}, \dots, \pi_n\}$ and $\{\pi_{m-1}, \pi_{m+1}, \dots, \pi_n\}$. Under Lemma 6, there exists an optimal tree t^s in both

$$(\{\pi_1, \dots, \pi_{m-1}\}, \{\pi_{m+1}, \dots, \pi_n\}, y^{r\pi}, c)$$

and

$$(\{\pi_1, \dots, \pi_{m-2}\}, \{\pi_{m-1}, \pi_{m+1}, \dots, \pi_n\}, y^{r\pi}, c)\,.$$

Hence,

$$\begin{aligned} y^{r\pi}_{\pi_m} =& C\left(\{\pi_1, \dots, \pi_m\}, \{\pi_{m+1}, \dots, \pi_n\}, y^{r\pi}, c\right) \\ &- C\left(\{\pi_1, \dots, \pi_{m-1}\}, \{\pi_{m+1}, \dots, \pi_n\}, y^{r\pi}, c\right) \\ =& c(t^*) - y^{r\pi}\left(\{\pi_{m+1}, \dots, \pi_n\}\right) \\ &- c(t^s) + y^{r\pi}\left(t^s(N) \cap \{\pi_{t+1}, \dots, \pi_n\}\right) \end{aligned} \tag{6}$$

and

$$\begin{aligned} y^{r\pi'}_{\pi_m} =& C\left(\{\pi_1, \dots, \pi_{m-2}, \pi_m\}, \{\pi_{m-1}, \pi_{m+1}, \dots, \pi_n\}, y^{r\pi}, c\right) \\ &- C\left(\{\pi_1, \dots, \pi_{m-2}\}, \{\pi_{m-1}, \pi_{m+1}, \dots, \pi_n\}, y^{r\pi}, c\right) \\ =& c(t^*) - y^{r\pi}\left(\{\pi_{m-1}, \pi_{m+1}, \dots, \pi_n\}\right) \\ &- c(t^s) + y^{r\pi}\left(t^s(N) \cap \{\pi_{m-1}, \pi_{m+1}, \dots, \pi_n\}\right). \end{aligned} \tag{7}$$

Since $\pi_{m-1} \in P_1$, we deduce that π_{m-1} does not belong to the same connected component as π_m, and hence $\pi_{m-1} \in t^s(N)$. This implies that (6) and (7) coincide and thus $y^{r\pi}_{\pi_m} = y^{r\pi'}_{\pi_m}$. An analogous reasoning leads to $y^{r\pi}_{\pi_{m-1}} = y^{r\pi'}_{\pi_{m-1}}$ and from it, $y^{r\pi}_{\pi_i} = y^{r\pi'}_{\pi_i}$ for all $i < m-1$, so that $y^{r\pi} = y^{r\pi'}$, as desired.

Assume now $S \subset t'(N)$. Let $T' = t'(N) \setminus S$. It is straightforward to check that $\emptyset \neq T' \subset P$. We will prove that (I$a$) and (I$b$) hold with this T'. As a previous step, we need to prove the following Claim:

Claim A: For all $i \in N \setminus \{p_1\}$, $i^* \in S \cup T' \Rightarrow i \in S \cup T'$.

Proof. Assume, on the contrary, that there exists some $i \in N \setminus \{p_1\}$ such that $i^* \in S \cup T'$ and $i \notin S \cup T'$. That is, $i^* \in t'(N)$ and $i \notin t'(N)$.

Let $Q = \left\{j \in N \setminus T' : \psi^*_{ij} \subseteq N \setminus T'\right\}$, where ψ^*_{ij} is the (unique) path form node i to node j in t^*. Thus, Q is the set of followers of node i without leaving $P \setminus T'$. Hence, $Q \subseteq P \setminus T'$. Since $p_1 \notin F^{*j}$ for all $j \in P'$, we deduce that $p_1 \notin Q$. Let $R = \{j \in N : j^* \in Q\}$ be the set of nodes in S that immediately follow some node in Q (case $R = \emptyset$ is also possible). Denote $R = \{r_1, \ldots, r_k\}$. We define the set of edges $E = \{e_1, \ldots, e_k\} \subset N^p$ inductively as follows: Let $e_1 = (i_1, i'_1) \in N^p$ be the first edge in the (unique) path in t' from r_1 to the source such that $i_1 \in F^{*r_1}$ and $i'_1 \notin F^{*r_1}$. Let $e_2 = (i_2, i'_2) \in N^p$ be the first edge in the (unique) path in t' from r_2 to the source such that $i_2 \in F^{*r_1} \cup F^{*r_2}$ and $i'_2 \notin F^{*r_1} \cup F^{*r_2}$. Let $e_3 = (i_3, i'_3) \in N^p$ be the first edge in the (unique) path in t' from r_3 to the source such that $i_3 \in F^{*r_1} \cup F^{*r_2} \cup F^{*r_3}$ and $i'_3 \notin F^{*r_1} \cup F^{*r_2} \cup F^{*r_3}$. And so on. Thus, E is the set of edges that connect R to the source in t'. Now,

$$t^1 = (t^* \setminus \{(j, j^*)\}_{j \in Q \cup R}) \cup E \tag{8}$$

is a tree that connects all the nodes in S to the source using nodes in $P \setminus Q$. Since t^* is optimal in $(S, P, y^{r\pi}, c)$, we deduce that

$$c(t^*) - y^{r\pi}(P) \leq c(t^1) - y^{r\pi}(P \setminus Q)$$

or, equivalently,

$$c(t^*) \leq c(t^1) + y^{r\pi}(Q). \tag{9}$$

Now,

$$t^2 = (t' \setminus E) \cup \{(j, j^*)\}_{j \in Q \cup R} \tag{10}$$

is a tree that connects all the nodes in S to the source using nodes in $T' \cup Q \subseteq P'$. Since t' is optimal in $(S, P', y^{r\pi}, c)$, we deduce that

$$c(t') - y^{r\pi}(T') \leq c(t^2) - y^{r\pi}(T' \cup Q)$$

or, equivalently,

$$c(t') \leq c\left(t^2\right) - y^{r\pi}(Q). \tag{11}$$

By applying (9) and (11),

$$\begin{aligned}
c\left(t^2\right) &\overset{(11)}{\geq} c(t') + y^{r\pi}(Q) \overset{(9)}{\geq} c(t') + c(t^*) - c(t^1) \\
&\overset{(8)}{=} c(t') + c(t^*) - c(t^*) + c(\{(j,j^*)\}_{j\in Q\cup R}) - c(E) \\
&= c\left(t'\right) + c\left(\{(j,j^*)\}_{j\in Q\cup R}\right) - c\left(E\right) \\
&\overset{(10)}{=} c\left(t^2\right).
\end{aligned}$$

This implies $c\left(t^2\right) = c\left(t'\right) + y^{r\pi}(Q)$, so

$$c\left(t^2\right) - y^{r\pi}(T' \cup Q) = c\left(t'\right) - y^{r\pi}(T')$$

and hence t^2 is also an optimal tree in $(S, P', y^{r\pi}, c)$ with $|t^2(N)| = |t'(N)| + |Q|$, which contradicts that $|t'(N)|$ is maximum among these optimal trees (notice that $i \in Q \neq \emptyset$). This contradiction completes the proof of Claim A. □

We can now prove that (Ia) holds with T'. Under the induction hypothesis on statement (II), we have $y^{r\pi^a}\left(N \setminus T'\right) = \widehat{C}(N \setminus T')$ for all $\pi^a \in \Pi$ order t^*-compatible with T'. On the other hand, if π is t^*-compatible with T', then we can also apply the induction hypothesis on (II) to deduce that $y^{r\pi}\left(N \setminus T'\right) = \widehat{C}(N \setminus T')$. Hence, it is enough to prove that π is indeed t^*-compatible with T'. We check both external and internal t^*-compatibility.

External t^*-compatibility: Let i, j such that $\pi_i \preceq_* \pi_j$ or $\pi_i^* = \pi_j$, $\pi_i \in T'$ and $\pi_j \in N \setminus T'$. We have three cases: If $\pi_j \in S$, then $j < i$ because π is externally t^*-compatible with P. If $\pi_j = p_1$ and $\pi_i \preceq_* p_1$, then $s < i$ because $\pi_i \in T' \subset P$ and $p_1 = \pi_s$ is the first element of P in π; by internal t^*-compatibility with P, we deduce $\pi_i \not\preceq_* \pi_s = \pi_j$, which is a contradiction. If $\pi_j = p_1$ and $\pi_i^* = \pi_j$, then $j < i$ is equivalent to $s < i$, which holds because $\pi_i \in T' \subset S$. Finally, if $\pi_j \in P \setminus (T' \cup \{p_1\})$, then $\pi_j \notin S \cup T' \cup \{p_1\}$; under Claim A, we have $\pi_j^* \notin S \cup T'$; by applying Claim A iteratively, and since $\pi_i \preceq_* \pi_j$, we conclude that $\pi_i \notin S \cup T'$, which is a contradiction because $\pi_i \in T'$.

Internal t^*-compatibility: Let i, j such that $\pi_i \preceq_* \pi_j$ and $\pi_i, \pi_j \in T'$. Since $T' \subset P$, we have $\pi_i, \pi_j \in P$. Hence, $i \leq j$ because π is internally t^*-compatible with P.

We now prove that (Ib) holds with T'. By definition of $y_{p_1}^{r\pi}$, both t^* and t' are optimal trees in $(N, P, y^{r\pi}, c)$. Hence, $C(N) - y^{r\pi}(P) = C(S \cup T') -$

$y^{r\pi}(T')$. Equivalently,

$$y^{r\pi}(P \setminus T') = C(N) - C(S \cup T'). \tag{12}$$

Let $\pi^b \in \Pi$ be an order t^*-compatible with $P \setminus T'$. We check that $y^{r\pi^b}(P \setminus T') = C(N) - C(S \cup T')$. By the induction hypothesis on statement (II), we can assume that nodes in $P \setminus T'$ follow the same order as in π. In particular, p_1 is still the first node in $P \setminus T'$ under π^b. Denote $P \setminus T' = \{p_1, \ldots, p_L\}$ following order π^b, i.e. when $p_i = \pi^b_{i^b}$ and $p_j = \pi^b_{j^b}$, then $i \leq j \Leftrightarrow i^b \leq j^b$. Internal t^*-compatibility assures that $p_i \preceq_* p_j \Rightarrow i \leq j$. For each $l \in \{1, \ldots, L\}$, let $G^l = \{i \in S \cup T' : i^* = p_l\}$ be the set of nodes in $S \cup T'$ that immediately follow p_l in t^* (case $G^l = \emptyset$ is also possible), and denote $G^l = \{g^l_1, \ldots, g^l_{k_l}\}$. We define the set of edges $E^l = \{e^l_1, \ldots, e^l_{k_l}\} \subset N^p$ inductively as follows: Let $e^l_1 = (i_{l1}, i'_{l1}) \in N^p$ be the first edge in the (unique) path in t' from g^l_1 to the source such that $i_{l1} \in F^{*g^l_1}$ and $i'_{l1} \notin F^{*g^l_1}$. Let $e^l_2 = (i_{l2}, i'_{l2}) \in N^p$ be the first edge in the (unique) path in t' from g^l_2 to the source such that $i_{l2} \in F^{*g^l_1} \cup F^{*g^l_2}$ and $i'_{l2} \notin F^{*g^l_1} \cup F^{*g^l_2}$. And so on. Thus, E^l is the set of nodes that connect each G^l to the source in t'. We now define the trees $t^0, t^1, \ldots, t^L$ inductively as $t^0 = t'$ and

$$t^l = \left(t^{l-1} \cup \{(i, i')\}_{i \in G^l \cup \{p_l\}}\right) \setminus E^l$$

for each $l = 1, \ldots, L$. By optimality of t^* and t', each t^l is also optimal in $(N, \{p_{l+1}, \ldots, p_L\}, y^{r\pi}, c)$. Hence, we have

$$\begin{aligned}
y^{r\pi}_{p_L} &= c\left(t^L\right) - c\left(t^{L-1}\right) = C(N) - C(N \setminus \{p_L\}) \\
y^{r\pi}_{p_{L-1}} &= c\left(t^{L-1}\right) - c\left(t^{L-2}\right) = C(N \setminus \{p_L\}) - C(N \setminus \{p_L, p_{L-1}\}) \\
&\vdots \\
y^{r\pi}_{p_1} &= c\left(t^1\right) - c\left(t^0\right) = C(S \cup T' \cup \{p_1\}) - C(S \cup T')
\end{aligned}$$

from where we deduce

$$\begin{aligned}
y^{r\pi^b}_{p_L} &= C(N) - C(N \setminus \{p_L\}) \\
y^{r\pi^b}_{p_{L-1}} &= C(N \setminus \{p_L\}) - C(N \setminus \{p_L, p_{L-1}\}) \\
&\vdots \\
y^{r\pi^b}_{p_1} &= C(S \cup T' \cup \{p_1\}) - C(S \cup T')
\end{aligned}$$

and, adding up these terms, we get

$$y^{r\pi^b}(P \setminus T') = C(N) - C(S \cup T') \tag{13}$$

so that (Ib) comes from (12) and (13).

We now prove statement (II), i.e. $y^{r\pi}(S) = \widehat{C}(S)$. Since $y^{r\pi} \in Core(C) = Core\left(\widehat{C}\right)$, we have $y^{r\pi}(S) \leq \widehat{C}(S)$. When $y^{r\pi}(S) = C(S)$, we have $\widehat{C}(S) \leq y^{r\pi}(S)$ because $\widehat{C}(S) \leq C(S)$. When $y^{r\pi}(S) \neq C(S)$, under statement (I) there exits $\emptyset \neq T' \subset P$ satisfying (Ia) and (Ib). We now apply the induction hypothesis on statement (II) to deduce that for all $\pi^a \in \Pi$ order t^*-compatible with T', and $\pi^b \in \Pi$ order t^*-compatible with $P \setminus T'$, we have $\widehat{C}(N \setminus T') = y^{r\pi^a}(N \setminus T')$ and $\widehat{C}(S \cup T') = y^{r\pi^b}(S \cup T')$. Under (I$a$) and (I$b$), we have $y^{r\pi^a}(T') = y^{r\pi}(T')$ and $y^{r\pi^b}(P \setminus T') = y^{r\pi}(P \setminus T')$, respectively. Hence, $\widehat{C}(N \setminus T') = y^{r\pi}(N \setminus T')$ and $\widehat{C}(S \cup T') = y^{r\pi}(S \cup T')$. Thus,

$$\begin{aligned}
\widehat{C}(S) &= \min\left\{C(S), \min_{\emptyset \neq T'' \subset P}\left\{\widehat{C}(S \cup T'') + \widehat{C}(N \setminus T'') - C(N)\right\}\right\} \\
&\leq \min_{\emptyset \neq T'' \subset P}\left\{\widehat{C}(S \cup T'') + \widehat{C}(N \setminus T'') - C(N)\right\} \\
&\leq \widehat{C}(S \cup T') + \widehat{C}(N \setminus T') - C(N) \\
&= y^{r\pi}(S \cup T') + y^{r\pi}(N \setminus T') - C(N) \\
&= y^{r\pi}(S \cup T') + y^{r\pi}(N) - y^{r\pi}(T') - C(N) \\
&= y^{r\pi}(S).
\end{aligned}$$

concluding the proof. ■

The next proposition will allow us to deduce the main result.

Proposition 3 *For all $c \in \Gamma$, $ExtCore(C) = \{y^{r\pi}\}_{\pi \in \Pi}$.*

Proof. We proceed by contradiction. Assume $ExtCore(C) \neq \{y^{r\pi}\}_{\pi \in \Pi}$. Since each $y^{r\pi}$ is a core allocation, we have that there exists an extreme core allocation x that does not belong to the convex hull of $\{y^{r\pi}\}_{\pi \in \Pi}$. From this, we deduce that there exists some S such that $x(S) - y^{r\pi}(S)$ has the same (nonzero) sign for each $\pi \in \Pi$. That is, either $x(S) - y^{r\pi}(S) > 0$ for all $\pi \in \Pi$, or $x(S) - y^{r\pi}(S) < 0$ for all $\pi \in \Pi$. Notice that S is one of the sets that determine a saturate constraint on a face of the convex hull of $\{y^{r\pi}\}_{\pi \in \Pi}$. Since x does not belong to this convex hull, then it should lay inside the opposite side of the half-space. We can assume $x(S) - y^{r\pi}(S) > 0$ for all $\pi \in \Pi$ because, in the opposite, we instead consider $T = N \setminus S$.

We now take $\pi \in \Pi$ such that $y^{r\pi}(S) = \widehat{C}(S)$. Existence of such a $\pi \in \Pi$ is guaranteed by Proposition 2. Then, $x(S) > \widehat{C}(S)$ and thus, by Lemma 5, x does not belong to $Core\left(\widehat{C}\right) = Core(C)$. Hence the contradiction. ■

As a direct implication of Proposition 3, we obtain our main result: a complete description of the core.

Theorem 1 *For all $c \in \Gamma$, $Core(C)$ is the convex hull of $\{y^{r\pi}\}_{\pi\in\Pi}$.*

We conclude this section by providing the full set of extreme points of the core for our running example.

Example 3 *Consider the mcst problem illustrated in Example 1. The table below shows the 6 distinct extreme points we obtain with our method. $|\pi|$ indicates the number of permutations that yield the given extreme point.*

$\lvert\pi\rvert$	x_1	x_2	x_3	x_4
8	*-3*	*5*	*5*	*5*
6	*1*	*5*	*4*	*2*
3	*-2*	*5*	*4*	*5*
3	*0*	*6*	*4*	*2*
3	*0*	*5*	*5*	*2*
1	*-2*	*6*	*4*	*4*

5 On the generalization of the results to general nonconcave balanced games

We devote this section to discuss how to generalize computation of the core for general nonconcave balanced games. The most relevant issue is the structure that serves as a basis for the solution. In the case of mcst problems, this structure is the (minimal) spanning tree; in case of assignment problems, this structure is the (optimal) matching, and so on. Next table lists these and some other examples with their respective underlying structure:

Class	Structure	Reference
Mcst problems	undirected tree	*(this paper)*
Assignment problems	matching	Núñez and Rafels (2003)
k-Hop problems	k-hop tree	Bergantiños et al. (2012)
Production games	coalition	Trudeau (2009).

Given the respective structure, we define a generalized class of games in which some players become public agents whose collaboration, via Maschler-Davis idea of reduced game, is optional. For mcst problems, these subgames are the mcst problems with priced nodes. In general, this gives rise to new optimal structures (trees, matching,...) that may or may not coincide with the optimal structure(s) of the original game. However, an equivalent condition for $y^{r\pi}$ to belong to the core is that any optimal structure of the original problem should also be optimal in the degenerate reduced game with no players. This is also equivalent to the empty structure being also optimal. For example, in Example 1, when nodes 1, 2, 3 and 4 are priced, respectively, 1, 5, 4, and 2, then no structure (spanning tree) can generate a positive return. On the other hand, several structures (spanning trees) can be built at zero cost, which is also the cost of the empty structure. These structures always include the optimal tree(s) of the original problem. Our results show that the optimal tree(s) of the original problem should be optimal also in each of the intermediate generalized problems that arise in the process. In Example 1, when node 4 is priced 2, then the original optimal tree is also optimal in the new reduced games with players 1, 2 and 3, and so on.

Minimal spanning trees in mcst problems are examples of such structures that satisfy this property. As an example of problems where this property is not satisfied, consider the case of totally balanced 2-hop mcst problems. These are similar to mcst problems but the maximum length of a path from any agent to the source cannot be larger than 2. For example, let $N = \{1,2,3,4\}$, $c_{01} = 3$, $c_{02} = c_{03} = c_{04} = 2$, $c_{12} = c_{13} = c_{14} = 0$, and $c_{ij} = 5$ otherwise. The corresponding cost game of the 2-hop mcst problem is as follows:

S	Optimal tree	$C(S)$	S	Optimal tree	$C(S)$
$\{1\}$	$\{(0,1)\}$	3	$\{2,3\}$	{(0,2),(0,3)}	4
$\{2\}$	{(0,2)}	2	$\{2,4\}$	{(0,2),(0,4)}	4
$\{3\}$	{(0,3)}	2	$\{3,4\}$	{(0,3),(0,4)}	4
$\{4\}$	{(0,3)}	2	$\{1,2,3\}$	{(0,1),(1,2),(1,3)}	3
$\{1,2\}$	{(0,2),(2,1)}	2	$\{1,2,4\}$	{(0,1),(1,2),(1,4)}	3
$\{1,3\}$	{(0,3),(3,1)}	2	$\{1,3,4\}$	{(0,1),(1,3),(1,4)}	3
$\{1,4\}$	{(0,4),(4,1)}	2	$\{2,3,4\}$	{(0,2),(0,3),(0,4)}	6
			N	{(0,1),(1,2),(1,3),(1,4)}	3.

This game is totally balanced. For example, $(0,1,1,1) \in Core(C)$. However, $y^{r[1234]} = (2,0,1,0) \notin Core(C)$.

In this case, the optimal tree $t^* = \{(0,1),(1,2),(1,3),(1,4)\}$ is also optimal when nodes 4 and 3 become priced nodes with prices 0 and 1 respectively. However, after node 2 becomes a priced node with price 0, t^* is no longer an optimal tree, because player 1 can connect through node 3 and the 2-hop restriction precludes node 2 to connect to the source through a path that uses both nodes 1 and 3.

Assume now we are dealing with a class of problems in which the optimal underlying structure is optimal in each of the sequential subgames that follows giving an order π. This will assure us that the resulting vector $y^{r\pi}$ belongs to the core and, in fact, is an extreme core allocation. The next step is to study under which circumstances each extreme core allocation can be obtained under this procedure. This is equivalent to prove that for each $S \subseteq N$, there exists an order π such that $y^{r\pi}(S) = \widehat{C}(S)$. In the case of mcst problems, this order is achieved by requiring that the underlying structure should be both internally and externally compatible with the complementary coalition $P = N \setminus S$. External compatibility states that the members of S should be the last ones to set their prices or, more precisely, no other relevant node should set her price before them. This is a natural requirement because $y^{r\pi}(S) = \widehat{C}(S)$ says that players in S get their most disfavorable allocation in the core. However, external compatibility is not enough. We also need these relevant nodes to set their prices in the most disfavorable order for players in S. Such an order arises when the first relevant nodes to set their prices are those closest to S. Assuring that is the role of internal compatibility.

We check this idea in the mcst problems described in Examples 1 and 2. As seen in Example 2, when $S = \{2\}$, the compatible orders are [2143] and [2134]. External compatibility states that node 2 should be before their relevant nodes, which are nodes 1 and 3, so that these nodes set their respective prices before agent 2. Internal compatibility states that node 1 should be before nodes 3 and 4, but the only relevant requirement is that node 1 goes before node 3. Node 3 should set her price before node 1 because she is closer to S. Assume, on the contrary, that node 3 comes before node 1, as for example in order $\pi^a = [2431]$. With this order, $y^{r\pi^a}(S) = 5 < 6 = \widehat{C}(S)$. Node 3 cannot extract all the surplus from coalition S because she has been previously limited by node 1's price. Internal compatibility is a sufficient condition[5] for node 3 to set her price before node 1.

[5]It is not a necessary condition though. Order $\pi^b = [2413]$ is not internally compatible with $N \setminus S$, yet it induces $y^{r\pi^b}(S) = 6 = \widehat{C}(S)$. Notice that node 4 is not relevant for S because it belongs to a different branch in t^*.

We see now an example of a game where this compatibility is not possible.

Trudeau (2009) considers the class of production games, where agents each need one unit of a common good and each coalition of agents owns a technology that allows their members to produce any amount of good at a constant marginal cost. There are no cost to transport goods from one agent to another. We consider a variant that we call the *public good production game*, in which all coalitions produce (the same amount of) a public good, with varying technologies. A public good production game is characterized by a vector $C \in \mathbb{R}_{+}^{2^N}$ such that $C(\emptyset) = 0$ and $C(S) \geq C(T)$ for all $\emptyset \neq S \subset T \subseteq N$. Let Γ^{PP} be the set of all such public good production games. Let C^n be such that for all $\emptyset \neq S \subseteq N \setminus \{n\}$, $C^n(S) = \min\{C(S), C(S \cup \{n\}) + C(N \setminus \{n\}) - C(N)\}$. For $y \in \mathbb{R}^N$, let $y_{-i} \in \mathbb{R}^{N \setminus \{i\}}$ be the same vector without the coordinate in position i.

In this case, the underlying structure is the optimal coalition.

We first show that the reduced game vectors do provide extreme core allocations. We need the following lemma:

Lemma 7 *(Núñez and Rafels, 2003) Let* $\pi = [1 \ldots n]$*. If* $y_{-n}^{r\pi} \in ExtCore(C^n)$ *and* $y_n^{r\pi} \leq C(\{n\})$*, then* $y^{r\pi} \in ExtCore(C)$*.*

In the next proposition, we keep notation $y^{r\pi}$ to name the reduced marginal cost vector of $C \in \Gamma^{PP}$ related to permutation π.

Proposition 4 *For all* $\pi \in \Pi$ *and all* $C \in \Gamma^{PP}$*,* $y^{r\pi} \in ExtCore(C)$*.*

Proof. The result is trivially true for $n = 1$. Assume $n > 1$ and the result holds for less than n agents. W.l.o.g., take $\pi = [1 \ldots n]$. Notice first that $y_n^{r\pi} = C(N) - C(N \setminus \{n\}) \leq 0 \leq C(\{n\})$. We next show that $C^n \in \Gamma^{PP}$. For $\emptyset \neq S \subseteq N \setminus \{n\}$ we have

$$\begin{aligned} C^n(S) &= \min\{C(S), C(S \cup \{n\}) + C(N \setminus \{n\}) - C(N)\} \\ &= \min\{C(S), C(S \cup \{n\}) - y_n^{r\pi}\}. \end{aligned}$$

If $C^n(S) = C(S)$, then $C^n(S) = C(S) \geq C(T) \geq C^n(T)$ for all $T \supset S$. If $C^n(S) = C(S \cup \{n\}) + C(N \setminus \{n\}) - C(N)$, then $C^n(S) = C(S \cup \{n\}) + C(N \setminus \{n\}) - C(N) \geq C(T \cup \{n\}) + C(N \setminus \{n\}) - C(N) \geq C^n(T)$ for all $T \supset S$. Also, since $y_n^{r\pi} \leq 0$, $C^n(S) \geq 0$ for all $\emptyset \neq S \subseteq N \setminus \{n\}$. Finally, we have that $C^n(\emptyset) = 0$ by definition. Thus, $C^n \in \Gamma^{PP}$. By induction hypothesis, $y^{r\pi^n}(C^n) \in ExtCore(C^n)$, where $\pi^n = [1 \ldots n-1]$.

Hence, under Lemma 7, it is enough to prove that $y^{r\pi^n}(C^n) = y^{r\pi}_{-n}(C)$. Given $i \in N \setminus \{n\}$, we need to prove that $y^{r\pi^n}(C^n)_i = y^{r\pi}_{-n}(C)_i$. We proceed by backward induction on i. For $i = n-1$,

$$\begin{aligned}
y^{r\pi^n}(C^n)_{n-1} &= C^n(N \setminus \{n\}) - C^n(N \setminus \{n-1, n\}) \\
&= \min\{C(N \setminus \{n\}), C(N) - y^{r\pi}_n\} \\
&- \min\{C(N \setminus \{n-1, n\}), C(N \setminus \{n-1\}) - y^{r\pi}_n\} \\
&= C(N \setminus \{n\}, \{n\}, y^{r\pi}) - C(N \setminus \{n-1, n\}, \{n\}, y^{r\pi}) \\
&= y^{r\pi}(C)_{n-1} = y^{r\pi}_{-n}(C)_{n-1}.
\end{aligned}$$

Assume now $i < n-1$ and $y^{r\pi^n}(C^n)_j = y^{r\pi^n}_{-n}(C)_j$ for all $j > i$. Then,

$$\begin{aligned}
y^{r\pi^n}(C^n)_i &= C^n(\{1, \ldots, i\}, \{i+1, \ldots, n-1\}, y^{r\pi}) \\
&- C^n(\{1, \ldots, i-1\}, \{i+1, \ldots, n-1\}, y^{r\pi}) \\
&= \min_{T \subseteq \{i+1, \ldots, n-1\}} \{C^n(\{1, \ldots, i\} \cup T) - y^{r\pi}(T)\} \\
&- \min_{T \subseteq \{i+1, \ldots, n-1\}} \{C^n(\{1, \ldots, i-1\} \cup T) - y^{r\pi}(T)\} \\
&= C(\{1, \ldots, i\}, \{i+1, \ldots, n\}, y^{r\pi}) \\
&- C(\{1, \ldots, i-1\}, \{i+1, \ldots, n\}, y^{r\pi}) \\
&= y^{r\pi}(C)_i = y^{r\pi}_{-n}(C)_i.
\end{aligned}$$

■

We can also see the result in terms of the optimal structure in the degenerate reduced game with no players. The optimal structure is to use the technology of the grand coalition. The empty coalition can always choose the empty structure at a cost of zero. Since any allocation divides the total cost among the agents, using the technology of the grand coalition and compensating all agents can also be done at zero cost. Since costs in public good production games are decreasing, only the marginal contribution when added to the empty set can be strictly positive. Hence, the reduced game method can only assign a strictly positive share to π_1. Using the technology of any coalition that does not include π_1 cannot be cheaper than zero. By definition, using the technology of π_1 (and compensating her) has a cost of zero. Using the technology of any other coalition will also generate a nonnegative cost.

We next show that in that case we do not obtain all extreme core allocations. Suppose a public good production game such that $N = \{1, 2, 3, 4, 5\}$

and C as follows:

$$C(S)=\begin{cases} 0 \text{ if } S=\emptyset \text{ or } \{3,4,5\}\subseteq S \\ 1 \text{ if } 5\in S \text{ and } \{3,4\}\not\subset S \\ 2 \text{ if } 5\notin S \text{ and } \{3,4\}\cap S\neq\emptyset \\ 3 \text{ if } S\neq\emptyset \text{ and } \{3,4,5\}\cap S=\emptyset. \end{cases}$$

It is clear that agents 1 and 2 are symmetric, as well as agents 3 and 4. The reduced game method then generates the following extreme points with permutations of allocations of agents 1 and 2, and 3 and 4, given their symmetry: $(0,0,2,0,-2)$, $(0,0,-1,2,-1)$, $(0,0,-1,0,1)$, $(2,0,0,0,-2)$ and $(2,0,-1,0,-1)$. However, the allocation $(1.25,1.25,-0.5,-0.5,-1.5)$ is in the core and cannot be obtained as a weighted average of other extreme points. In particular, agents 1 and 2 pay a combined share of 2.5, while they pay at most 2 in all extreme points generated by the reduced game method.

In this case, there exists no equivalence to an order compatible with coalition $\{3,4,5\}$. The equivalence to external compatibility would imply that agents $1,2$ are the first in the order (they are the last to price themselves). But agents 3, 4 and 5 are all relevant for $1,2$, as its maximal core allocation depends on each of its stand-alone costs with agents 3, 4 and 5 individually. Given how these constraints interact with each other, it is impossible to obtain the maximal core allocation of $1,2$ by sequentially fixing prices for agents 3, 4 and 5. We thus have no equivalence to internal compatibility in this class of games.

The next table summarizes these results:

Class	$y^{r\pi}\in Core(C)$	$Core(C)=<\{y^{r\pi}\}>$
Mcst problems	yes	yes
Assignment problems	yes	yes
k-Hop problems	NO	-
Public good production games	yes	NO.

6 Links with the core of the public game and with known cost sharing solutions

In this section, we show that our method allows to bridge many gaps in the mcst literature.

First, we consider a variant of the mcst problem, called the public mcst problem, where a coalition can use all nodes, including those belonging to agents outside of the coalition, to connect to the source. We show that the core of the public mcst problem is easily obtained from the result of the previous section.

Second, we can define a new cost-sharing solution as the permutation-weighted average of extreme core allocations. We show that it has close links with three of the most famous cost sharing solutions in the literature on mcst.

6.1 The core of the public mcst problem

The public mcst problem allows a coalition S to connect to the source using the nodes of agents in $N \backslash S$. We thus obtain the following characteristic cost function. For all $S \subseteq N$, we have

$$\overline{C}(S,c) = \min_{T \subseteq N \backslash S} C(S \cup T, c).$$

While most of the literature uses the private version of the game, the distinction between the two games has been noted in Bogomolnaia and Moulin (2010) and Trudeau (2013). We obtain the following result.

Theorem 2 *For all $c \in \Gamma$, $Core\left(\overline{C}\right) = Core(C) \cap \mathbb{R}^N_+$.*

Proof. We first show that $Core\left(\overline{C}\right) \subseteq Core(C) \cap \mathbb{R}^N_+$. Let $x \in Core\left(\overline{C}\right)$. Then

$$x(S) \leq \min_{T \subseteq N \backslash S} C\left(S \cup T, c\right) \leq C(S,c)$$

for all $S \subseteq N$, so $x \in Core(C)$. It remains to show that $x \in \mathbb{R}^N_+$. Suppose that $x_i < 0$ for some $i \in N$. Since $\overline{C}$ is monotonically increasing, we have that $\overline{C}\left(N \setminus \{i\}\right) \leq \overline{C}(N)$ and thus $x(N \setminus \{i\}) > \overline{C}(N \setminus \{i\}, c)$, contradicting the assumption that $x \in Core\left(\overline{C}\right)$.

We next show that $Core(C) \cap \mathbb{R}^N_+ \subseteq Core\left(\overline{C}\right)$. Let $x \in Core(C) \cap \mathbb{R}^N_+$. Since $x \in Core(C)$, we have, for any $S \subset N$ and $T \subseteq N \setminus S$, that

$$x(S \cup T) \leq C(S \cup T)$$

which, with budget balance, implies that

$$x(N \setminus S) - x(T) \geq C(N) - C(S \cup T).$$

This inequality can be rewritten as

$$x(N \setminus S) \geq x(T) + C(N) - C(S \cup T) \geq C(N) - C(S \cup T)$$

where the last inequality comes from the fact that $x \in \mathbb{R}_{+}^{N}$. Under budget balance, $x(S) \leq C(S \cup T)$. Since this inequality was obtained for an arbitrary $T \subseteq N \setminus S$, we obtain $x(S) \leq \min_{T \subseteq N \setminus S} C(S \cup T)$, which is the condition for $Core\left(\overline{C}\right)$. ■

Thus, restricting $Core(C)$ to its elements in $\mathbb{R}_{+}^{N}$ yields $Core\left(\overline{C}\right)$. Therefore, once we have the extreme points of $Core(C)$, we have a trivial way to obtain the extreme points of $Core\left(\overline{C}\right)$.

Example 4 *Consider the mcst problem illustrated in Example 1. The table below shows the four distinct extreme points we obtain for* $Core\left(\overline{C}\right)$.

x_1	x_2	x_3	x_4
1	5	4	2
0	6	4	2
0	5	5	2
0	5	4	3.

Notice that the first three extreme allocations are the extreme allocations of Example 3 that are in $\mathbb{R}_{+}^{N}$. *The fourth allocation is on the line between the extreme core allocations (of C)* $(-2, 5, 4, 5)$ *and* $(1, 5, 4, 2)$, *at the intersection with* $\mathbb{R}_{+}^{N}$.

We complete this subsection by showing how to extend the definition of $y^{r\pi}$ to the public mcst game.

A *public mcst problem with priced nodes* is a tupla (N, P, y, c) where $P \subseteq N$ are nodes that do not need to be connected and $y \in \mathbb{R}_{+}^{P}$ is the vector whose coordinates are the prices that nodes in P pay to agents in $N \setminus P$ if they are actually connected. In the public mcst problem with priced nodes, a coalition $S \subseteq N \setminus P$ can use the nodes of agents in $N \setminus (S \cup P)$ to connect without paying anything, while it collects the payoff in y if it connects through the nodes of agents in P. Hence, the cost of (N, P, y, c) is defined as

$$\overline{C}(N, P, y, c) = \min_{T \subseteq P} \{C((N \setminus P) \cup T) - y(T)\}$$

and the cost of a subset $S \subseteq N \setminus P$ is given by

$$\overline{C}(S, P, y, c) = \min_{T \subseteq N \setminus S} \{C(S \cup T) - y(T \cap P)\}.$$

In particular, $\overline{C}(N, P, y, c) = \overline{C}(N \setminus P, P, y, c)$. We define the reduced marginal cost vector of the public game related to permutation π, denoted as $\overline{y}^{r\pi}(c)$, or simply $\overline{y}^{r\pi}$, in a recursive manner, starting with $\overline{y}^{r\pi}_{\pi_n}$ and making our way down to $\overline{y}^{r\pi}_{\pi_1}$:

$$\overline{y}^{r\pi}_{\pi_i} = \overline{C}\left(\{\pi_1 \ldots \pi_i\}, P^{\pi i+1}, \overline{y}^{r\pi}, c\right) - \overline{C}\left(\{\pi_1 \ldots \pi_{i-1}\}, P^{\pi i+1}, \overline{y}^{r\pi}, c\right).$$

Notice that, since $\overline{C}$ is a monotonically increasing function, $\overline{y}^{r\pi}_{\pi_n} \geq 0$. By the definition of the public mcst problem with priced nodes, $\overline{y}^{r\pi}_{\pi_n} \geq 0$ guarantees that $\overline{y}^{r\pi}_{\pi_{n-1}}(c) \geq 0$. Applying the argument successively, we obtain that $\overline{y}^{r\pi} \in \mathbb{R}^N_+$.

Proposition 5 *For all $c \in \Gamma$, $ExtCore\left(\overline{C}\right) = \{\overline{y}^{r\pi}\}_{\pi \in \Pi}$.*

The proof is identical to the proof of Proposition 3 and is omitted.

6.2 Links with well-known cost-sharing solutions

A cost sharing solution (or rule) assigns a cost allocation $y(c)$ to any admissible cost matrix c. We start by building a solution from the allocations defined in the previous sections. This solution is the selective value (Vidal-Puga, 2004) defined as the average of the reduced marginal cost vectors, i.e.

$$y^s(c) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^{r\pi}(c).$$

By our previous section, $y^s(c)$ is the permutation-weighted average of the extreme allocations of $Core(C)$. It is also corresponds to the Alexia value, as the permutation-weighted average of leximals (Tijs, 2005; Funaki et al., 2007; Kongo et al., 2010; Tijs et al., 2011). We define $\overline{y}^s(c)$ analogously.

6.2.1 The Bird solution

The Bird solution is defined as follows. Let $\Pi^*(c)$ be the set of orders obtained in Prim's algorithm. For all $\pi \in \Pi^*(c)$, let $y^{b\pi}(c)_{\pi_i} = \min_{k=0,\ldots,i-1} c_{\pi_k \pi_i}$ where $\pi_0 = 0$. The Bird solution is

$$y^b(c) = \frac{1}{|\Pi^*(c)|} \sum_{\pi \in \Pi^*(c)} y^{b\pi}(c).$$

Proposition 6 *For all $\pi \in \Pi^*(c)$, $y^{b\pi}(c) = y^{r\pi}(c)$.*

Proof. Observe that for all $\pi \in \Pi^*(c)$ and all $S \in N \setminus \ P^{\pi i+1}$, we have

$$C\left(S, P^{\pi i+1}, y^{r\pi}, c\right) = C(S, c).$$

Therefore, for all $\pi \in \Pi^*(c)$ and all $i = 1 \ldots, n$, we have

$$\begin{aligned} y^{r\pi}_{\pi_k}(c) &= C(\{\pi_1, \ldots, \pi_j\}, c) - C(\{\pi_1, \ldots, \pi_{j-1}\}, c) \\ &= \min_{k=0,\ldots,j-1} c_{\pi_k \pi_j} \\ &= y^{b\pi}_{\pi_j}(c). \end{aligned}$$

Hence, $y^{b\pi}(c) = y^{r\pi}(c)$. ■

The Bird allocation is thus a very special case of our method, as it only uses the reduced marginal cost vector of C related to permutation π if π is a permutation corresponding to an order in which we construct the mcst using Prim's algorithm.

We thus obtain an alternative proof of the stability of the Bird allocation. In addition, we can see that the permutations used for the Bird allocation are such that there are no modifications to do on C before computing the marginal cost vector. For other permutations, we usually obtain that the corresponding marginal cost vector (without modifying C) is not in the core.

Notice that if $|\Pi^*(c)| = 1$ the Bird allocation consists of a single extreme allocation of the core. In terms of fairness, this is hard to justify, providing additional ammunition to the criticisms of Dutta and Kar (2004) and Bergantiños and Vidal-Puga (2007).

6.2.2 The cycle-complete solution

To define the cycle-complete solution, we need to define the irreducible and cycle-complete cost matrices.

Given $l, m \in N_0$, a path ψ_{lm} between l and m is a set of K edges (i_k, i_{k+1}), with $k \in \{0, \ldots, K-1\}$ such that $i_0 = l$ and $i_K = m$. Let $\Psi_{lm}(N_0)$ be the set of all such paths between nodes l and m.

From any cost matrix c, we can define the irreducible cost matrix c^* as follows:

$$c^*_{ij} = \min_{\psi_{ij} \in \Psi_{ij}(N_0)} \max_{e \in \psi_{ij}} c_e \text{ for all } i, j \in N_0.$$

From any cost matrix c, we can define the cycle-complete cost matrix c^{**} as follows:

$$\begin{aligned} c^{**}_{ij} &= \max_{k \in N \setminus \{i,j\}} \left(c^{N \setminus \{k\}}\right)^{*}_{ij} \text{ for all } i,j \in N \\ c^{**}_{0i} &= \max_{k \in N \setminus \{i\}} \left(c^{N \setminus \{k\}}\right)^{*}_{0i} \text{ for all } i \in N \end{aligned}$$

where $\left(c^{N \setminus \{k\}}\right)^{*}$ indicates the matrix that we first restricted to agents in $N \setminus \{k\}$ before transforming into an irreducible matrix.

The cycle complete matrix can also be defined using cycles (Trudeau, 2012): for edge (i,j), we look at cycles that go through i and j. If there is one such cycle such that its most expensive edge is cheaper than a direct connection on edge (i,j), we assign this cost to edge (i,j).

Let C^{**} be the characteristic cost function associated with the mcst problem (N, c^{**}). The cycle-complete solution $y^{cc}(c)$ is the Shapley value of C^{**}.

Let Γ^e be the set of elementary cost matrices: for any $c \in \Gamma^e$ and any $i,j \in N_0$, $c_{ij} \in \{0,1\}$. We show that for elementary cost matrices, the cycle-complete solution corresponds to the selective value.

Proposition 7 *For any elementary cost matrix* $c \in \Gamma^e$, $y^s(c) = y^{cc}(c)$.

Proof. Trudeau (2012) showed that for elementary problems, $Core(C) = Core\left(C^{**}\right)$ and that (N, C^{**}) is concave. By the properties of the Shapley value for concave games, y^{cc} is the average over the set of permutations of incremental cost allocations, with each of them being an extreme core allocation. It is obvious that the incremental cost vector corresponding to order π is exactly $y^{r\pi}$. We thus have that $y^s(c) = y^{cc}(c)$. ■

An alternative explanation of the above result is that the changes from the original to the cycle-complete cost matrix are the same as those imposed by our method. If node i has two distinct free paths to node j, say with the help of S and T, she will obtain the cost savings with both coalitions. This will result in $\widehat{C}\left(\{i,j\}\right) = 0$, the same result as if we modified directly the matrix into a cycle-complete matrix.

6.2.3 The folk solution

The folk solution, labelled $y^f(c)$, is the Shapley value of C^*, the cost game associated with the irreducible cost matrix c^* defined in the previous subsection. As for the cycle-complete solution, we show a link between our method and the folk solution in elementary mcst problems.

Proposition 8 *For any elementary cost matrix* $c \in \Gamma^e$, $\overline{y}^s(c) = y^f(c)$.

Proof. Consider a connected component $T \subseteq N_0$. If $0 \in T$, then $C(T \backslash \{0\}) = 0$. Otherwise, $C(T \setminus \{0\}) = 1$. In such a case, c^* is such that $c^*_{ij} = 0$ for all $i, j \in T$. By the properties of the public mcst problem, we must have that $\overline{C}(S \setminus \{0\}) = \overline{C}(T \setminus \{0\})$ for all $S \subseteq T$. Therefore, in both cases, the changes are identical.

We next show that there are no other changes. If i, j belong to different connected components, say T_1 and T_2, then $c^*_{ij} = c_{ij}$. We also have that for all $R \subseteq T_1$ and $S \subseteq T_2$, $\overline{C}(R \cup S) = \overline{C}(R) + \overline{C}(S)$. Therefore, we do not need to make any other changes.

After the modifications, to obtain $y^f(c)$ we take the Shapley value of (N, C^*), the cost game associated with c^*. Given that (N, C^*) is concave (Bergantiños and Vidal-Puga, 2007), it is immediate that $\overline{y}^s(c) = y^f(c)$. ■

We thus obtain the new result that for elementary mcst problems, the folk solution is the permutation-weighted average of the extreme core allocations of $Core\left(\overline{C}\right)$.

Therefore, we obtain, for elementary cost matrices, a clear distinction between the folk and cycle-complete solutions. They both are selective values: the former for the core of the public mcst problem and the latter for the core of the (private) mcst problem.

In Trudeau (2013), it was already noted that the folk solution was axiomatized by properties that fit better with the interpretation of the public game. Our result clearly shows why.

We conclude this section with a computation of the various solutions for our running example.

Example 5 *Consider the mcst problem illustrated in Example 1. The table below shows the shares for the various cost sharing solutions discussed in this section.*

	$y^s(c)$	$y^{cc}(c)$	$\overline{y}^s(c)$	$y^f(c)$
y_1	-1.08	-1.00	0.25	1.00
y_2	5.17	5.00	5.25	5.00
y_3	4.46	4.50	4.25	4.00
y_4	3.46	3.50	2.25	2.00.

7 Discussion

The result of the previous section on the folk and cycle-complete solutions does not hold for non-elementary cost matrices. For those, we can compute the folk and cycle-complete solutions by decomposing the cost matrix into a series of elementary cost matrices and summing up. While that approach is computationally advantageous, one of the disadvantage is that, in general, we have that $Core\,(C^{**})$ is a strict subset of $Core(C)$ and $Core\,(C^{*})$ is a strict subset of $Core\,\left(\overline{C}\right)$ (which is itself also a strict subset of $Core(C)$); the cycle-complete and folk solutions are no longer the permutation-weighted averages of the extreme allocations of, respectively, the core of the private mcst game and the core of the public mcst game.

If we are willing to forego the piecewise linearity property, we can use $y^s(c)$ and $\overline{y}^s(c)$ as non-piecewise linear extensions of, respectively, the cycle-complete and folk solutions.

Given that the reduced game approach allows us to find the full sets of extreme points of the core just as for the assignment problem (Núñez and Rafels, 2003), it is worth exploring if the two problems share more characteristics. In particular, the assignment problem satisfies the CoMa property, meaning that all of the extreme points of its core are marginal vectors (Hamers et al., 2002). This is not true for mcst problems. An example is provided in appendix.

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A Appendix - Counter-examples for some properties

In this section we show that known results for similar methods cannot be applied for mcst problems. We use the following example:

Example 6 *Let* $N = \{1, 2, 3, 4\}$ *and* c *be as described in the following (i horizontally, j vertically) and illustrated in Figure 3:*

c_{ij}	*1*	*2*	*3*	*4*
0	*5*	*7*	*8*	*9*
1		*7*	*7*	*5*
2			*4*	*2*
3				*5*

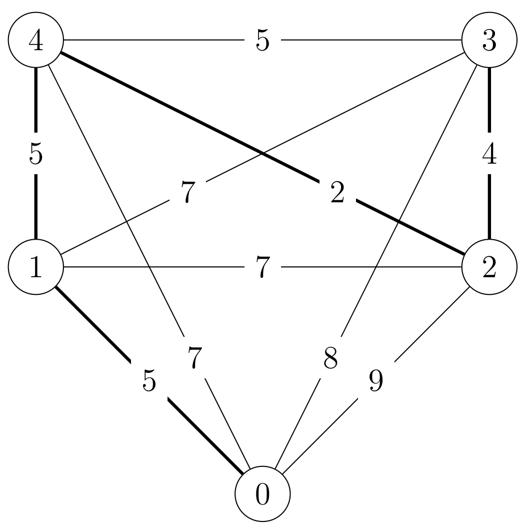

Figure 3: Example of a minimal cost spanning tree problem.

The unique mcst uses edges $(0, 1), (1, 4), (2, 4)$ *and* $(2, 3)$*, for a total cost of 16.*

Our method provides 10 distinct extreme points of the core: $(3, 1, 5, 7)$, $(3, 1, 8, 4)$, $(3, 2, 4, 7)$, $(3, 3, 8, 2)$, $(3, 7, 4, 2)$, $(4, 1, 8, 3)$, $(5, 1, 7, 3)$, $(5, 2, 4, 5)$, $(5, 4, 7, 0)$ *and* $(5, 7, 4, 0)$.

A.1 Almost-concavity

Núñez and Rafels (1998) prove that almost-concavity is a sufficient condition for the set $\{y^{r\pi}\}_{\pi\in\Pi}$ to be the set of extreme points of the core. Almost-concavity is defined as follows: $C(S) + C(T) \geq C(S \cup T) + C(S \cap T)$ for all $S, T \subset N$ such that $S \cup T \neq N$.

In our example, it is easy to see that $C(\{3\}) = 8, C(\{1,3\}) = 12, C(\{2,3\}) = 11$ and $C(\{1,2,3\}) = 16$. We thus have that $C(\{1,3\}) + C(\{2,3\}) < C(\{1,2,3\}) + C(\{3\})$, which contradicts the almost-concavity condition.

A.2 Concavity of the corresponding exact game

It is known that for each balanced game there exists a unique exact game with the same core as the original game. By construction, in our case, that exact game is $\widehat{C}$. As shown by Funaki et al. (2007), a sufficient condition for the lexicore to be equal to the core is for that exact game to be concave. The condition for concavity is $C(S) + C(T) \geq C(S \cup T) + C(S \cap T)$ for all $S, T \subset N$.

We go back to our example. It is easy to see that $\widehat{C}(\{3\}) = 8, \widehat{C}(\{1,3\}) = 12, \widehat{C}(\{2,3\}) = 11$ and $\widehat{C}(\{1,2,3\}) = 16$. We thus have that $\widehat{C}(\{1,3\}) + \widehat{C}(\{2,3\}) < \widehat{C}(\{1,2,3\}) + \widehat{C}(\{3\})$, which contradicts the concavity condition.

A.3 The CoMa property

The marginal vector of C with respect to order π is defined as follows. For all $i = 1, \ldots, n$, $m^{\pi}_{\pi_i} = C(\{\pi_1, \ldots, \pi_i\}) - C(\{\pi_1, \ldots, \pi_{i-1}\})$.

A game C has the CoMa property if $ExtCore(C) \subseteq \{m^{\pi}\}_{\pi\in\Pi}$. i.e. all extreme core allocations are also marginal vectors.

Núñez and Rafels (2003) show that the assignment problem, which shares many characteristics of the mcst problem, has the CoMa property. We show that it is not the case for mcst problems.

In our example, the marginal vectors are $(1,1,5,9)$, $(1,2,4,9)$, $(2,1,4,9)$, $(2,1,8,5)$, $(3,0,4,9)$, $(3,0,8,5)$, $(3,3,8,2$, $(3,7,4,2)$, $(4,1,8,3)$, $(4,4,8,0)$, $(5,1,7,3)$, $(5,2,4,5)$, $(5,3,3,5)$, $(5,3,8,0)$, $(5,4,7,0)$ and $(5,7,4,0)$.

We thus have that $(3,1,5,7)$, $(3,1,8,4)$ and $(3,2,4,7)$ are extreme points of the core but not marginal vectors.

A.4 The public game

It is easy to see that in our example the public and the private games are the same, i.e. $C(S,c) = \overline{C}(S,c)$ for all $S \subseteq N$. Thus, all examples in this section carry over to the public game.