Duality in land rental problems*

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Abstract

Assume there is a single tenant who needs to rent a particular amount of land from several lessors. Moreover, lessors own more than this particular amount. The way lessors rent their land can be seen as equivalent to a bankruptcy problem where the estate is the required land and each claim is the amount of land available from each lessor. In this paper, we explore the idea of self-duality in bankruptcy problems to these land rental situations. Roughly speaking, self-duality says that it is equivalent to share the rented land than sharing the land that is not rented. We provide a complete characterization of the family of rules that satisfy this self-duality. In particular, self-duality is enough to assure that the amount of land is shared under the proportional rule, as opposed to bankruptcy problems, where other properties are also required. We also provide the characterization of a unique rule satisfying self-duality and other reasonable properties.

Keywords: *Self-duality, land rental problems, bankruptcy problems, proportional rule*

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1 Introduction

Duality is an important concept in mathematics which usually implies the involution operation, i.e. given x and its dual y, then the dual of y is x. Even though duality has different interpretations depending on the area, we focus on the interpretation in bankruptcy problems (O'Neill, 1982), which is consequently extended to a land rental problem context.

A land rental problem involves several lessors, and a single tenant, who needs to rent land, but less than all land that is owned by the lessors. The main question is knowing how much land each lessor should rent and the price per unit of land (Valencia-Toledo and Vidal-Puga, 2015).

In order to set the concept of self-duality in land rental problems, let us introduce bankruptcy problems as follows: these are situations where a group of agents (N) have some claims ($c \in \mathbb{R}_+^N$) that sum up more than the particular amount of land demanded by the tenant (E), which is a perfectly divisible estate to share among them, i.e. $\sum_{i \in N} c_i > E$. A bankruptcy rule is a way to share the total estate among the claimants. For each claimant, her share can be seen either as an award of part of her claim or as a loss of the other part.

The self-duality principle in bankruptcy problems was firstly studied by Aumann and Maschler (1985). It states that the criterion used to grant awards should coincide with the one used to grant losses. Many important rules in the context of bankruptcy problems satisfy self-duality, for example, the proportional rule (Young, 1988; Herrero and Villar, 2001, 2002; Herrero, 2003) the Talmut rule (Chun, 1988), the Contested Garment principle (Dagan, 1996), and the reverse Talmud rule (van den Brink et al., 2013). For a survey, we refer to Thomson (2003, 2015) and Moulin (2002).

We extend the idea of self-duality to land rental problems. The lessors can be seen as claimants in a bankruptcy problem. Lessors' claims are the land they own, and the estate is the amount of land that the tenant needs. Moreover, land renting problems have other elements to take into account, as for example the fact that the tenant is also an agent. There is a revenue, or reservation price, that every lessor obtains by each unit of non-rented land, which is denoted by r. As opposed, the revenue that the tenant can get by each unit of rented land is denoted by R. We always assume that there exists benefit of cooperation, i.e. $R > r$.

We apply the concept of self-duality to the land share. We can think in terms of how to share the amount of land that is not rented instead of the amount of rented land, since they are equivalent problems.

In order to study self-duality, let us define the dual problem of a land rental problem as follows. The agents and the amount of available land remain unchanged. Meanwhile, the dual of the amount of land that the tenant needs is given by $E^d = c(N) - E$. The dual of the reservation price per unit of land (denoted by r^d) satisfies $r^d E^d = rE$. This is done so that the revenue for the lessors are equal in both the original problem and its dual problem. Also, the dual of the revenue per unit of land for the tenant (denoted by R^d) satisfies $R^d E^d = RE$. This means that the revenue for the tenant are equal in both the original problem and its dual problem.

We provide the complete characterization of the family of rules that satisfy self-duality. Under these rules, the amount of land is shared by the proportional rule. As opposed, in bankruptcy problems, it is necessary at least to consider some additional properties apart from self-duality.

In order to characterize a unique rule as a main result, we introduce two reasonable properties of the literature.

Price independence property says that the price does not depend on the distribution among the lessors of the available land. This property is related to the idea of land reassignment. If it does not hold, lessors could redistribute their land in order to increase the price and end up better off than they were initially. The property assures that even if they redistribute their land, so that the aggregate land does not change with respect to the initial land rental problem, the price does not change either. Consequently, the tenant and lessors generate and define a new land rental problem in which at least a pair of lessors changes her available land. Then, a proposal is said to be price independence if they obtain the same price in both land rental problems.

Standard for two-person property was introduced by Hart and Mas-Colell (1989). The idea is that whether there are two agents, the total benefit is shared as follows: the corresponding to each one plus half of the benefit of cooperation that they generate. We make an extension and study this principle in land rental problems. This means that whenever there are a tenant and a unique lessor, they share equally the final benefit of cooperation.

Properties that provide conditions on how the share should be for the 2-person case have played an important role in the axiomatization of many solutions in different contexts. For example, in cooperative games (Hart and Mas-Colell, 1989; Huettner, 2015), in bankruptcy problems (Thomson, 2008), and in land renting problems (Valencia-Toledo and Vidal-Puga, 2015).

The remainder of this paper is organized as follows. In Section 2, we present the model. In Section 3, we define self-duality and we provide a complete characterization of the family of rules that satisfy self-duality. Also, we present charaterizations of a subfamily of rules, adding price independence, and a unique rule, adding standard for two. In Section 4, we give some concluding remarks.

2 Notation and model

We denote the set of nonnegative real numbers as $\mathbb{R}_+$. We denote the set of positive real numbers as $\mathbb{R}_{++}$. Let $\mathbb{N}_+ = \{1, 2, \dots\}$ be the set of potential lessors and let $\mathcal{N}$ be the set of non-empty finite subsets of $\mathbb{N}_+$. Let $N = \{1, 2, \dots, n\} \in \mathcal{N}$ be a generic set of lessors, and let S be a generic subset of N. Let $\mathbb{R}^S$ be the $|S|$-dimensional Euclidean space, whose coordinates are indexed by the elements of S. Given $y \in \mathbb{R}^S$, we write $y(S) = \sum_{i \in S} y_i$. Given $x, y \in \mathbb{R}^S$, we write $x \leq y$ when $x_i \leq y_i$ for all $i \in S$. Given $x \in \mathbb{R}^N$ and $S \subset N$, we denote the restriction of x on S as x_S i.e. $x_S = (x_i)_{i \in S}$. Moreover, the vectors $(0, \dots, 0) \in \mathbb{R}^S$ and $(1, \dots, 1) \in \mathbb{R}^S$ are denoted by 0_S and 1_S, respectively.

A *land rental problem* is a tuple $L = (N_0, c, r, K, E)$ where $N_0 = \{0\} \cup N$ is the set of agents with 0 the unique tenant and N the set of lessors, $c \in \mathbb{R}^N_{++}$ is the vector whose coordinates represent the amount of available land for each lessor, $r \in \mathbb{R}_{++}$ is the reservation price per unit of land for the lessors, $E \in]0, c(N)[$ is the amount of land that the tenant needs to rent, and $K \in \mathbb{R}_{++}$ is the revenue that the tenant can obtain from that amount of rented land. In order to assure that there exists benefit of cooperation, we assume $K > rE$. We define $R = \frac{K}{E}$ as the revenue per unit of land that the tenant gets with her activity. For convenience, we denote (N, c, r, R, E) instead of (N_0, c, r, K, E). Let $\mathcal{L}$ be the set of all land rental problems.

An efficient feasible agreement is a pair $(p, x) \in \mathbb{R}_+ \times \mathbb{R}^N_+$ where p denotes the price per unit of land, and $x \leq c$ and $x(N) = E$, so that x_i denotes the land rented by lessor $i \in N$. We denote A^L as the set of feasible agreements on a land rental problem L. Let $\mathscr{A} = \bigcup_{L \in \mathcal{L}} A^L$ be the set of all potential feasible agreements.

Given $(p, x) \in A^L$, the utility for tenant and each lessor $i \in N$ are $u^L_0(p, x) = (R - p)E$ and $u^L_i(p, x) = (p - r)x_i$, respectively. Notice that with a slight notation abuse and for simplicity, we write $u^L_j(p, x)$ instead of $u^L_j((p, x))$ for all $j \in N_0$.

We define a *rule* as a function $\psi : \mathcal{L} \longrightarrow \mathscr{A}$ that assigns to each problem $L = (N, c, r, R, E) \in \mathcal{L}$ a pair $\left(p^{\psi}(L), x^{\psi}(L)\right) = \psi(L) \in A^L$ satisfying the following conditions:

(i) *Scale invariance* says that the final price and the amount of rented land are independent of changes of scale i.e. $p^{\psi}\left(N, \beta c, \frac{\alpha}{\beta} r, \frac{\alpha}{\beta} R, \beta E\right) = \frac{\alpha}{\beta} p^{\psi}(N, c, r, R, E)$ and $x^{\psi}\left(N, \beta c, \frac{\alpha}{\beta} r, \frac{\alpha}{\beta} R, \beta E\right) = \beta x^{\psi}(N, c, r, R, E)$ for all $\alpha, \beta > 0$.

(ii) *Strict individual rationality* says that the price is greater than r and less than R, i.e. $r < p^{\psi}(N, c, r, R, E) < R$, in order to assure the benefit of the lessors and the tenant respectively.

(iii) *Continuity* says that the price and the amount of land are continuous functions.

Let Ψ be the set of all rules ψ.

3 A characterization result with self-duality

Given a land rental problem $L = (N, c, r, R, E)$, we define its dual as $L^d = \left(N, c, r^d, R^d, E^d\right) \in \mathcal{L}$ where $E^d = c(N) - E$ and r^d and R^d are such that $r^d E^d = rE$ and $R^d E^d = RE$, respectively. Hence,

$$r^d = \frac{rE}{c(N) - E}$$

and

$$R^d = \frac{RE}{c(N) - E}.$$

We define self-duality as follows.

Definition 1. *A rule ψ satisfies self-duality if for all $L = (N, c, r, R, E) \in \mathcal{L}$, $u^L\left(p^{\psi}(L), x^{\psi}(L)\right) = u^{L^d}\left(p^{\psi}\left(L^d\right), x^{\psi}\left(L^d\right)\right)$ and $x^{\psi}(L) = c - x^{\psi}\left(L^d\right)$.*

In order to characterize the rules that satisfy self-duality, we define

$$\Theta = \left\{(e, t) \in \left]0, \frac{1}{2}\right] \times \left]0, 1\right[\right\},$$

which is illustrated in Figure 1, and

$$\Delta^{\mathcal{N}} = \left\{ \hat{c} \in \mathbb{R}^N_{++} : N \in \mathcal{N}, \hat{c}(N) = 1 \right\}.$$

Let $\mathcal{F}$ be the set of continuous functions $f : \Theta \times \Delta^{\mathcal{N}} \to \mathbb{R}$ with $f(e,t,\hat{c}) \in]t,1[$ for all $(e,t) \in \Theta$ and $\hat{c} \in \Delta^{\mathcal{N}}$.

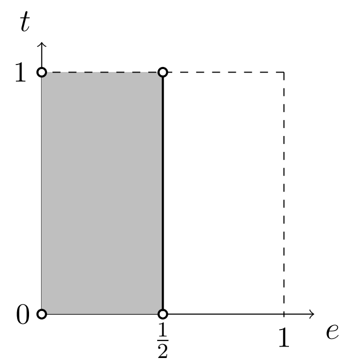

Figure 1: Dual area (Θ).

In Proposition 1, we state the characterization result.

Proposition 1. *A rule ψ satisfies self-duality iff there exists $f \in \mathcal{F}$ such that for all $L = (N, c, r, R, E) \in \mathcal{L}$, the price is given by*

$$p^{\psi}(L) = \begin{cases} Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right), & \text{if } c(N) > 2E \\ Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right), & \text{if } c(N) \leq 2E, \end{cases}$$

and the amount of land is given by $x^{\psi}(L) = \frac{E}{c(N)} c$.

Proof. ($\Rightarrow$) Let ψ be a rule that satisfies self-duality. Let $L = (N, c, r, R, E) \in \mathcal{L}$. Preliminarily, we prove that the dual price satisfies

$$p^{\psi}(L^d) = \frac{E}{c(N) - E} p^{\psi}(L). \qquad (1)$$

By self-duality, the tenant's utility is $\left(R - p^{\psi}(L)\right) E = \left(R^d - p^{\psi}(L^d)\right) E^d$. This is equivalent to write $\left(R - p^{\psi}(L)\right) E = \left(\frac{RE}{c(N)-E} - p^{\psi}(L^d)\right) (c(N) - E)$, from which we deduce that (1) holds.

Now, we prove that there exists a function $f \in \mathcal{F}$ such that the price is given as $p^{\psi}(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and $p^{\psi}(L) = Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \leq 2E$.

By scale invariance,

$$p^{\psi}\left(N, c, r, R, E\right) = Rp^{\psi}\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, \frac{E}{c(N)}\right). \tag{2}$$

We define $f\left(e, t, \hat{c}\right) = p^{\psi}(N, \hat{c}, t, 1, e)$ for all $(e, t, \hat{c}) \in \Theta \times \Delta^{\mathcal{N}}$. By strict individual rationality, $t < f(e, t, \hat{c}) < 1$. Since the price (p^{ψ}) is continue function, f is continue. Hence, $f \in \mathcal{F}$.

On the one hand, when $c(N) > 2E$,

$$p^{\psi}\left(L\right) \overset{(2)}{=} Rp^{\psi}\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, \frac{E}{c(N)}\right) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right).$$

On the other hand, when $c(N) \leq 2E$,

$$\begin{aligned}
p^{\psi}(L) &\overset{(1)}{=} \frac{c(N) - E}{E} p^{\psi}\left(L^d\right) \\
&= \frac{c(N) - E}{E} p^{\psi}\left(N, c, \frac{rE}{c(N) - E}, \frac{RE}{c(N) - E}, c(N) - E\right) \\
&\overset{(2)}{=} \frac{c(N) - E}{E} \frac{RE}{c(N) - E} p^{\psi}\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, \frac{c(N) - E}{c(N)}\right) \\
&= Rp^{\psi}\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, 1 - \frac{E}{c(N)}\right) \\
&= Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right).
\end{aligned}$$

Finally, we prove that the amount of land is given by $x^{\psi}(L) = \frac{E}{c(N)}c$. By self-duality, for all $i \in N$, $\left(p^{\psi}(L) - r\right) x_i^{\psi}(L) = \left(p^{\psi}\left(L^d\right) - r^d\right) x_i^{\psi}\left(L^d\right)$. By (1) and the last part of self-duality ($x^{\psi}(L) = c - x^{\psi}(L^d)$), it is equivalent to write

$$\left(p^{\psi}(L) - r\right) x_i^{\psi}(L) = \left(\frac{E}{c(N) - E} p^{\psi}(L) - \frac{rE}{c(N) - E}\right)\left(c_i - x_i^{\psi}(L)\right). \tag{3}$$

Since $p^{\psi}(L) > r$, (3) is simplified to $\left(c(N) - E\right) x_i^{\psi}(L) = E\left(c_i - x_i^{\psi}(L)\right)$, from where we deduce $x_i^{\psi}(L) = \frac{Ec_i}{c(N)}$ for all $i \in N$.

($\Leftarrow$) Let ψ be a rule given by $p^{\psi}(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and $p^{\psi}(L) = Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \leq 2E$, for some $f \in \mathcal{F}$, and $x^{\psi}(L) = \frac{E}{c(N)}c$.

We prove that ψ satisfies self-duality. Since the definition of self-duality has two parts, we proceed as follows.

i We check first that $x^{\psi}(L) = c - x^{\psi}(L^d)$. Since the land is shared proportionally, i.e. $x^{\psi}(L) = \frac{E}{c(N)}c$, it is checked straightforward.

ii We check that $u^L(\psi(L)) = u^{L^d}\left(\psi(L^d)\right)$.

First, $c(N) > 2E$. For every $i \in N$, $u_i^L(\psi(L)) = \left(p^{\psi}(L) - r\right) x_i^{\psi}(L)$. Since $c(N) > 2E$, for every $i \in N$, we have

$$u_i^L(\psi(L)) = \left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^{\psi}(L).$$

Furthermore, for every $i \in N$, we have $u_i^{L^d}\left(\psi(L^d)\right) = \left(p^{\psi}(L^d) - r^d\right) x_i^{\psi}(L^d)$. Since $c(N) > 2E$ or $c(N) - 2E > 0$, $2E^d = 2(c(N) - E) = c(N) + c(N) - 2E > c(N)$, i.e. $c(N) < 2E^d$, so that $u_i^{L^d}\left(\psi(L^d)\right) =$

$$\begin{aligned}
&= \left[R^d f\left(1 - \frac{E^d}{c(N)}, \frac{r^d}{R^d}, \frac{c}{c(N)}\right) - r^d\right]\left[c_i - x_i^{\psi}(L)\right] \\
&= \left[\frac{RE}{c(N) - E} f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - \frac{Er}{c(N) - E}\right]\left[c_i - \frac{Ec_i}{c(N)}\right] \\
&= \frac{E}{c(N) - E}\left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{(c(N) - E)c_i}{c(N)} \\
&= \left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{Ec_i}{c(N)} \\
&= \left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^{\psi}(L).
\end{aligned}$$

Second, $c(N) < 2E$. For every $i \in N$, $u_i^L(\psi(L)) = \left(p^{\psi}(L) - r\right) x_i^{\psi}(L)$. Since $c(N) < 2E$, for every $i \in N$, we have

$$u_i^L(\psi(L)) = \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^{\psi}(L).$$

Furthermore, for every $i \in N$, we have $u_i^{L^d}\left(\psi(L^d)\right) = \left(p^{\psi}(L^d) - r^d\right) x_i^{\psi}(L^d)$. Since $c(N) < 2E$ or $c(N) - 2E < 0$, $2E^d = 2(c(N) - E) = c(N) + c(N) - 2E < c(N)$, i.e. $c(N) > 2E^d$, so that $u_i^{L^d}\left(\psi(L^d)\right) =$

$$
\begin{aligned}
&= \left[R^d f\left(\frac{E^d}{c(N)}, \frac{r^d}{R^d}, \frac{c}{c(N)}\right) - \frac{E}{c(N)-E} r\right]\left[c_i - x_i^{\psi}(L)\right] \\
&= \left[\frac{RE}{c(N)-E} f\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - \frac{rE}{c(N)-E}\right]\left[c_i - \frac{c_i E}{c(N)}\right] \\
&= \frac{E}{c(N)-E}\left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{(c(N)-E)c_i}{c(N)} \\
&= \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{Ec_i}{c(N)} \\
&= \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^{\psi}(L).
\end{aligned}
$$

Third, $c(N) = 2E$, for every $i \in N$, we have $u_i^L(\psi(L)) =$

$$
\begin{aligned}
&= \left(p^{\psi}(L) - r\right) x_i^{\psi}(L) \\
&= \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{Ec_i}{c(N)} \\
&= \left[Rf\left(1 - \frac{E}{2E}, \frac{r}{R}, \frac{c}{2E}\right) - r\right]\frac{Ec_i}{2E} \\
&= \left[\frac{RE}{2E-E} f\left(1 - \frac{2E-E}{2E}, \frac{r}{R}, \frac{c}{2E}\right) - \frac{rE}{2E-E}\right]\left[c_i - \frac{c_i E}{2E}\right] \\
&= \left[\frac{RE}{c(N)-E} f\left(1 - \frac{c(N)-E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - \frac{rE}{c(N)-E}\right]\left[c_i - \frac{c_i E}{c(N)}\right] \\
&= \left[R^d f\left(1 - \frac{E^d}{c(N)}, \frac{r^d}{R^d}, \frac{c}{c(N)}\right) - r^d\right]\left[c_i - x_i^{\psi}(L)\right] \qquad (4)
\end{aligned}
$$

Since $c(N) = 2E$, $2E^d = 2(c(N) - E) = 2(2E - E) = 2E = c(N)$, i.e. $c(N) = 2E^d$. Then, (4) is equal to $u_i^L(\psi(L)) = \left(p^{\psi}(L^d) - r^d\right) x_i^{\psi}(L^d)$. Hence, $u_i^L(\psi(L)) = u_i^{L^d}\left(\psi(L^d)\right)$.

By efficiency, for the utility of tenant, the proof is analogous as before.

■

The rule associated to $f \in \mathcal{F}$ is denoted as ψ^f.

The next property states that the price does no depend on the distribution of the land.

We formalize the property as follows.

Definition 2. *A rule ψ satisfies price independence if for all $L = (N, c, r, R, E) \in \mathcal{L}$ and $c' \in \mathbb{R}^N_{++}$ such that $c'(N) = c(N)$, we have $p^\psi(L') = p^\psi(L)$ where $L' = (N, c', r, R, E)$.*

In order to characterize the rules that satisfy self-duality and price independence, we define $\mathcal{G}$ as the set of continuous functions $g : \Theta \to \mathbb{R}$ with $g(e,t) \in]t,1[$ for all $(e,t) \in \Theta$.

In Proposition 2, we state the corresponding characterization result.

Proposition 2. *A rule ψ satisfies self-duality and price independence iff there exists $g \in \mathcal{G}$ such that for all $L = (N, c, r, R, E) \in \mathcal{L}$, the price is given by*

$$p^\psi(L) = \begin{cases} Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right), & \text{if } c(N) > 2E \\ Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right), & \text{if } c(N) \le 2E, \end{cases}$$

and the amount of land is given by $x^\psi(L) = \frac{E}{c(N)}c$.

Proof. ($\Rightarrow$) Let ψ be a rule that satisfies self-duality and price independence. Let $L = (N, c, r, R, E) \in \mathcal{L}$. Under self-duality, by Proposition 1 there exits $f \in \mathcal{F}$ such that $p^\psi(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and $p^\psi(L) = Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \le 2E$, and $x^\psi(L) = \frac{E}{c(N)}c$. Under price independence, for every $c' \in \mathbb{R}^N_{++}$ with $c'(N) = c(N)$, we have $p^\psi(L') = p^\psi(L)$. We prove the existence of $g \in \mathcal{G}$ so that $p^\psi(L) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) > 2E$ and $p^\psi(L) = Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) \le 2E$. First, assume that $c(N) > 2E$. Since $c(N) = c'(N)$,

$$Rf\left(\frac{E}{c'(N)}, \frac{r}{R}, \frac{c'}{c'(N)}\right) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c'}{c(N)}\right).$$

Then, $f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c'}{c(N)}\right) = f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$. Consequently, f does not depend on $\frac{c}{c(N)}$, and hence $g\left(\frac{E}{c(N)}, \frac{r}{R}\right) = f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ is well-defined. Second, assume now that $c(N) \leq 2E$. Since $c'(N) = c(N)$,

$$Rf\left(1-\frac{E}{c'(N)}, \frac{r}{R}, \frac{c'}{c'(N)}\right) = Rf\left(1-\frac{E}{c(N)}, \frac{r}{R}, \frac{c'}{c(N)}\right).$$

Then, $f\left(1-\frac{E}{c(N)}, \frac{r}{R}, \frac{c'}{c(N)}\right) = f\left(1-\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$. Consequently, f does not depend on $\frac{c}{c(N)}$ and hence $g\left(1-\frac{E}{c(N)}, \frac{r}{R}\right) = f\left(1-\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ is well-defined.

($\Leftarrow$) Let ψ be a rule given by, for all $L \in \mathcal{L}$, $p^{\psi}(L) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) > 2E$ and $p^{\psi}(L) = Rg\left(1-\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) \leq 2E$ for some $g \in \mathcal{G}$, and $x^{\psi}(L) = \frac{E}{c(N)}c$. Let $f \in \mathcal{F}$ defined as $f(e,t,\hat{c}) = g(e,t)$ for all $(e,t) \in \Theta$ and $\hat{c} \in \Delta^{\mathcal{N}}$. Then, we write $p^{\psi}(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and $p^{\psi}(L) = Rf\left(1-\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \leq 2E$. By Proposition 1, ψ satisfies self-duality. Let $L' \in \mathcal{L}$ be given as in the definition of price independence. We prove $p^{\psi}(L') = p^{\psi}(L)$. First, assume that $c(N) > 2E$. Since $c'(N) = c(N)$. Then, we deduce

$$\begin{aligned} p^{\psi}(L') &= Rg\left(\frac{E}{c'(N)}, \frac{r}{R}\right) \\ &= Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right) = p^{\psi}(L). \end{aligned}$$

Second, assume that $c(N) \leq 2E$. Since $c'(N) = c(N)$. Then, we deduce

$$\begin{aligned} p^{\psi}(L') &= Rg\left(1-\frac{E}{c'(N)}, \frac{r}{R}\right) \\ &= Rg\left(1-\frac{E}{c(N)}, \frac{r}{R}\right) = p^{\psi}(L). \end{aligned}$$

■

The rule corresponding to $g \in \mathcal{G}$ is denoted (with a slight abuse of notation) as ψ^{g}.

Let $\mathcal{L}^2 \subset \mathcal{L}$ be the set of all rental problems with a unique lessor. A rule is standard for two if the tenant and the unique lessor share equally the benefit of cooperation. We formalise this property as follows.

Definition 3. *A rule ψ satisfies standard for two if for all $L \in \mathcal{L}^2$, $u_0^L(\psi(L)) = u_i^L(\psi(L))$ where $N = \{i\}$.*

In Theorem 1, we state the characterization of the unique rule that satisfies self-duality, consistency and standard for two.

Theorem 1. *A rule ψ satisfies self-duality, price independence and standard for two iff for all $L \in \mathcal{L}$, $p^{\psi}(L) = \frac{R+r}{2}$ and $x^{\psi}(L) = \frac{E}{c(N)}c$.*

Proof. ($\Rightarrow$) Let ψ be a rule that satisfies self-duality, price independence and standard for two. Let $L = (N, c, r, K, E) \in \mathcal{L}$. By Proposition 2, there exists $g \in \mathcal{G}$ such that $p^{\psi}(L) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) > 2E$ and $p^{\psi}(L) = Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) \leq 2E$, and $x^{\psi}(L) = \frac{E}{c(N)}c$. We need to prove that $Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right) = \frac{R+r}{2}$ if $c(N) > 2E$ and $Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right) = \frac{R+r}{2}$ if $c(N) \leq 2E$, or equivalently $g(e,t) = \frac{1+t}{2}$ if $\frac{1}{2} > e$ and $g(1-e,t) = \frac{1+t}{2}$ if $\frac{1}{2} \leq e$ for all $(e,t) \in \Theta$. Let $e = \frac{E}{c(N)} \in \left]0, \frac{1}{2}\right]$ and $t = \frac{r}{R} \in [0,1[$. By standard for two, for every $i \in N$, $u_0^{L'}(\psi(L')) = u_i^{L'}(\psi(L'))$. Assume first $\frac{1}{2} > e$. By efficiency, we have $(R - Rg(e,t))E = (Rg(e,t) - r)E$. Then, we deduce that $g(e,t) = \frac{1+t}{2}$. Assume now $\frac{1}{2} \leq e$. By efficiency, we have $(R - Rg(1-e,t))E = (Rg(1-e,t) - r)E$. Then, we deduce that $g(1-e,t) = \frac{1+t}{2}$.

($\Leftarrow$) Let ψ a rule given by $p^{\psi}(L) = \frac{R+r}{2}$ and $x^{\psi}(L) = \frac{E}{c(N)}c$. It is straightforward to check that $\psi = \psi^g$ for $g(e,t) = \frac{1+t}{2}$ for all $(e,t) \in \Theta$. By Proposition 2, ψ satisfies self-duality and price dependence. Then, we prove that ψ satisfies standard for two. It is equivalent to prove that $u_0^L(\psi(L)) = u_i^L(\psi(L))$. The left side of the equality is equal to $u_0^L(\psi(L)) = \left(R - \frac{R+r}{2}\right)E = \frac{R-r}{2}E$. Analogously, the right side of the equality is equal to $u_i^L(\psi(L)) = \left(\frac{R+r}{2} - r\right)E = \frac{R-r}{2}E$. ■

We denote by ψ^2 the unique rule which is characterized in Theorem 1.

We present the discussion about the independence of properties as follows. Whether we remove one the properties, we obtain more rules. First, let us consider the rule given by $p^{\psi}(L) = \frac{R+\alpha r}{1+\alpha}$ for $\alpha \in]0,1[$ and $x^{\psi}(L) = \frac{E}{c(N)}c$, which satisfies self-duality and price independence, but fails standard for two. Second, let us consider the rule given by $p^{\psi}(L) = r + \frac{R-r}{2c(N)}\max_{i\in N} c_i$ and $x^{\psi}(L) = \frac{E}{c(N)}c$, which satisfies self-duality and standard for two, but fails price independence. Finally, let us consider the rule given by $p^{\psi}(L) = \frac{R+r}{2}$ and for every $i \in N$, $x_i^{\psi}(L) = \min\{c_i, \lambda\}$ where λ solves $\sum_{i\in N}\min\{c_i, \lambda\} = E$, which is known as CEA rule in bankruptcy problems. This rule satisfies standard for two and price independence, but fails self-duality. Therefore, these three properties are independent.

4 Concluding remarks

In this paper, we have studied the duality concept in land rental problems. We have presented the self-duality concept based on the idea of self-duality in bankruptcy problems. In bankruptcy problems, for every agent, the final outcome coincides with the amount obtained from the share of the estate. But, in land rental problems, we have considered that for every agent, the utility (final outcome) depends on the share of the rented land (estate) and the price by each unit of land.

The proportional land share is only obtained by self-duality in land rental problems. However, for example, in bankruptcy problems, it is necessary at least to consider some other properties. Furthermore, we have characterized the family of rules that satisfy self-duality. A rule in this family is denoted as ψ^f. We have characterized the subfamily of rules that satisfies self-duality and price independence. A rule in this subfamily is denoted as ψ^g . Finally, We have characterized the unique rule that satisfies self-duality, price independence and standard for two, which is denoted as ψ^2. We also provided the discussion about independence of properties.

Table 1 shows a summary of these results.

Property	ψ^f	ψ^g	ψ^2
Self-duality	Y	Y	Y
Price independence	-	Y	Y
Standard for two	-	-	Y

Table 1: Properties of rules in land rental problems.

References

Aumann, R. J. and Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, 36(2):195 – 213.

Chun, Y. (1988). The proportional solution for rights problems. *Mathematical Social Sciences*, 15(3):231 – 246.

Dagan, N. (1996). New characterizations of old bankruptcy rules. *Social Choice and Welfare*, 13(1):51–59.

Hart, S. and Mas-Colell, A. (1989). Potential, value, and consistency. *Econometrica*, 57(3):589–614.

Herrero, C. (2003). Equal awards vs. equal losses: Duality in bankruptcy. In Sertel, M. and Koray, S., editors, *Advances in Economic Design*, Studies in Economic Design, pages 413–426. Springer Berlin Heidelberg.

Herrero, C. and Villar, A. (2001). The three musketeers: four classical solutions to bankruptcy problems. *Mathematical Social Sciences*, 42(3):307 – 328.

Herrero, C. and Villar, A. (2002). Sustainability in bankruptcy problems. *Top*, 10(2):261–273.

Huettner, F. (2015). A proportional value for cooperative games with a coalition structure. *Theory and Decision*, 78(2):273–287.

Moulin, H. (2002). Axiomatic cost and surplus sharing. In Arrow, K., Sen, A., and Suzumura, K., editors, *Handbook of Social Choice and Welfare*, volume I, chapter 6, pages 289–357. North-Holland, Amsterdam.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 2(4):345 – 371.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, 45(3):249 – 297.

Thomson, W. (2008). Two families of rules for the adjudication of conflicting claims. *Social Choice and Welfare*, 31(4):667–692.

Thomson, W. (2015). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update. *Mathematical Social Sciences*, 74:41 – 59.

Valencia-Toledo, A. and Vidal-Puga, J. (2015). Non-manipulable rules for land rental problems. Working paper, Universidade de Vigo.

van den Brink, R., Funaki, Y., and van der Laan, G. (2013). Characterization of the reverse Talmud bankruptcy rule by exemption and exclusion properties. *European Journal of Operational Research*, 228(2):413 – 417.

Young, H. (1988). Distributive justice in taxation. *Journal of Economic Theory*, 44(2):321 – 335.