Additive rules in discrete allocation problems[1]

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Abstract

In this paper, we study allocation problems and other related problems where a discrete estate should be divided among agents who have claims on it. We characterize the set of rules satisfying additivity on the estate along with additivity on the estate and the claims. These results complete the characterizations given by Bergantiños and Vidal-Puga (Mathematical Social Sciences) in the continuous case.

Keywords: Game theory, Allocation problems.

1 Introduction

Many economic and political situations can be modelled as a problem of how to divide a resource among agents who have claims on it. The way in which this division should take place may depend on the particular problem to be studied.

In some problems agents have legitimate rights over a scarce good. In this case no agent should receive more than his claim, but neither less than nothing. These problems are called bankruptcy problems (O'Neill, 1982; Aumann and Maschler, 1985; Young, 1987).

Other problems are those which arise when the agents collectively contribute in a common project which generates a surplus. The claims of the agents should then be interpreted as their contribution to the project. For example, a group of families may collaborate to build a row of terraced houses for living. In these circumstances, the only restriction is that nobody should receive less than nothing. We call these problems surplus problems (Moulin, 1987).

In allocation problems (Chun, 1988; Maschler, Herrero and Villar, 1999) there are no restrictions in what an agent may receive. In loss problems (Bergantiños and Vidal-Puga, 2004) nobody should receive more than his claim.

All these papers are devoted to the continuous problem, *i.e.* the resource is perfectly divisible (for instance, money). Nevertheless, there are some practical situations in which the resources come in indivisible units. We call the class of such problems discrete problems.

We now give some examples of discrete problems. The assignment of seats in a parliament, queuing problems where the demands consist of finite number of "jobs", or secretaries to departments in a university, can be modelled as a discrete bankruptcy problem[1]. Consider a group of families that collaborate to build a row of terraced houses for living. This situation corresponds to a discrete surplus problem. Assume now the terraced houses in the previous example are built not by families but by building contractors. They just need to distribute the houses but they may have other (real or projected)

[1] These examples appear in Moulin (2000) and Herrero and Martínez (2004).

houses on their own. In this case, the assignment may be such that some contractors cede some of their own houses. This situation corresponds to a discrete allocation problem.

Finally, assume that the joint project by the contractors fails, and there are less houses than claimed. Since the joint project failed, no contractor should get more houses than those he has rights on. Thus, we can model this situation as a discrete loss problem.

Even though most of the literature is devoted to the continuous problem, during the last years many papers study the discrete problem. We can mention, for instance, Moulin (2000), Moulin (2002a), Moulin and Stong (2002), and Herrero and Martínez (2004). Also, in Balinski and Young (1982) it is possible to find an extensive literature on discrete allocation problems in other settings.

We can study these problems from two different approaches. One of them is the axiomatic characterizations of rules. The idea is to propose desirable properties and find out which of them characterize every rule. Properties often help agents to compare different rules and to decide which rule is preferred for a particular situation. Another approach is to study what the rules satisfying a set of properties are. For instance, Young (1988) characterizes the rules satisfying continuity, symmetry, and consistency; de Frutos (1999) characterizes the rules satisfying non-manipulability; and Moulin (2000) characterizes the rules satisfying consistency, composition up, composition down, and scale invariance. Thomson (2003) and Moulin (2002b) give a survey of this literature. In this paper, we follow the second approach and concentrate on the property of additivity.

Additivity is a widely used property. For instance, the Shapley value, the most important value in cooperative games with transferable utility, is characterized by additivity and other properties. If we compare the Shapley value with other prominent values (for example the nucleolus) we realize that these values satisfy all the properties characterizing the Shapley value (efficiency, null player, and symmetry) except additivity.

Bergantiños and Vidal-Puga (2004) study the property of additivity in the continuous problem. In this paper we study this property in the discrete problem. We must note that even though the results of this paper are related with those of Bergantiños and Vidal-Puga (2004), the proofs are completely different and this paper is not a consequence of the previous one. We use two definitions of additivity: additivity on the estate ($A1$) (Moulin, 1987; and Chun, 1988), and additivity on the estate and the claims ($A2$) (Bergantiños and Méndez-Naya, 2001).

In this paper we characterize the rules satisfying $A1$ and $A2$ in each of the four problems mentioned before. The rules satisfying $A1$ are as follows: In allocation problems they are characterized by the product of the estate and a claims-dependent function. In surplus problems all the estate is given to an agent, who is selected depending on the claims. In loss and bankruptcy problems there are no rules.

The rules satisfying $A2$ are as follows: In allocation problems they are characterized by the sum of two parts, one depending on the estate and the other depending on the claims. In surplus problems, the estate is always given to a fixed agent. In loss problems, all the loss is always suffered by a fixed agent. There is no bankruptcy rule satisfying this additivity.

The paper is organized as follows. Section 2 introduces the problems studied in this paper. Section 3 studies the rules which satisfy these additivity properties.

2 Preliminaries

Let $\mathbb{Z}$ denote the set of integer numbers and let $\mathbb{N}$ denote the set of non-negative integer numbers. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_+$ denote the set of non-negative real numbers.

We also denote the set of potential agents as $\mathbb{N}$. Let N be any finite subset of $\mathbb{N}$. Given $x, y \in \mathbb{R}^N$, $x \geq y$ means $x_i \geq y_i$ for all $i \in N$; and $x + y$ is the vector $(x_i + y_i)_{i \in N}$. Moreover, $0_N = (0, ..., 0) \in \mathbb{R}^N$. Given $S \subset N$, 1_S is the vector $(x_i)_{i \in N}$ such that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$.

We study problems where an estate $E \in \mathbb{N}$ must be divided among a group of agents N. Let c_i be the claim of agent $i \in N$, let $c = (c_i)_{i \in N}$ be the vector of claims, and let $C = \sum_{i \in N} c_i$ be the sum of the claims. We assume that the estate and the claims are non-negative. The question that arises is: how to divide the estate among agents? This question is answered by means of defining rules. A rule, f, is a map which assigns to any problem (c, E) a vector $f(c, E)$ in $\mathbb{Z}^N$, where $f_i(c, E)$ denotes the part of the estate received by agent $i \in N$.

The estate to be distributed and the claims come in indivisible units, *i.e.* $(c, E) \in \mathbb{N}^N \times \mathbb{N}$. It is not difficult to check that our results are still valid if we take $(c, E) \in \mathbb{R}_+^N \times \mathbb{N}$. Allocation of organs for transplants, college admissions, and in general queuing problems where individual claims consisting of a finite number of jobs, could be some problems in which the claims are indivisible. In Moulin (2000, 2002a) and Moulin and Stong (2002) some of these discrete problems are studied.

We now give a list of problems that fit in our general framework. Notice that the difference among these problems is, mainly, in the definition of what a rule is.

A *bankruptcy problem*, BP, is a pair $(c, E) \in \mathbb{N}^N \times \mathbb{N}$ where $C \geq E$. We denote the set of all bankruptcy problems as $\mathcal{B}$. A *bankruptcy rule* is a function $f^B : \mathcal{B} \to \mathbb{Z}^N$ satisfying that for all $(c, E) \in \mathcal{B}$, $\sum_{i \in N} f_i^B(c, E) = E$ and $0 \leq f_i^B(c, E) \leq c_i$ for all $i \in N$.

A *surplus problem*, SP, is a pair $(c, E) \in \mathbb{N}^N \times \mathbb{N}$. We denote the set of all surplus problems as $\mathcal{S}$. A *surplus rule* is a function $f^S : \mathcal{S} \to \mathbb{Z}^N$ satisfying that for all $(c, E) \in \mathcal{S}$, $\sum_{i \in N} f_i^S(c, E) = E$ and $0 \leq f_i^S(c, E)$ for all $i \in N$.

An *allocation problem*, AP, is a pair $(c, E) \in \mathbb{N}^N \times \mathbb{N}$. We denote the set of all allocation problems as $\mathcal{A}$. An *allocation rule* is a function $f^A : \mathcal{A} \to \mathbb{Z}^N$ satisfying that for all $(c, E) \in \mathcal{A}$, $\sum_{i \in N} f_i^A(c, E) = E$.

A *loss problem*, LP, is a pair $(c, E) \in \mathbb{N}^N \times \mathbb{N}$ where $C \geq E$. We denote the set of all loss problems as $\mathcal{L}$. A *loss rule* is a function $f^L : \mathcal{L} \to \mathbb{Z}^N$ satisfying that for all $(c, E) \in \mathcal{L}$, $\sum_{i \in N} f_i^L(c, E) = E$ and $f_i^L(c, E) \leq c_i$ for all $i \in N$.

Remark 1 *These four problems cover all possible definitions of a rule. Given $i \in N$, a bankruptcy rule satisfies $0 \leq f_i^B(c, E) \leq c_i$, a surplus rule satisfies $0 \leq f_i^S(c, E)$, a loss rule satisfies $f_i^L(c, E) \leq c_i$, and an allocation rule has no restrictions. Notice that every bankruptcy rule is a loss rule and every surplus rule is an allocation rule. A bankruptcy rule f^B is not a surplus rule because f^B is not defined for surplus problems with $C < E$. Similarly, a loss rule is not an allocation rule.*

In BP and LP we need to impose the condition $C \geq E$ because otherwise it is not possible to find f satisfying $\sum_{i \in N} f_i(c, E) = E$ and $f_i(c, E) \leq c_i$ for all $i \in N$.

We now present the two additivity properties considered in this paper.

Additivity on E (A1). For all (c, E) and (c, E'),

$$f(c, E + E') = f(c, E) + f(c, E').$$

Moulin (1987) and Chun (1988) use this property in surplus problems and in allocation problems, respectively. Additivity on E indicates that dividing the estate among the agents is the same as dividing, first, one part of the estate, and afterwards, the remaining estate.

Additivity on (c, E) (A2). For all (c, E) and (c', E'),

$$f(c + c', E + E') = f(c, E) + f(c', E').$$

Bergantiños and Méndez-Naya (2001) introduced this property in bankruptcy problems and in allocation problems. Suppose that the product sold by a firm depends on several parts (quality and marketing, for instance). The manager wants to award the employees with $E + E'$ days off. This revenue can be divided into two parts: one motivated by quality (E) and the other by marketing (E'). We can also determine the contribution of every agent of the firm to quality (c) and marketing (c'). Now we can allocate the revenue according to two procedures. First, we allocate the total revenue $(E + E')$ according to the total contribution $(c + c')$. Second, we allocate the revenue motivated by quality (E) according to the contribution to quality (c), and the revenue of marketing (E') according to the contribution to marketing (c'). $A2$ guarantees that both procedures coincide.

Usually it is not very difficult to determine the contribution of the agents to each part (for instance, hours worked) and to the total revenue. But sometimes it seems impossible to know exactly the contribution of each part to the total revenue. Under these circumstances it appears that the second procedure cannot be applied. However, if the allocation rule satisfies $A2$, its application is possible since both procedures coincide.

There is no logical relation between $A1$ and $A2$. Later, we will see examples of rules satisfying $A1$ but not $A2$ and rules satisfying $A2$ but not $A1$.

Remark 2 *Another possibility is to define additivity on* c. *For all* (c, E) *and* (c', E), $f(c + c', E) = f(c, E) + f(c', E)$. *Nevertheless, no rule satisfies this property because* $\sum_{i \in N} f_i(c + c', E) = E$, $\sum_{i \in N} f_i(c, E) = E$, *and* $\sum_{i \in N} f_i(c', E) = E$.

Three properties will now be considered. Symmetry is a standard property that could be defined in each of the four problems studied in this paper.

Symmetry (SYM). For every problem (c, E), if $c_i = c_j$, then

$$f_i(c, E) = f_j(c, E).$$

Assume that agent $i's$ claim is larger than agent $j's$ claim. Weak order preservation says that agent i must receive at least the same amount as agent j.

Weak Order Preservation (WOP). For every problem (c, E), if $c_i > c_j$, then

$$f_i(c, E) \geq f_j(c, E).$$

Assume that agent $i's$ claim is at least as large as agent $j's$ claim. Order preservation says that agent i must receive at least the same amount as agent j.

Order Preservation (OP). For every problem (c, E), if $c_i \geq c_j$, then

$$f_i(c, E) \geq f_j(c, E).$$

Of course, OP implies both SYM and WOP.

Even though SYM and OP are appealing properties, no rule satisfies them, as the following Lemma shows:

Lemma 3 *There is no rule satisfying* SYM *or* OP *for discrete problems.*

Proof. Take two agents with equal claims and $E = 1$. Since the one unit cannot be split equally between the agents, SYM and OP must fail. ■

3 The additive rules

In this section we characterize the set of additive rules in the four problems. In Theorem 4 we characterize the rules satisfying $A1$ and in Theorem 6 the rules satisfying $A2$. Finally, we compare, briefly, the rules satisfying $A1$ with the rules satisfying $A2$.

Theorem 4 *a) An allocation rule f^A satisfies $A1$ if and only if for all $(c, E) \in \mathcal{A}$,*

$$f_i^A(c, E) = E\alpha_i(c) \text{ for all } i \in N$$

where $\alpha : \mathbb{N}^N \to \mathbb{Z}^N$ satisfies $\sum_{i \in N} \alpha_i(c) = 1$ for all $c \in \mathbb{N}^N$.

b) A surplus rule f^S satisfies $A1$ if and only if for all $(c, E) \in \mathcal{S}$,

$$f_i^S(c, E) = \begin{cases} E & \text{if } i = s(c) \\ 0 & \text{otherwise} \end{cases}$$

where $s : \mathbb{N}^N \to N$.

c) There is no loss rule satisfying $A1$.

d) There is no bankruptcy rule satisfying $A1$.

Proof. a) It is trivial to prove that, if $f^A(c, E) = E\alpha(c)$, then f^A satisfies $A1$.

We now prove the converse. Suppose that f^A is an allocation rule satisfying $A1$. Let $\alpha : \mathbb{N}^N \to \mathbb{Z}^N$ be such that $\alpha(c) = f^A(c, 1)$. Since f is an allocation rule, we conclude that $\sum_{i \in N} \alpha_i(c) = 1$. Given $(c, E) \in \mathcal{A}$ and $i \in N$, by $A1$, $f_i^A(c, E) = E f_i^A(c, 1) = E\alpha_i(c)$.

b) Since all surplus rules are allocation rules, the allocation rules of part a) satisfying $f(c, E) \geq 0_N$ for all $(c, E) \in \mathcal{S}$ are all the surplus rules satisfying $A1$.

Assume that f^S is defined as $f^S(c, E) = E\alpha(c)$ with $\sum_{i \in N} \alpha_i(c) = 1$ and $f^S(c, E) \geq 0_N$ for all $(c, E) \in \mathcal{S}$. Since $\sum_{i \in N} \alpha_i(c) = 1$ for all $c \in \mathbb{N}^N$, it is not difficult to conclude that, given $c \in \mathbb{N}^N$, there exists $i_c \in N$ such that $\alpha_{i_c}(c) = 1$ and $\alpha_i(c) = 0$ if $i \neq i_c$. Considering $s : \mathbb{N}^N \to N$ such that $s(c) = i_c$ the result holds trivially.

c) Using arguments similar to those used in part a) we can conclude that if f^L is a loss rule satisfying $A1$. Then, for all $(c, E) \in \mathcal{L}$, $f^L(c, E) = E\alpha(c)$

where $\sum_{i\in N}\alpha_i(c) = 1$. Then, given $c \in \mathbb{N}^N$, there exists $j \in N$ such that $\alpha_j(c) \geq 1$. If $E > c_j$, then $f_j^L(c, E) = E\alpha_j(c) > c_j$. Hence, f^L is not a loss rule.

d) Since every bankruptcy rule is a loss rule, part d) is a consequence of part c). ■

The allocation and surplus rules satisfying $A1$ have a special structure. Given a vector of claims c, what really matters is the way in which we divide one unit among the agents. When there are E units, any agent receives E times what he received when there is one unit.

Notice that in SP all the estate is received by one agent, meanwhile the rest receive zero. This agent is selected depending on c.

Consider the family F^S of surplus rules where an agent with the highest claim receives all the estate and the rest of agents receive nothing. Formally,

$$F^S = \left\{ \begin{array}{c} f^S \mid \forall c \in \mathbb{N}^N,\ \exists s(c) \in N \text{ such that } c_{s(c)} = \max\limits_{j\in N} c_j, \\ f^S_{s(c)}(c, E) = E \text{ and } f^S_i(c, E) = 0 \text{ if } i \neq s(c) \end{array} \right\}.$$

This kind of rules frequently appears in several real-life situations where a prize is given to a single person. For example, the presidency of a chamber may be given to the candidate with more votes and, in case of tie, to the eldest candidate from among those with maximum number of votes. In this case the set of agents is the set of candidates, the estate is the presidency, and the claim of each candidate is the number of votes he receives.

The next corollary characterizes F^S as the set of rules satisfying $A1$ and WOP.

Corollary 5 *A surplus rule f^S satisfies $A1$ and WOP if and only if $f^S \in F^S$.*

Proof. It is clear that the rules of F^S satisfy $A1$ and WOP.

Assume that f^S is a surplus rule satisfying $A1$ and WOP. By Theorem 4, we know that there exists $s(c) \in N$ such that $f^S_{s(c)}(c, E) = E$ and $f^S_i(c, E) = 0$ if $i \neq s(c)$. Since f^S satisfies WOP we conclude that $c_{s(c)} = \max_{j\in N}\{c_j\}$. Hence, $f^S \in F^S$. ■

This corollary is a tight characterization result. Let g^S be given by

$$g_i^S(c,E)=\begin{cases} E & \text{if } i=s(c) \\ 0 & \text{otherwise} \end{cases}$$

where $s:\mathbb{N}^N\to N$ satisfies $c_{s(c)}=\min_{i\in N}\{c_i\}$ for all $c\in\mathbb{N}^N$. By Theorem 4b), g^S satisfies $A1$. However, it does not satisfy WOP. For example,

$$g^S((2,1),1)=(0,1).$$

Let h^S be given by

$$h_i^S(c,E)=\begin{cases} E & \text{if } i=s(c) \text{ and } C\neq E \\ 0 & \text{if } i\neq s(c) \text{ and } C\neq E \\ c_i & \text{if } C=E \end{cases}$$

where $s:\mathbb{N}^N\to N$ satisfies $c_{s(c)}=\max_{i\in N}\{c_i\}$ for all $c\in\mathbb{N}^N$. It is clear that h^S satisfies WOP. However, it does not satisfy $A1$. Let $N=\{1,2\}$ be and assume $s((1,1))=1$. Then,

$$\begin{aligned} h^S((1,1),1)+h^S((1,1),2) &= (1,0)+(1,1)=(2,1) \\ h^S((1,1),3) &= (3,0). \end{aligned}$$

The next theorem characterizes the rules satisfying $A2$.

Theorem 6 *a) An allocation rule f^A satisfies $A2$ if and only if, for all $(c,E)\in\mathcal{A}$,*

$$f_i^A(c,E)=\beta_i(c)+Ex_i \text{ for all } i\in N$$

where $\beta:\mathbb{N}^N\to\mathbb{Z}^N$ satisfies $\sum_{i\in N}\beta_i(c)=0$ for all $c\in\mathbb{N}^N$ and $\beta(c+c')=\beta(c)+\beta(c')$ for all $c,c'\in\mathbb{N}^N$. Moreover, $x\in\mathbb{Z}^N$ and $\sum_{i\in N}x_i=1$.

b) A surplus rule f^S satisfies $A2$ if and only if there exists $i_0\in N$ such that for all $(c,E)\in\mathcal{S}$,

$$f_i^S(c,E)=\begin{cases} E & \text{if } i=i_0 \\ 0 & \text{otherwise.} \end{cases}$$

c) A loss rule f^L satisfies $A2$ if and only if there exists $i_0\in N$ such that for all $(c,E)\in\mathcal{L}$,

$$f_i^L(c,E)=\begin{cases} c_i-(C-E) & \text{if } i=i_0 \\ c_i & \text{otherwise.} \end{cases}$$

d) There is no bankruptcy rule satisfying $A2$.

Proof. a) It is straightforward to prove that, if $f^A(c,E) = \beta(c) + Ex$, then f^A satisfies $A2$.

We now prove the converse. Suppose that f^A is an allocation rule satisfying $A2$. Given an allocation problem (c,E) and a rule f^A satisfying $A2$,

$$f^A(c,E) = f^A(c,0) + f^A(0_N, E).$$

Since f^A satisfies $A2$, $f^A(0_N, E) = Ef^A(0_N, 1)$. Consider $x = f^A(0_N, 1)$. Then, $x \in \mathbb{Z}^N$ and $\sum_{i\in N} x_i = \sum_{i\in N} f_i^A(0_N, 1) = 1$.

Consider $\beta : \mathbb{N}^N \to \mathbb{Z}^N$ given by $\beta(c) = f^A(c,0)$ for all $c \in \mathbb{N}^N$. Then, $\sum_{i\in N} \beta_i(c) = \sum_{i\in N} f_i^A(c,0) = 0$. Moreover, for all $c, c' \in \mathbb{N}^N$, $\beta(c+c') = f^A(c+c',0) = f^A(c,0) + f^A(c',0) = \beta(c) + \beta(c')$.

b) Since every surplus rule is an allocation rule, the allocation rules of part a) satisfying $f(c,E) \geq 0_N$ for all $(c,E) \in \mathcal{S}$ are all the surplus rules satisfying $A2$.

Assume that f^S is defined as in part a) and $f^S(c,E) \geq 0_N$ for all $(c,E) \in \mathcal{S}$. Given $c \in \mathbb{N}^N$, $\beta(c) = f^S(c,0) \geq 0_N$ and $\sum_{i\in N} f_i^S(c,0) = 0$. Hence, $\beta(c) = 0_N$. Since $x = f^S(0_N, 1) \geq 0_N$ and $\sum_{i\in N} x_i = 1$, there exists $i_0 \in N$ such that $x_{i_0} = 1$ and $x_i = 0$ for all $i \in N \setminus \{i_0\}$. Now the result holds trivially.

c) It is trivial to prove that, if $f^L(c,E) = c - (C-E)\,1_{\{i_0\}}$, then f^L satisfies $A2$.

We now prove the converse. Assume that f^L satisfies $A2$. Since $f^L(c,E) \leq c$ and $\sum_{i\in N} f_i^L(c,E) = E$ we conclude that, for all $i \in N$,

- $f^L\left(1_{\{i\}}, 1\right) = 1_{\{i\}}$, and
- $f^L\left(1_{\{i\}}, 0\right) = 1_{\{i\}} - 1_{\{i'\}}$ where $i' \in N$.

Consider $i \in N$ such that $f^L\left(1_{\{i\}}, 0\right) = 1_{\{i\}} - 1_{\{i'\}}$. Consider $j \in N \setminus \{i\}$ such that $f^L\left(1_{\{j\}}, 0\right) = 1_{\{j\}} - 1_{\{j'\}}$. Since f satisfies $A2$,

$$\begin{aligned}
f^L\left(1_{\{i,j\}}, 1\right) &= f^L\left(1_{\{i\}}, 1\right) + f^L\left(1_{\{j\}}, 0\right) = 1_{\{i\}} + 1_{\{j\}} - 1_{\{j'\}} \\
f^L\left(1_{\{i,j\}}, 1\right) &= f^L\left(1_{\{j\}}, 1\right) + f^L\left(1_{\{i\}}, 0\right) = 1_{\{j\}} + 1_{\{i\}} - 1_{\{i'\}}
\end{aligned}$$

which means that $i' = j'$.

We have proved that there exists $i_0 \in N$ such that $f^L\left(1_{\{i\}}, 0\right) = 1_{\{i\}} - 1_{\{i_0\}}$ for all $i \in N$.

Given $(c, E) \in \mathcal{L}$, we can find a partition $\{N_1, \{i\}, N_2\}$ of N such that $c_i = c_i^1 + c_i^2$, $c_i^1 \in \mathbb{N}$, $c_i^2 \in \mathbb{N}$, and $E = \sum_{j \in N_1} c_j + c_i^1$. Since f^L satisfies $A2$, $f^L(c, E) =$

$$\begin{aligned}
& \sum_{j \in N_1} f^L\left(c_j 1_{\{j\}}, c_j\right) + f^L\left(c_i^1 1_{\{i\}}, c_i^1\right) + f^L\left(c_i^2 1_{\{i\}}, 0\right) + \sum_{j \in N_2} f^L\left(c_j 1_{\{j\}}, 0\right) \\
= & \sum_{j \in N_1} c_j f^L\left(1_{\{j\}}, 1\right) + c_i^1 f^L\left(1_{\{i\}}, 1\right) + c_i^2 f^L\left(1_{\{i\}}, 0\right) + \sum_{j \in N_2} c_j f^L\left(1_{\{j\}}, 0\right) \\
= & \sum_{j \in N_1} c_j 1_{\{j\}} + c_i^1 1_{\{i\}} + c_i^2\left(1_{\{i\}} - 1_{\{i_0\}}\right) + \sum_{j \in N_2} c_j\left(1_{\{j\}} - 1_{\{i_0\}}\right) \\
= & \; c - (C - E) 1_{\{i_0\}}.
\end{aligned}$$

d) Suppose that $N = \{1, 2\}$, $E = 5$, and $c = (7, 7)$. We can find $i \in N$ such that $f_i^B(c, E) \geq 3$. Assume without loss of generality that $i = 1$.

Since f^B satisfies $A2$, $f_1^B(c, E) = f_1^B((6, 1), 1) + f_1^B((1, 6), 4) \leq 1 + 1 = 2$, which is a contradiction. ■

Remark 7 *Part c) of Theorem 6 cannot be proved using part a) because the set of allocation problems* $(\mathcal{A})$ *is different from the set of loss problems* $(\mathcal{L})$. *If* $(c, E) \in \mathcal{L}$, *then* $C \geq E$, *but in* $\mathcal{A}$, $C < E$ *is also possible. This means that an allocation rule could satisfy* $A2$ *in* $\mathcal{L}$ *but not in* $\mathcal{A}$. *For example, let* $i_0, i_1 \in N$, $i_0 \neq i_1$ *and consider the rule* $f_i(c, E) = c_i - (C - E)$ *if* $i = i_0$ *and* $C \geq E$, $f_i(c, E) = c_i - (C - E)$ *if* $i = i_1$ *and* $C < E$, *and* $f_i(c, E) = c_i$ *otherwise. This rule satisfies* $A2$ *in* $\mathcal{L}$ *but not in* $\mathcal{A}$.

The allocation rules satisfying $A2$ can be divided into two terms: the first term depending on c $(\beta(c))$ and the second term depending on E (Ex). In the first term, given a vector of claims c, the function β reassigns units of the indivisible good among agents in such a way that some agents must provide to other agents some units of this good. In the second term any agent receives E times the amount he would receive when all agents claim 0 and only 1 unit is available. Notice that the class of allocation rules satisfying $A2$ is unrelated to the class of allocation rules satisfying $A1$ (there are allocation rules satisfying $A2$ but not $A1$ and vice versa). As a consequence of Theorems

4 and 6, an allocation rule f^A satisfies $A1$ and $A2$ if and only if for all $(c, E) \in \mathcal{A}$, $f^A(c, E) = Ex$ where $x \in \mathbb{Z}^N$ and $\sum_{i \in N} x_i = 1$.

The surplus rules satisfying $A2$ are those in which an agent receives all the estate and the rest receive nothing. Notice that the class of surplus rules satisfying $A2$ is a subset of the class of surplus rules satisfying $A1$.

The loss rules satisfying $A2$ are those in which an agent loses the total loss $(C - E)$ and the rest lose nothing. Notice that there is no loss rule satisfying $A1$. Moreover, it is not difficult to check that the surplus rules satisfying $A2$ are dual[2] of the loss rules satisfying $A2$, when we consider both as allocation rules.

In BP there is no bankruptcy rule satisfying $A1$ or $A2$.

Our main conclusions are: In bankruptcy problems, both $A1$ and $A2$ are very restrictive properties. In loss and surplus problems with $A2$, we characterize the "everything for one agent" rules. Moreover, the rules satisfying $A2$ in loss problems are dual of the rules satisfying $A2$ in surplus problems. Notice that the results obtained with $A1$ are not so "homogeneous" as with $A2$. For instance, the comments about duality are not true with $A1$.

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[2] Given a bankruptcy rule f^B, Aumann and Maschler (1985) define the *dual rule* $\widehat{f^B}$ as

$$\widehat{f^B}(c, E) = c - f^B(c, C - E)$$

for all $(c, E) \in \mathcal{B}$.

Notice that $\widehat{f^B}$ assigns "awards" (E) in the same way as f^B assigns "losses" $(C - E)$. When $C - E \geq 0$, we can extend this definition to AP, SP, and LP.

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