Realizing fair outcomes in minimum cost spanning tree problems through non-cooperative mechanisms

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In the context of minimum cost spanning tree problems, we present a bargaining mechanism for connecting all agents to the source and dividing the cost amongst them. The basic idea is very simple: we ask each agent the part of the cost he is willing to pay for an arc to be constructed. We prove that there exists a unique payoff allocation associated with the subgame perfect Nash equilibria of this bargaining mechanism. Moreover, this payoff allocation coincides with the rule defined in Bergantiños and Vidal-Puga (2007a) [3].

Key words: minimum cost spanning tree problems, cost sharing, implementation
MSC2000 Subject Classification: Primary: 91A40; secondary: 90B
OR/MS subject classification: Primary: cost analysis; secondary: noncooperative games

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). Consider that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence they have to share cost of the distribution network. This example appears in Dutta and Kar (2004) [8]. Bergantiños and Lorenzo (2004, 2005) [1] [2] studied a real situation where villagers should pay the cost of constructing pipes from their houses to the water supplier.

An important issue addressed in the literature is how to allocate the connection cost amongst the agents. Several rules are proposed in the literature. We can mention, for instance, the papers by Bird (1976) [6], Feltkamp *et al* (1994) [9], Kar (2002) [10], Dutta and Kar (2004) [8], and Bergantiños and Vidal-Puga (2007a) [3].

In this paper we obtain a fair outcome through a non-cooperative mechanism. Even though the non-cooperative approach is quite standard in the literature, there are no many papers with this approach in *mcstp*. We mention two papers.

Bergantiños and Lorenzo (2004, 2005) [1] [2] studied a non-cooperative mechanism inspired by a real situation. Many equilibria may exist. Some of them are Pareto optimal (the cost of connecting all the agents to the source is minimal) but others are not. Moreover, some of the equilibria are extremely unfair. For example, the payoff of symmetric agents could be different.

Mutuswami and Winter (2002) [12] introduced a general mechanism of network formation and pointed out that this mechanism can be applied to *mcstp*. They studied *mcstp* in a more general framework where the agents have some private benefits which affect the final outcome. We study *mcstp* in the classical framework (private benefits are not taken into account). Moreover, their mechanism is not easy to interpret in terms of *mcstp*. For instance, it is not clear what the role of the source is.

We present a bargaining mechanism for connecting all the agents to the source and dividing the cost amongst them. The basic idea is very simple, we ask each agent the part of the cost he is willing to pay for an arc to be constructed. We apply this idea following a sequential protocol. Initially, nobody is connected to the source. At each step, at least one agent is connected to the source. Unconnected agents can connect to the source through the agents connected in previous steps.

This bargaining mechanism has three stages.

Stage 1. Each agent i submits a vector $\left(x_j^i\right)_{j\in N\in\setminus\{i\}}$ where $x_j^i \geq 0$ is the amount that agent i is willing to pay to agent j if agent j connects to the source. The agent with the highest net offer (the difference between what other agents offer to him and what he offers to the others) is selected. Ties are solved randomly.

Stage 2. The selected agent, which we denote as α, proposes to the agents in $S \subset N$ ($\alpha \in S$) to construct a tree t (in which each agent in S will be connected to the source) and to divide the cost according to $y \in \mathbb{R}^S$.

Stage 3. If some agent in S rejects $\alpha's$ proposal, α connects to the source and each agent $i \in N \setminus \{\alpha\}$ pays x_α^i to α. The agents in $N \setminus \{\alpha\}$ continue bargaining among themselves. However, they can now connect to the source directly or through agent α.

If all the agents in S accept α's proposal, t is constructed. The agents in $N \setminus S$, if any, continue bargaining among themselves. However, they can now connect to the source directly or through the agents in S.

We prove that there exists a unique payoff allocation associated with the subgame perfect Nash equilibria of this mechanism. Moreover, this payoff allocation coincides with the rule φ defined in Bergantiños and Vidal-Puga (2007a) [3]. Thus, we have obtained a fair outcome through a non-cooperative mechanism.

We also prove that there are strong Nash equilibria whose payoff coincides with the rule φ. These results are robust to many small modifications of the bargaining mechanism.

Pérez-Castrillo and Wettstein (2001) [13] defined the bidding mechanism for *TU* games. They proved that the payoff equilibria of the bidding mechanism coincides with the Shapley value. We apply the ideas of the bidding mechanism to *mcstp*. We prove that the payoff associated with each equilibria coincides with the Shapley value of the cooperative game associated with the *mcstp*, as defined by Bird (1976) [6].

The paper is organized as follows. In Section 2 we introduce *mcstp*. In Section 3 we introduce the bargaining mechanism and present our main results. In Section 4 we discuss several modifications of the mechanism. In Section 5 we apply the mechanism of Pérez-Castrillo and Wettstein (2001) [13] to *mcstp*. In the Appendix we present the proofs of the results.

2 The minimum cost spanning tree problem

In this section we introduce minimum cost spanning tree problems and some results which are relevant for this paper.

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$. Our interest lies on networks where each node of N is (directly or indirectly) connected to the source.

Let Π_N denote the set of all orders over N. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of elements of N which come before i in the order given by π. Namely, $Pre\,(i, \pi) = \{j \in N : \pi\,(j) < \pi\,(i)\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* (N_0, C), we denote the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : \ i, j \in N_0\}$. The elements of g are called *arcs*. Given a network g and a pair of nodes i and j, a *path* form i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$ and $j = i_l$.

A *tree* is a network satisfying that for all $i \in N$ there exists a unique path from agent i to the source. If t is a tree, we usually write $t = \left\{\left(i^0, i\right)\right\}_{i \in N}$ where $i^0 \in N_0$ represents the first node in the unique path in t from agent i to 0.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimal tree* for (N_0, C), briefly an *mt*, is a tree t such that $c(t) = \min\{c(t') : t' \text{ is a tree}\}$. It is well-known in the literature of *mcstp* that an *mt* exists, even though it does not necessarily have to be unique. Given an *mcstp* (N_0, C) we denote the cost associated with any *mt* t in (N_0, C) as $m(N_0, C)$.

One of the most important issues addressed in the literature about *mcstp* is how to divide the connection cost amongst the agents. A (*cost allocation*) *rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each *mcstp* (N_0, C) and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usually, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

A Transferable Utility (*TU*) game is a pair (N, v) where $N \subset \mathcal{N}$ is finite and $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. We denote the *Shapley value* (Shapley, 1953 [15]) of the *TU* game (N, v) as $Sh(N, v)$.

Bird (1976) [6] associated a *TU* game (N, v_C) with each *mcstp* (N_0, C). For each coalition $S \subset N$, $v_C(S) = m(S_0, C)$. Usually, we write v instead of v_C. Thus, we can define rules in *mcstp* through v. For instance, Kar (2002) [10] studied the Shapley value.

Feltkamp *et al* (1994) [9] defined a rule called Equal Remaining Obligations (*ERO*) through Kruskal's algorithm (Kruskal, 1956 [11]). *ERO* was later called the *P-value* in Branzei *et al* (2004) [7].

In Bergantiños and Vidal-Puga (2007a) [3] we defined the rule φ. A cost matrix C is irreducible if reducing the cost of any arc, the minimal cost of connecting all agents is also reduced. We defined

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

where C^* is the unique irreducible matrix associated with C.

In Bergantiños and Vidal-Puga (2007c) [5] we proved that φ coincides with *ERO*.

In Bergantiños and Vidal-Puga (2007a, 2007b) [3] [4] we proved that φ satisfies several properties. We mention three of them because we will use them in several proofs.

Solidarity (*SOL*). For all *mcstp* (N_0, C) and (N_0, C') such that $C \leq C'$, $\psi(N_0, C) \leq \psi(N_0, C')$.

Separability (*SEP*). For all *mcstp* (N_0, C) and $S \subset N$ such that $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$,

$$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

Given an *mcstp* (N_0, C), (S_0, C^{+T}) denotes the *mcstp* obtained from (N_0, C) assuming that the agents in S have to be connected and the agents in T are already connected. Formally, $c_{ij}^{+T} = c_{ij}$ for all $i, j \in S$ and $c_{i0}^{+T} = \min_{j \in T_0} c_{ij}$ for all $i \in S$. When $T = \{i\}$, we write C^{+i} instead of C^{+T}.

Given $S \subset N$ and $i \in S$, we denote $S^{-i} = S \setminus \{i\}$.

Equal Contributions (EC). For all $i, j \in N$, $i \neq j$,

$$\varphi_i (N_0, C) - \varphi_i \left(N_0^{-j}, C^{+j} \right) = \varphi_j (N_0, C) - \varphi_j \left(N_0^{-i}, C^{+i} \right).$$

3 The bargaining mechanism

For each *mcstp* (N_0, C), we introduce the bargaining mechanism $B(N_0, C)$. We prove that there exists a unique payoff allocation associated with the subgame perfect Nash equilibria of $B(N_0, C)$. This payoff allocation coincides with $-\varphi(N_0, C)$. Thus, we have obtained a fair outcome through a non-cooperative mechanism.

The objective of a fair rule is to divide the cost associated with an *mt*. We believe that a natural approach is to decide, for each arc of the *mt*, which part of the cost should be paid by each agent.

We present a bargaining mechanism for constructing a tree and dividing the cost. The basic idea is very simple, we ask each agent the amount he is willing to pay for an arc to be constructed. We apply this idea following a sequential protocol. Initially nobody is connected to the source. At each stage, at least one agent connects to the source. Unconnected agents can connect to the source through the agents connected in previous stages.

We explain it in the following example.

Example 1. Let (N_0, C) be such that $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 10 & 90 \\ 10 & 0 & 2 \\ 90 & 2 & 0 \end{pmatrix}$$

In this example the *mt* is $t = \{(0,1), (1,2)\}$ and $m(N_0, C) = 12$.

We present a bargaining mechanism for constructing a tree sequentially. At each step, at least an arc is constructed.

We first decide the agent who connects to the source. We ask to each agent, say agent i, the amount he is willing to pay to the other agent, say j, if agent j connects to the source. Let x^1 $\left(x^2\right)$ denote the amount offered by agent 1 (2). This amount must be non-negative.

Assume $x^1 > x^2$. This means that agent 1 offers agent 2 more than agent 2 offers to agent 1. Thus, agent 2 is selected and he must connect to the source. If $x^1 < x^2$, agent 1 is selected. If $x^1 = x^2$, each agent is selected with probability 0.5.

At this point there is a status quo in which agent 2 connects to the source (*i.e.* the arc $(0, 2)$ is constructed), agent 1 pays x^1 to agent 2 and the rest of the cost of arc $(0, 2)$ is paid by agent 2.

Now agent 2 can make a proposal (t, y) to agent 1, where t is a tree and $y = (y_1, y_2)$ satisfies $y_1 + y_1 = c(N_0, C, t)$.

If agent 1 accepts the proposal, the tree t is constructed, each agent i pays y_i, and the bargaining mechanism finishes.

If agent 1 rejects the proposal, the status quo is implemented. In the next stage agent 1 connects to the source directly or through agent 2.

A natural question that arises is the following: what happens if we directly implement the status quo? This means that the selected agent must connect to the source and he cannot make a proposal.

Assume $x^1 = 20$. This offer seems to be extremely bad because if agent 2 is selected, agent 1 must pay 20 to agent 2. Later, agent 1 can connect to the source through agent 2 by paying 2. This means that he must pay 22. Nevertheless if $x^1 = 0$, in the worst case agent 1 connects to the source directly paying only 10.

We now argue that offering $x^1 = 20$ can be a strategic offer for agent 1. If $x^2 < 20$, agent 2 must connect to the source and agent 1 pays 20 to agent 2. Thus, agent 2 pays $90 - 20 = 70$. But if $x^2 = 21$, agent 1 connects to the source. Later, agent 2 can connect to the source through agent 1 paying the cost of arc $(1, 2)$. In this case agent 2 pays $21 + 2 = 23$.

Thus, 20 can be a good offer for agent 1 because agent 2 can react offering him a larger amount. In this case, agent 1 will not pay 20.

This example suggests that if we directly implement the status quo, agents in a good situation (with relatively small connection costs to the source) can take advantage of the agents in a bad situation by offering them very high and unreasonable amounts. Since we try to obtain fair outcomes, we do not directly implement the status quo.

We allow the agents to make proposals in order to force them to offer something reasonable. Now, if $x^1 = 20$ and $x^2 < 20$, agent 2 is selected. Agent 2 proposes (t, y) where t is the mt and $y = (12, 0)$. If agent 1 accepts, he pays 12. If he rejects, he pays 22. Thus, his unreasonable offer goes against him.

In the next section we discuss this issue in a more formal way, obtaining some theoretical results.

We now introduce the bargaining procedure $B(N_0, C)$ for the general case.

If there is only one agent ($N = \{i\}$), agent i connects to the source and pays c_{0i}. Assume we have defined the bargaining procedure for $n-1$ agents. We now define it for the *mcstp* (N_0, C).

1. Stage 1. Simultaneously, each agent $i \in N$ submits a vector $x^i = \left(x^i_j\right)_{j \in N^{-i}}$ where $x^i_j \in \mathbb{R}_+$ for all $i, j \in N$, $i \neq j$. We denote $x = \left(x^i\right)_{i \in N}$.

   x^i_j is the amount that agent i is willing to pay to agent j if agent j connects to the source.

We define the *net offer* $X(i)$ of agent i as

$$X(i) = \sum_{j \in N^{-i}} x_i^j - \sum_{j \in N^{-i}} x_j^i.$$

$X(i)$ is the difference between what other agents offer to agent i and what he offers to the others.

We define the set of winners according with x as

$$W(x) = \left\{ i \in N : X(i) = \max_{j \in N} \{X(j)\} \right\}.$$

We randomly select one agent out of $W(x)$. Assume that α is chosen.

2. Stage 2. Agent α proposes (S, t, y) where $\alpha \in S \subset N$, t is a tree in (S_0, C), and $y = (y_i)_{i \in S}$ satisfies $\sum_{i \in S} y_i = c(S_0, C, t)$.

   Agent α proposes to the agents in S to construct the tree t and to divide the cost according with y.

3. Stage 3. Sequentially, each agents in $S^{-\alpha}$ either accepts or rejects (S, t, y). Two cases are possible.

   (a) Some agent in $S^{-\alpha}$ rejects (S, t, y).
   Each agent $i \in N^{-\alpha}$ pays x_α^i to agent α, who connects to the source. Agent α's payoff is $-c_{0\alpha} + \sum_{i \in N^{-\alpha}} x_\alpha^i$.
   The agents in $N^{-\alpha}$ continue bargaining among them but now they can connect to the source directly or through agent α, *i.e.* agents of $N^{-\alpha}$ play $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$. The payoff for each agent $i \in N^{-\alpha}$ is

   $$-x_\alpha^i + p_i\left(B\left(N_0^{-\alpha}, C^{+\alpha}\right)\right)$$

   where $p_i\left(B\left(N_0^{-\alpha}, C^{+\alpha}\right)\right)$ denotes agent i's payoff in $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

   (b) All the agents in $S^{-\alpha}$ accept (S, t, y).
   The tree t is constructed. The payoff for each agent $i \in S$ is $-y_i$.
   The agents in $N \setminus S$ continue bargaining among them but now they can connect to the source directly or through agents in S, *i.e.* they play $B\left((N \setminus S)_0, C^{+S}\right)$.
   The payoff for each agent $i \in N \setminus S$ is $p_i\left(B\left((N \setminus S)_0, C^{+S}\right)\right)$.

Remark 1. In Stage 3 we say that agents of $S^{-\alpha}$, sequentially, either accept or reject (S, t, y). We do not specify the order because the results of the paper are independent of the order. The only relevant issue is that agents answer sequentially.

Nash Equilibria (NE) and Subgame Perfect Nash Equilibria ($SPNE$) are well-known concepts. Next proposition shows that it is possible to find an $SPNE$ in $B(N_0, C)$.

Proposition 1. There exists an $SPNE$ for $B(N_0, C)$ whose payoff allocation coincides with $-\varphi(N_0, C)$.

Proof. See the Appendix.

We now describe, in the *mcstp* explained in Example 1, the behavior of agents when they play $s = (s_1, s_2)$ as in the proof of Proposition 1.

Stage 1. Both agents submit 4 $\left(x_2^1 = x_1^2 = 4\right)$. Hence, $X(1) = X(2) = 0$, $W(x) = N$, and each agent is selected with probability 0.5.

Stage 2. Agent α proposes (N, t, y) where $t = \{(0,1), (1,2)\}$ and $y = (6, 6)$.

Stage 3. The other agent accepts this proposal.

Assume that agent 1 is selected. Should his proposal be accepted, his payoff is -6. Should his proposal be rejected, agent 2 pays 4 to agent 1 and agent 1 connects to the source. Thus, agent 1's payoff is $-10 + 4 = -6$.

Assume that agent 2 is selected. Should his proposal be accepted, his payoff is also -6. However, should his proposal be rejected, agent 1 pays 4 to agent 2 and agent 2 connects to the source. Thus, agent 2's payoff is -86.

The situation of agents 1 and 2 in this $SPNE$ is completely asymmetric. Agent 1 can guarantee himself a payoff of -6, independently of what agent 2 does. This is not the case for agent 2 because his worst-case payoff is -86.

$B(N_0, C)$ can have several $SPNE$. In Example 1, let $s' = (s'_1, s'_2)$ be such that s'_2 is the strategy defined in the proof of Proposition 1, and s'_1 is defined as follows:

Stage 1. $x_2^1 = 4$.

Stage 2. If $\alpha = 1$, agent 1 proposes (N, t, y) where $t = \{(0,1), (1,2)\}$ and $y = (0, 12)$.

Stage 3. Agent 1 accepts y if and only if $y_1 \leq x_2^1 + 2$.

If agent 1 rejects agent 2's proposal, then he connects to the source through agent 2 and he pays the cost of arc $(1, 2)$. If agent 1 accepts agent 2's proposal, then $B(N_0, C)$ finishes.

It is not difficult to check that s' is an $SPNE$ of $B(N_0, C)$. Moreover, the final payoff allocation is $u(s') = (-6, -6)$.

If agents play s', with probability 0.5 agent 1 is selected, he proposes (N, t, y) where t is the *mt* and $y = (0, 12)$, and his proposal is rejected. Thus, in $B(N_0, C)$ exist $SPNE$ paths with probability larger than 0 in which some Stage 2 proposals are rejected.

However, Proposition 2 guarantees that there exists a unique equilibrium payoff allocation.

Proposition 2. In any $SPNE$ of $B(N_0, C)$, the final payoff allocation is $-\varphi(N_0, C)$.

Proof. See the Appendix.

The next theorem is a trivial consequence of Propositions 1 and 2.

Theorem 1. Given an *mcstp* (N_0, C), $B(N_0, C)$ has $SPNE$. Moreover, the payoff allocation for each $SPNE$ coincides with $-\varphi(N_0, C)$.

Given a strategy profile $s = (s_i)_{i\in N}$, we say that s is an Strong Nash Equilibria (SNE) if for all $T \subset N$ we cannot find $(s'_i)_{i\in T}$ such that $u_j\left((s_i)_{i\in N\setminus T}, (s'_i)_{i\in T}\right) > u_j(s)$ for all $j \in T$.

We now present a kind of equilibria, which is even stronger than SNE. We say that s is a Very Strong Nash Equilibria ($VSNE$) if for all $T \subset N$ and all $(s'_i)_{i\in T}$, $\sum\limits_{j\in T} u_j\left((s_i)_{i\in N\setminus T}, (s'_i)_{i\in T}\right) \leq \sum\limits_{j\in T} u_j(s)$.

Notice that $VSNE \Rightarrow SNE \Rightarrow NE$.

In the next theorem we prove that $B(N_0, C)$ has $VSNE$ and hence SNE.

Theorem 2. There exists a $VSNE$ for $B(N_0, C)$ whose payoff allocation coincides with $-\varphi(N_0, C)$.

Proof. See the Appendix.

4 Small modifications of the bargaining mechanism

In this section we discuss some small modifications of $B(N_0, C)$.

4.1 Modification 1

We modify $B(N_0, C)$ in such a way that the agent chosen at Stage 1 (α) only can make proposals to the grand coalition N. Namely, let $B^1(N_0, C)$ be such that

1. Stage 1. As in $B(N_0, C)$.
2. Stage 2. As in $B(N_0, C)$ but $S = N$.
3. Stage 3. As in $B(N_0, C)$.

Propositions 1 and 2 and Theorem 1 also hold true for $B^1(N_0, C)$. In fact, the proofs of Propositions 1 and 2 are simpler under $B^1(N_0, C)$ than under $B(N_0, C)$.

4.2 Modification 2

We modify Stage 2 allowing agent α to connect to the source receiving what the other agents offer to him in Stage 1. Namely, let $B^2(N_0, C)$ be such that

1. Stage 1. As in $B(N_0, C)$.
2. Stage 2. Agent α has two options. He can connect to the source or he can propose (S, t, y) as in $B(N_0, C)$.

   If he connects to the source, each agent $i \in N^{-\alpha}$ pays x_α^i to agent α. Moreover, agents of $N^{-\alpha}$ play $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

   If he proposes (S, t, y) we go to Stage 3.
3. Stage 3. As in $B(N_0, C)$.

It is not difficult to see that Propositions 1 and 2 and Theorem 1 also hold true for $B^2(N_0, C)$.

4.3 Modification 3

Assume that α must connect to the source receiving what the other agents offer to him in Stage 1. Namely, let $B^3(N_0, C)$ be such that

1. Stage 1. As in $B(N_0, C)$.
2. Stage 2. Each agent $i \in N^{-\alpha}$ pays x_α^i to agent α, who connects to the source. Agent α's payoff is $-c_{0\alpha} + \sum\limits_{i \in N^{-\alpha}} x_\alpha^i$.
3. Stage 3. The agents in $N^{-\alpha}$ play $B^3\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Agent i's payoff, $i \in N^{-\alpha}$, is
$$-x_\alpha^i + p_i\left(B^3\left(N_0^{-\alpha}, C^{+\alpha}\right)\right)$$
   where $p_i\left(B^3\left(N_0^{-\alpha}, C^{+\alpha}\right)\right)$ is agent i's payoff in $B^3\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

Before introducing $B(N_0, C)$, we already discussed this bargaining mechanism in Example 1. We now discuss $B^3(N_0, C)$ more carefully.

First, we must note that $B^3(N_0, C)$ could not have NE.

Consider the *mcstp* (N_0, C) of Example 1.

Assume that agents submit $\left(x_2^1, x_1^2\right)$ in Stage 1. It is straightforward to prove that the final payoff for agents 1 and 2 are:

$$u_1\left(x_2^1, x_1^2\right) = \begin{cases} -10 + x_1^2 & \text{if } x_2^1 < x_1^2 \\ -2 - x_2^1 & \text{if } x_2^1 > x_1^2 \\ \frac{1}{2}\left(-10 + x_1^2\right) + \frac{1}{2}\left(-2 - x_2^1\right) & \text{if } x_2^1 = x_1^2 \end{cases} \quad \text{and}$$
$$u_2\left(x_2^1, x_1^2\right) = \begin{cases} -2 - x_1^2 & \text{if } x_2^1 < x_1^2 \\ -90 + x_2^1 & \text{if } x_2^1 > x_1^2 \\ \frac{1}{2}\left(-2 - x_1^2\right) + \frac{1}{2}\left(-90 + x_2^1\right) & \text{if } x_2^1 = x_1^2. \end{cases}$$

We now prove that $B^3(N_0, C)$ has no NE. We consider several cases:

- $x_2^1 < x_1^2$. Assume that agent 2 submits y with $x_2^1 < y < x_1^2$. Thus,

$$u_2\left(x_2^1, y\right) = -2 - y > -2 - x_1^2 = u_2\left(x_2^1, x_1^2\right).$$

- $x_2^1 > x_1^2$. It is not a NE. It is analogous to the previous case.
- $x_2^1 = x_1^2 \leq 4$. In this case, $u\left(x_2^1, x_1^2\right) = (-6, -46)$. Assume that agent 2 submits 5. Hence,

$$u_2\left(x_2^1, 5\right) = -2 - 5 = -7 > -46 = u_2\left(x_2^1, x_1^2\right).$$

- $x_2^1 = x_1^2 > 4$. In this case, $u\left(x_2^1, x_1^2\right) = (-6, -46)$. Assume that agent 1 submits 4. Hence,

$$u_1\left(4, x_1^2\right) = -10 + x_1^2 > -6 = u_1\left(x_2^1, x_1^2\right).$$

There is no equilibria in $B^3(N_0, C)$ because each agent i can submit any vector $x^i = \left(x_j^i\right) \in \mathbb{R}_+^{N^{-i}}$. Assume that in Stage 1 we restrict agent i's strategies so that player i can only submit a vector from a finite subset of $\mathbb{R}_+^{N^{-i}}$. For instance, for each $i \in N$, $j \in N^{-i}$, $0 \leq x_j^i \leq m(N_0, C)$ and $100x_j^i \in \mathbb{Z}_+$. This means that, if the offered amounts are measured in dollars, the agents must offer an integer number of cents.

Under this quite realistic assumption there are $SPNE$ in $B^3(N_0, C)$. However, some $SPNE$ payoff allocations are unfair. The agents who are close to the source may take advantage of the agents who are far away from the source. We see it in Example 1.

Assume that $x_j^i \in \mathbb{Z}_+$ for each $i, j \in N$.

Consider the strategy profile (s_1, s_2) in Example 1 as follows: Stage 1. $x_2^1 = 20$, $x_1^2 = 21$. Stage 3. Agent $i \neq \alpha$ connects to agent α.

Under (s_1, s_2): In Stage 1, $\alpha = 1$. In Stage 2, agent 2 pays 21 to agent 1 and agent 1 connects to the source. In Stage 3 agent 2 connects to agent 1. Thus,

$$u(s_1, s_2) = (21 - 10, -21 - 2) = (11, -23).$$

We now prove that (s_1, s_2) is an $SPNE$ of $B^3(N_0, C)$.

- Of course, (s_1, s_2) induces an NE in the subgame of Stage 3.
- If agent 1 plays s_1' with $x_2'^1 < 20$, $u_1(s_1', s_2) = 11$.
- If agent 1 plays s_1' with $x_2'^1 = 21$, $u_1(s_1', s_2) = \frac{1}{2}11 + \frac{1}{2}(-21 - 2) = -6$.
- If agent 1 plays s_1' with $x_2'^1 > 21$, $u_1(s_1', s_2) = -x_2'^1 - 2 < 11$.
- If agent 2 plays s_2' with $x_1'^2 < 20$, $u_2(s_1, s_2') = 20 - 90 = -70$.
- If agent 2 plays s_2' with $x_1'^2 = 20$, $u_2(s_1, s_2') = \frac{1}{2}(-22) + \frac{1}{2}(-70) = -46$.

- If agent 2 plays s'_2 with $x'^2_1 > 21$, $u_2(s_1, s'_2) = -x'^2_1 - 2 < -23$.

Consider the strategy profile (s_1, s_2) where: Stage 1. $x^1_2 = x$, $x^2_1 = x + 1$. Stage 3. For each $i \in N$, $i \neq \alpha$, agent i connects to agent α.

It is not difficult to prove that (s_1, s_2) is an $SPNE$ of $B^3(N_0, C)$ for all $x = 3, ..., 43$.

4.4 Modification 4

Assume that a group of agents must select a project from a given set. Pérez-Castrillo and Wettstein (2002) [14] proposed the multibidding mechanism. In the multibidding mechanism each agent i announces k bids, one for each project, that are constrained to sum up zero. Furthermore, he chooses one of the projects. The project with the highest aggregate bid is chosen as the winner. In case of a tie, the winning project is randomly chosen among those with the highest aggregate bid that have been selected by at least one agent. Once the winning project is identified, the bids corresponding to it are paid (or received).

Pérez-Castrillo and Wettstein (2002) [14] proved that the equilibria of this mechanism have interesting properties. For instance, the selected project is efficient. The efficiency of the project crucially depends on the following facts. Agents must select one of the projects. The project is selected among those selected by at least one agent. Without this facts, the selected project could be inefficient.

If we come back to $B^3(N_0, C)$ we realize that there are some similarities with the multibidding mechanism. In both cases agents select something by submitting bids. In $B^3(N_0, C)$ agents select the first agent who connects to the source.

Of course, there are important differences. We mention two. Firstly, in $B^3(N_0, C)$ each agent's bids are positive and unrelated. In the multibidding mechanism, the bids could be positive or negative but they are related (they sum up zero). Secondly, in $B^3(N_0, C)$ the agents do not choose any agent for connecting to the source, whereas in the multibidding mechanism the agents select one project.

We believe that in *mcstp* it is more natural to force agents to submit positive bids. Nevertheless, it makes sense that agents select the agent who connects to the source.

We now modify $B^3(N_0, C)$ allowing the agents to select one agent in Stage 1. We then select α randomly among the agents with the highest aggregate bid who have been selected for at least one agent.

Formally, we define $B^4(N_0, C)$ as follows:

1. Stage 1. Simultaneously, each agent $i \in N$ submits a pair (x^i, i^*) where $x^i = (x^i_j)_{j \in N^{-i}} \in \mathbb{R}^{N^{-i}}_+$ and $i^* \in N$.

   x^i_j has the same interpretation as in $B(N_0, C)$. i^* is the agent that i prefers to be first connected to the source.

x, $X(i)$ with $i \in N$, and $W(x)$ are defined as in $B(N_0, C)$. Moreover,

$$W^*(x) = \{j \in W(x) : j = i^* \text{ for some } i \in N\}.$$

We randomly select $\alpha \in W^*(x)$ when $W^*(x) \neq \varnothing$ and $\alpha \in W(x)$ when $W^*(x) = \varnothing$.

2. Stage 2. As in $B^3(N_0, C)$.
3. Stage 3. As in $B^3(N_0, C)$.

In the next proposition we prove that $-\varphi(N_0, C)$ can be obtained as the payoff of an $SPNE$ of $B^4(N_0, C)$.

Proposition 3. There exists an $SPNE$ of $B^4(N_0, C)$ whose payoff coincides with $-\varphi(N_0, C)$.

Proof. See the Appendix.

Nevertheless, there are $SPNE$ with a payoff different from $-\varphi(N_0, C)$.

Consider, in Example 1, the strategy combination $s = (s_1, s_2)$ where $x_2^1 = x_1^2 = 20$ and $1^* = 2^* = 1$. If the agents play s, agent 1 connects to the source and agent 2 pays 20 to agent 1. Later, agent 2 connects to agent 1. Thus, $u(s) = (10, -22)$. This means that agent 2 pays all the cost of the tree constructed $(\{(0,1), (0,2)\})$ plus 10 units to agent 1.

We now prove that s is an $SPNE$ of $B^4(N_0, C)$. Because of the definition of $B^4(N_0, C)$, it is enough to prove that if agent i submits $s_i' = \left(x_j'^i, i'^*\right)$ in Stage 1, agent i does not improve. We consider several cases.

- If agent 1 submits s_1' with $x_2'^1 > 20$, $u_1(s_1', s_2) = -2 - x_2'^1 < 10$.
- If agent 1 submits s_1' with $x_2'^1 < 20$, $u_1(s_1', s_2) = 10$.
- If agent 1 submits s_1' with $x_2'^1 = 20$ and $1'^* = 2$, $u_1(s_1', s_2) = \frac{1}{2}(20 - 10) + \frac{1}{2}(-20 - 2) = -6 < 10$.
- If agent 2 submits s_2' with $x_1'^2 > 20$, $u_2(s_1, s_2') = -2 - x_1'^2 < -22$.
- If agent 2 submits s_2' where $x_1'^2 < 20$, $u_2(s_1, s_2') = -70 < -22$.
- If agent 2 submits s_2' where $x_1'^2 = 20$ and $2'^* = 2$, $u_2(s_1, s_2') = \frac{1}{2}(20 - 90) + \frac{1}{2}(-20 - 2) = -46 < -22$.

The idea behind this $SPNE$ is the following. Agent 2's connection cost to the source is very large. He prefers agent 1 to connect first. Knowing that, agent 1 offers 20 to agent 2. Agent 2 must offer the same to agent 1 because otherwise he will be forced to connect to the source, which is worse. If both of them submit 20, both of them prefer agent 1 to be first connected to the source. Thus, both of them select agent 1 in Stage 1.

Using similar arguments to those used before we can prove that $s = (s_1, s_2)$ where $x_2^1 = x_1^2 = x$, $4 \le x \le 44$, and $1^* = 2^* = 1$ is an $SPNE$ of $B^4(N_0, C)$. Moreover, $u(s) = (10 - x, x + 2)$.

5 A non-cooperative approach to the Shapley value of Bird's game

Pérez-Castrillo and Wettstein (2001) [13] defined, in TU games, the bidding mechanism. They proved that the unique $SPNE$ payoff allocation of the bidding mechanism coincides with the Shapley value. In this section we define $B^5(N_0, C)$ applying the ideas of the bidding mechanism to *mcstp*. We prove that $B^5(N_0, C)$ has $SPNE$. Moreover, the payoff allocation associated with each $SPNE$ coincides with $-Sh(N, v_C)$.

Given the *mcstp* (N_0, C) we define the non-cooperative game $B^5(N_0, C)$ as follows:

1. Stage 1. Simultaneously, each agent $i \in N$ submits a vector $x^i = \left(x^i_j\right)_{j \in N^{-i}} \in \mathbb{R}^{N^{-i}}$.

   $x^i_j \in \mathbb{R}$ represents the payoff that agent i is willing to pay to agent j in order to be the proposer.

   x, $X(i)$ with $i \in N$, $W(x)$ and α are defined as in $B(N_0, C)$.

   Agent α pays x^α_i to each agent $i \in N^{-\alpha}$.

2. Stage 2. Agent α proposes (t, y) where t is a tree in (N_0, C), $y = (y_i)_{i \in N}$, and $\sum_{i \in N} y_i = c(N_0, C, t)$. This means that α proposes to construct the tree t, in which each agent $i \in N$ will pay y_i.

3. Stage 3. Sequentially, the agents in $N^{-\alpha}$ either accept or reject the offer. Two cases are possible.

   (a) Some agent in $N^{-\alpha}$ rejects (t, y).
   Agent α connects to the source. His payoff is $-c_{0\alpha} - \sum_{i \in N^{-\alpha}} x^\alpha_i$.
   The agents in $N^{-\alpha}$ play $B^5\left(N_0^{-\alpha}, C\right)$. Agent i's payoff, $i \in N$, is

   $$x^\alpha_i + p_i\left(B^5\left(N_0^{-\alpha}, C\right)\right)$$

   where $p_i\left(B^5\left(N_0^{-\alpha}, C\right)\right)$ is agent i's payoff in $B^5\left(N_0^{-\alpha}, C\right)$.

   (b) All the agents in $N^{-\alpha}$ accept (t, y).
   The tree t is constructed. Agent α's payoff is $-\sum_{i \in N^{-\alpha}} x^\alpha_i - y_\alpha$. Each agent i's payoff, $i \in N^{-\alpha}$, is $x^\alpha_i - y_i$.

We now discuss the main differences between $B(N_0, C)$ and $B^5(N_0, C)$.

Stage 1. In both mechanisms each agent $i \in N$ submits a vector $\left(x^i_j\right)_{j \in N^{-i}}$. However, agents are submitting very different things. In $B(N_0, C)$, x^i_j represents the payoff that agent i is willing to pay to agent j so that agent j connects to the source. In $B^5(N_0, C)$, x_j^i represents the payoff that agent i is willing to pay to agent j in order to be the proposer.

In $B(N_0, C)$, x_j^i must be non-negative. In $B^5(N_0, C)$, x_j^i could be negative. This is fundamental. If we force $x_j^i \geq 0$ for all $i, j \in N$ in $B^5(N_0, C)$, the result stated in Proposition 4 below does not hold.

Stage 2. In $B(N_0, C)$, agent α can make a proposal to a subset of N. In $B^5(N_0, C)$ the proposal must be made to all agents of N. As me mention before, the result of Theorem 1 also holds for $B(N_0, C)$ if the proposal is made to all agents of N. In $B^5(N_0, C)$, Proposition 4 below does not hold if we allow agent α to make proposals to a proper subset of N.

Stage 3. Assume that agent α's proposal is rejected. In both mechanisms agent α connects to the source. In $B(N_0, C)$ agents of $N^{-\alpha}$ can connect to the source through agent α. In $B^5(N_0, C)$ they cannot. Both mechanisms are defined in order to be coherent with the previous stages. In $B(N_0, C)$, x_α^i represents the amount that agent i is willing to pay to agent α so that agent α connects to the source. Thus, the agents in $N^{-\alpha}$ can connect to the source through agent α because they are paying for that. In $B^5(N_0, C)$, x_i^α represents the payoff that agent α is willing to pay to agent i in order to be the proposer. Agent α is the proposer because he paid for that. The agents in $N^{-\alpha}$ should not take advantage of the connection of agent α.

We have the following result for $B^5(N_0, C)$.

Proposition 4. Given an *mcstp* (N_0, C), $B^5(N_0, C)$ has $SPNE$. Moreover, the payoff allocation for each $SPNE$ coincides with $-Sh(N, v_C)$.

Proof. See the Appendix.

6 Appendix

In this section we prove the main results of the paper.

6.1 Proof of Proposition 1

We proceed by induction on the number of agents n. If $n = 1$ the result is trivial. Assume that for all $n \leq p$ there exists an $SPNE$ whose payoff coincide with $-\varphi(N_0, C)$. We will prove it when $n = p + 1$.

We define the strategy profile $s = (s_i)_{i \in N}$ in $B(N_0, C)$ as follows:

Stage 1. For all $i \in N$ and $j \in N^{-i}$, $x_j^i = \varphi_i(N_0, C) - \varphi_i\left(N_0^{-j}, C^{+j}\right)$.

Stage 2. Agent α proposes (N, t, y) where t is an mt in (N_0, C) and $y_i = x_\alpha^i + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $i \in N^{-\alpha}$.

Stage 3. Each agent $i \in S \setminus \{\alpha\}$ accepts y if and only if $y_i \leq x_\alpha^i + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

If (S,t,y) is rejected, $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$ is played. By induction hypothesis we know that there exists an $SPNE$ $s^{+\alpha} = \left(s_i^{+\alpha}\right)_{i\in N^{-\alpha}}$ in $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$ such that $u\left(s^{+\alpha}\right)$, the payoff allocation associated with $s^{+\alpha}$ in $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$, coincides with $-\varphi\left(N_0^{-\alpha}, C^{+\alpha}\right)$. We define s as $s^{+\alpha}$ in $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

If (S,t,y) is accepted, the agents in $N\backslash S$ play $B\left((N\setminus S)_0, C^{+S}\right)$. Under the induction hypothesis we know that there exists an $SPNE$ $s^{+S} = \left(s_i^{+S}\right)_{i\in N\backslash S}$ in $B\left((N\setminus S)_0, C^{+S}\right)$ such that $u\left(s^{+S}\right) = -\varphi\left((N\setminus S)_0, C^{+S}\right)$. We define s as s^{+S} in $B\left((N\setminus S)_0, C^{+S}\right)$.

We first prove that s is well-defined. We only need to prove that $x_j^i \geq 0$ for all $i,j\in N$, $i\neq j$.

Take $i,j\in N$, $i\neq j$. Let C' be such that

$$c'_{kl} = \begin{cases} c_{kl} & \text{if } k,l\in N_0\setminus\{j\} \\ c_{0l}^{+j} & \text{if } k=j. \end{cases}$$

For all $l\in N_0^{-j}$, $c'_{jl} = c_{0l}^{+j} = \min\{c_{0l}, c_{jl}\} \leq c_{jl}$. Thus, $C'\leq C$. Since φ satisfies SOL, $\varphi_i(N_0,C') \leq \varphi_i(N_0,C)$ and hence, $x_j^i \geq \varphi_i(N_0,C') - \varphi_i\left(N_0^{-j}, C^{+j}\right)$.

Notice that $c'_{0j} = c_{0j}^{+j} = \min\{c_{0j}, c_{jj}\} = c_{jj} = 0$. Thus, $m(N_0,C') = m\left(N_0^{-j}, C'\right)$ and $m\left(\{j\}_0, C'\right) = 0$. Since φ satisfies SEP, $\varphi_i(N_0,C') = \varphi_i\left(N_0^{-j}, C'\right)$.

Since $c'_{kl} = c_{kl}^{+j}$ for all $k,l\in N^{-j}$, we have $\varphi_i(N_0,C') = \varphi_i\left(N_0^{-j}, C^{+j}\right)$ and hence $x_j^i\geq 0$.

We now prove that s is an $SPNE$ of $B(N_0,C)$ whose payoff allocation coincides with $-\varphi(N_0,C)$. We prove several claims:

Claim 1. Under s, $X(i)=0$ for all $i\in N$ and $u(s) = -\varphi(N_0,C)$.

Assume that agents play s. For each $i\in N$,

$$\begin{aligned} X(i) &= \sum_{j\in N^{-i}} x_i^j - \sum_{j\in N^{-i}} x_j^i \\ &= \sum_{j\in N^{-i}} \left[\left(\varphi_j(N_0,C) - \varphi_j\left(N_0^{-i}, C^{+i}\right)\right) - \left(\varphi_i(N_0,C) - \varphi_i\left(N_0^{-j}, C^{+j}\right)\right)\right]. \end{aligned}$$

Since φ satisfies EC we conclude that $X(i)=0$. Thus, α is randomly selected among all the agents.

The final payoff for each agent $i\in N$ can be written as $\frac{1}{n}\sum_{j\in N} p_i(j)$ where $p_i(j)$ is the payoff obtained by agent i when $\alpha = j$.

We now prove that $p_i(j) = -\varphi_i(N_0,C)$ for all $j\in N$. Since agents play s, agent α proposes (N,t,y) where for each $i\in N^{-\alpha}$, $y_i = x_\alpha^i + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) = \varphi_i(N_0,C)$. Moreover, all the agents in $N^{-\alpha}$ accept (N,t,y).

If $j \neq i$,

$$p_i(j) = -x_j^i - \varphi_i\left(N_0^{-j}, C^{+j}\right) = -\varphi_i(N_0, C).$$

If $j = i$,

$$\begin{aligned} p_i(i) &= -y_i = -m(N_0, C) + \sum_{k \in N^{-i}} y_k \\ &= -m(N_0, C) + \sum_{k \in N^{-i}} \varphi_k(N_0, C) = -\varphi_i(N_0, C). \end{aligned}$$

Claim 2. For all $S \subsetneq N$, s induces an $SPNE$ in $B\left((N \setminus S)_0, C^{+S}\right)$.
It is a trivial consequence of the induction hypothesis.

Claim 3. For all $S \subset N$ and $i \in S$,

$$\begin{aligned} m(N_0, C) &\leq m(S_0, C) + m\left((N \setminus S)_0, C^{+S}\right) \\ &\leq m(S_0, C) + \sum_{j \in N \setminus S} \varphi_j\left(N_0^{-i}, C^{+i}\right). \end{aligned}$$

Let $\left(N_0^{-i}, C'\right)$ such that

$$c'_{jk} = \begin{cases} c_{jk} & \text{if } j,k \in N \setminus S \\ c_{0k}^{+S} & \text{if } j \in S_0^{-i},\ k \in N \setminus S \\ 0 & \text{if } j,k \in S_0^{-i}. \end{cases}$$

It is straightforward to prove that $m\left(S_0^{-i}, C'\right) = 0$ and moreover

$$m\left(N_0^{-i}, C'\right) = m\left((N \setminus S)_0, C'\right).$$

Since $\left((N \setminus S)_0, C'\right) = \left((N \setminus S)_0, C^{+S}\right)$ and φ satisfies SEP, $\varphi_j\left((N \setminus S)_0, C^{+S}\right) = \varphi_j\left(N_0^{-i}, C'\right)$ for all $j \in N \setminus S$.

Since $C' \leq C^{+i}$ and φ satisfies SOL, $\varphi_j\left(N_0^{-i}, C'\right) \leq \varphi_j\left(N_0^{-i}, C^{+i}\right)$ for all $j \in N \setminus S$. Hence,

$$m\left((N \setminus S)_0, C^{+S}\right) = \sum_{j \in N \setminus S} \varphi_j\left((N \setminus S)_0, C^{+S}\right) \leq \sum_{j \in N \setminus S} \varphi_j\left(N_0^{-i}, C^{+i}\right).$$

Let t and t' be two mt in (S_0, C) and $\left((N \setminus S)_0, C^{+S}\right)$, respectively. Because of the definition of C^{+S}, it is possible to find a graph g in (N_0, C) such that $m\left((N \setminus S)_0, C^{+S}\right) = c(N_0, C, g)$ and $t \cup g$ is a tree in (N_0, C). Thus,

$$\begin{aligned} m(N_0, C) &\leq c(N_0, C, t \cup g) = c(S_0, C, t) + c(N_0, C, g) \\ &= m(S_0, C) + m\left((N \setminus S)_0, C^{+S}\right) \\ &\leq m(S_0, C) + \sum_{j \in N \setminus S} \varphi_j\left(N_0^{-i}, C^{+i}\right). \end{aligned}$$

Claim 4. The strategy profile s induces a NE in the subgames that begin in Stage 3.

Let (S,t,y) be α's proposal in Stage 2 and $i \in S^{-\alpha}$. We prove that agent i does not improve by deviating.

Assume that (S,t,y) is rejected. Agent i's final payoff is

$$-x_\alpha^i + p_i\left(B\left(N_0^{-\alpha}, C^{+\alpha}\right), s\right)$$

where $p_i\left(B\left(N_0^{-\alpha}, C^{+\alpha}\right), s\right)$ is agent i's payoff in $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$ when the agents in $N \setminus \{i, \alpha\}$ play s.

Under the induction hypothesis, $p_i\left(B\left(N_0^{-\alpha}, C^{+\alpha}\right), s\right) \leq -\varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Thus, if (S,t,y) is rejected, agent i obtains at most

$$-x_\alpha^i - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right).$$

We consider several cases.

1. There exists $j \in S \setminus \{\alpha, i\}$ such that $y_j > x_\alpha^j + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

   Since j plays s_j, agent j rejects (S,t,y). If agent i plays s_i he obtains $-x_\alpha^i - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$, which is the maximum he can obtain when (S,t,y) is rejected. Thus, agent i cannot improve.

2. $y_j \leq x_\alpha^j + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $j \in S \setminus \{\alpha, i\}$ and $y_i > x_\alpha^i + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

   All the agents in $S \setminus \{\alpha, i\}$ accept (S,t,y). If agent i plays s_i, he rejects (S,t,y) and plays an $SPNE$ of $\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Thus, his payoff is $-x_\alpha^i - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

   Assume that agent i deviates. Two cases are possible.

   (a) He rejects (S,t,y). In this case, he obtains at most $-x_\alpha^i - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Thus, he cannot improve.

   (b) He accepts (S,t,y). In this case, his payoff is $-y_i$, which is smaller than $-x_\alpha^i - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Thus, he does not improve.

3. $y_j \leq x_\alpha^j + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $j \in S^{-\alpha}$.

   All the agents in $S \setminus \{\alpha, i\}$ accept (S,t,y). If agent i plays s_i, he accepts (S,t,y). Thus, his payoff is $-y_i$.

   Assume that agent i does not play s_i. If he accepts (S,t,y), his payoff is also $-y_i$. If he rejects (S,t,y), he obtains at most $-x_\alpha^i - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Thus, he does not improve.

Claim 5. The strategy profile s induces a NE in the subgames that begin in Stage 2.

By playing s, agent α gets

$$-m\left(N_0, C\right) + \sum_{i \in N^{-\alpha}} x_\alpha^i + \sum_{i \in N^{-\alpha}} \varphi_j\left(N_0^{-\alpha}, C^{+\alpha}\right).$$

Assume α makes a different proposal (S, t', y'). We will prove that he does not improve. We consider two cases.

1. There exists $j \in S^{-\alpha}$ such that $y'_j > x^j_\alpha + \varphi_j\left(N_0^{-\alpha}, C^{+\alpha}\right)$. In this case, α's proposal is rejected and α's payoff is $-c_{0\alpha} + \sum_{i \in N^{-\alpha}} x^i_\alpha$.

   Under Claim 3,

$$
\begin{aligned}
m(N_0, C) &\leq m(\{\alpha\}_0, C) + \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\
&= c_{0\alpha} + \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right).
\end{aligned}
$$

   Hence,

$$
-c_{0\alpha} + \sum_{i \in N^{-\alpha}} x^i_\alpha \leq -m(N_0, C) + \sum_{i \in N^{-\alpha}} x^i_\alpha + \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)
$$

   and α does not improve.

2. $y'_i \leq x^i_\alpha + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $i \in S^{-\alpha}$.
   In this case, (S, t, y) is accepted and α's payoff is

$$
\begin{aligned}
-y'_\alpha &= -c(S_0, C, t) + \sum_{i \in S^{-\alpha}} y'_i \\
&\leq -m(S_0, C) + \sum_{i \in S^{-\alpha}} \left(x^i_\alpha + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)\right).
\end{aligned}
$$

   Under Claim 3,

$$
\begin{aligned}
-y'_\alpha &\leq -m(N_0, C) + \sum_{i \in N \setminus S} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) + \sum_{i \in S^{-\alpha}} \left(x^i_\alpha + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)\right) \\
&= -m(N_0, C) + \sum_{i \in S^{-\alpha}} x^i_\alpha + \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right).
\end{aligned}
$$

   Since $x^i_\alpha \geq 0$ for all $i \in N^{-\alpha}$,

$$
-y_\alpha \leq -m(N_0, C) + \sum_{i \in N^{-\alpha}} x^i_\alpha + \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)
$$

   and α does not improve.

Claim 6. The strategy profile s induces a NE in the subgames that begin in Stage 1.

Under Claim 1, $u(s) = -\varphi(N_0, C)$.

Assume that agent i submits $x'^i = \left(x'^i_j\right)_{j \in N^{-i}}$ instead of x^i. We take $x' = \left\{x'^i, \left\{x^j\right\}_{j \in N^{-i}}\right\}$.

For all $j \in N^{-i}$,

$$\begin{aligned} X'(j) &= \sum_{k \in N^{-j}} x'^k_j - \sum_{k \in N^{-j}} x'^j_k = \sum_{k \in N \setminus \{i,j\}} x^k_j + x'^i_j - \sum_{k \in N^{-j}} x^j_k \\ &= X(j) - x^i_j + x'^i_j. \end{aligned}$$

Under Claim 1, $X(j) = 0$ for all $j \in N$. Thus, $X'(j) = x'^i_j - x^i_j$.

Since α is randomly chosen out of $W(x')$, the payoff obtained by agent i when he deviates is

$$\frac{1}{|W(x')|} \sum_{j \in W(x')} p_i(j)$$

where $p_i(j)$ denotes the payoff obtained by agent i when $\alpha = j$.

We consider two cases:

1. There exists some $j \in N^{-i}$ such that $x'^i_j > x^i_j$. In this case, $X'(j) > 0$. Moreover, for all $k \in W(x')$, $k \neq i$, we have that $X'(k) \geq X'(j) > 0$. Hence, $x'^i_k > x^i_k$ for all $k \in W(x')$, $k \neq i$.

    (a) Assume that $\alpha \neq i$. Since α plays s, he proposes (N, t, y) where $y_j = x'^j_\alpha + \varphi_j\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $j \in N^{-\alpha}$.
    All the agents in $N \setminus \{\alpha, i\}$ accept (N, t, y) because they play s. Under Claim 4,

    $$p_i(\alpha) \leq -x'^i_\alpha - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) < -x^i_\alpha - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) = -\varphi_i(N_0, C)$$

    and hence agent i is worse.

    (b) Assume that $\alpha = i$. Since the agents in N^{-i} play s, under Claim 5, $p_i(i) \leq -\varphi_i(N_0, C)$.

    Thus, agent i gets

    $$\frac{1}{|W(x')|} \sum_{j \in W(x')} p_i(j) \leq \frac{1}{|W(x')|} \sum_{j \in W(x')} \left(-\varphi_i(N_0, C)\right) = -\varphi_i(N_0, C)$$

    and does not improve.

2. $x'^i_j \leq x^i_j$ for all $j \in N^{-i}$.
    In this case,

    $$X'(j) = x'^i_j - x^i_j \leq 0$$

    for all $j \in N^{-i}$.

Since $x'^i \neq x^i$, there exists $j \in N^{-i}$ such that $x'^i_j < x^i_j$. Thus,

$$\begin{aligned} X'(i) &= \sum_{k \in N^{-i}} x'^k_i - \sum_{k \in N^{-i}} x'^i_k = \sum_{k \in N^{-i}} x^k_i - \sum_{k \in N^{-i}} x'^i_k \\ &> \sum_{k \in N^{-i}} x^k_i - \sum_{k \in N^{-i}} x^i_k = X(i) = 0, \end{aligned}$$

which means that $W(x') = \{i\}$. Under Claim 5, agent i gets, at most, $-\varphi_i(N_0, C)$.

6.2 Proof of Proposition 2

We proceed by induction on the number of agents n. If $n = 1$ the result is trivial. Assume that the result holds when $n \leq p$. We will prove it when $n = p + 1$.

Let $s = (s_i)_{i \in N}$ be an $SPNE$ for $B(N_0, C)$. We prove several claims.

Claim 1. If α's offer is rejected, each agent $i \in N^{-\alpha}$ gets $-x^i_\alpha - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$. It is an immediate consequence of the induction hypothesis.

Claim 2. If α proposes (S, t, y) such that $S = \{i_1, ..., i_q, \alpha\}$ and $y_i < x^i_\alpha + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $i \in S^{-\alpha}$, then (S, t, y) is accepted.

Assume that the agents in $S^{-\alpha}$ answer in the order $\{i_1, ..., i_q\}$. Assume that the agents in $\{i_1, ..., i_{q-1}\}$ have already accepted (S, t, y). If agent i_q accepts (S, t, y), his payoff is $-y_{i_q}$. If agent i_q rejects (S, t, y), under Claim 1, his payoff is $-x^{i_q}_\alpha - \varphi_{i_q}\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Since the agents play an $SPNE$, agent i_q accepts (S, t, y).

Assume now that the agents in $\{i_1, ..., i_{q-2}\}$ have already accepted (S, t, y). If agent i_{q-1} accepts, agent i_q must decide if he accepts or not. Since agent i_q is bound to accept (S, t, y), agent i_{q-1} will get $-y_{i_{q-1}}$ if he accepts. If he rejects, under Claim 1 he gets $-x^{i_{q-1}}_\alpha - \varphi_{i_{q-1}}\left(N_0^{-\alpha}, C^{+\alpha}\right)$. Since the agents play an $SPNE$, agent i_{q-1} accepts (S, t, y).

Repeating this argument we can conclude that all the agents in $S^{-\alpha}$ accept (S, t, y).

Claim 3. $u_i(s) \geq -\varphi_i(N_0, C)$ for all $i \in N$.

Given $i \in N$ and $\varepsilon > 0$, let s'_i be agent i's strategy defined as follows:

1. Stage 1. Agent i submits x^i with $x^i_j = \varphi_i(N_0, C) - \varphi_i\left(N_0^{-j}, C^{+j}\right)$ for all $j \in N^{-i}$.
2. Stage 2. If $\alpha = i$, agent i proposes (N, t, y) where t is an mt in (N_0, C) and for all $j \in N^{-i}$, $y_j = x^j_i + \varphi_j\left(N_0^{-i}, C^{+i}\right) - \frac{\varepsilon}{n-1}$.
3. Stage 3. If $\alpha \neq i$, agent i always rejects the proposal of agent α.

   Given $S \subset N$, $i \notin S$, s'_i coincides with s_i in $B\left((N \setminus S)_0, C^{+S}\right)$.

Let $x' = \left(x'^j\right)_{j \in N}$ be such that $x'^j = x^j$ for all $j \in N^{-i}$. It is clear that

$$u_i\left(s \setminus s'_i\right) = \frac{1}{\left|W\left(x'\right)\right|} \sum_{j \in W(x')} p_i\left(j\right)$$

where $s \backslash s'_i$ is the strategy profile that results from s after replacing agent i's strategy from s_i to s'_i.

We now prove that $u_i\left(s \setminus s'_i\right) \geq -\varphi_i\left(N_0, C\right) - \varepsilon$.

We consider two cases:

1. $i \notin W\left(x'\right)$.

   Take $j \in W\left(x'\right)$. Agent i rejects agent j's proposal. Since s'_i coincides with s_i in $B\left(N_0^{-j}, C^{+j}\right)$, under the induction hypothesis

$$p_i\left(j\right) = -x'^i_j - \varphi_i\left(N_0^{-j}, C^{+j}\right) = -\varphi_i\left(N_0, C\right).$$

   Thus, $u_i\left(s \setminus s'_i\right) = -\varphi_i\left(N_0, C\right) > -\varphi_i\left(N_0, C\right) - \varepsilon$.

2. $i \in W\left(x'\right)$.

   Following the same reasoning as in the previous cases, we know that $p_i\left(j\right) = -\varphi_i\left(N_0, C\right)$ for all $j \in W\left(x'\right) \setminus \{i\}$.

   Under Claim 2, agent i's proposal is accepted. Thus,

$$p_i\left(i\right) = -m\left(N_0, C\right) + \sum_{j \in N^{-i}} \left(x_i^j + \varphi_j\left(N_0^{-i}, C^{+i}\right) - \frac{\varepsilon}{n-1}\right).$$

   It is not difficult to see that $\sum\limits_{j \in N} X\left(j\right) = 0$. Since $i \in W\left(x'\right)$, we deduce that $X\left(i\right) \geq 0$. Hence,

$$\sum_{j \in N^{-i}} x_i^j \geq \sum_{j \in N^{-i}} x_j^i = \left(n-1\right)\varphi_i\left(N_0, C\right) - \sum_{j \in N^{-i}} \varphi_i\left(N_0^{-j}, C^{+j}\right).$$

   Thus,

$$\begin{aligned}\sum_{j \in N^{-i}} \left(x_i^j + \varphi_j\left(N_0^{-i}, C^{+i}\right)\right) \quad &\geq \quad \left(n-1\right)\varphi_i\left(N_0, C\right) \\ &\quad - \sum_{j \in N^{-i}} \left(\varphi_i\left(N_0^{-j}, C^{+j}\right) - \varphi_j\left(N_0^{-i}, C^{+i}\right)\right).\end{aligned}$$

   Since φ satisfies EC, the last expression coincides with

$$\left(n-1\right)\varphi_i\left(N_0, C\right) - \sum_{j \in N^{-i}} \left(\varphi_i\left(N_0, C\right) - \varphi_j\left(N_0, C\right)\right) = \sum_{j \in N^{-i}} \varphi_j\left(N_0, C\right).$$

Hence,

$$\begin{aligned} u_i(s \setminus s_i') &\geq \frac{1}{|W(x)|}\left(-m(N_0,C) + \sum_{j \in N^{-i}} \varphi_j(N_0,C) - \varepsilon\right) \\ &\quad - \frac{|W(x)|-1}{|W(x)|}\varphi_i(N_0,C) \\ &= \frac{1}{|W(x)|}\left(-\varphi_i(N_0,C) - \varepsilon\right) - \frac{|W(x)|-1}{|W(x)|}\varphi_i(N_0,C) \\ &= -\varphi_i(N_0,C) - \frac{\varepsilon}{|W(x)|} \geq -\varphi_i(N_0,C) - \varepsilon. \end{aligned}$$

Since s is an $SPNE$, $u_i(s) \geq u_i(s \setminus s_i')$. Thus, for all $\varepsilon > 0$, $u_i(s) \geq -\varphi_i(N_0,C) - \varepsilon$, which means that $u_i(s) \geq -\varphi_i(N_0,C)$.

Claim 4. $u_i(s) = -\varphi_i(N_0,C)$ for all $i \in N$.

Under Claim 3, $u_i(s) \geq -\varphi_i(N_0,C)$ for all $i \in N$. Since $\sum_{i \in N} \varphi_i(N_0,C) = m(N_0,C)$, it is enough to prove that $\sum_{i \in N} u_i(s) \leq -m(N_0,C)$.

$$\sum_{i \in N} u_i(s) = \sum_{i \in N} \frac{1}{|W(x)|} \sum_{j \in W(x)} p_i(j) = \frac{1}{|W(x)|} \sum_{j \in W(x)} \sum_{i \in N} p_i(j).$$

It is enough to prove that $\sum_{i \in N} p_i(j) \leq -m(N_0,C)$ for all $j \in W(x)$.

Given $j \in W(x)$, two cases are possible.

1. Agent j's proposal (S,t,y) is accepted. Thus,

$$\sum_{i \in S} p_i(j) = -c(S_0,C,t) \leq -m(S_0,C).$$

   Under the induction hypothesis,

$$\sum_{i \in N \setminus S} p_i(j) = \sum_{i \in N \setminus S} \left(-\varphi_i\left((N \setminus S)_0, C^{+S}\right)\right) = -m\left((N \setminus S)_0, C^{+S}\right).$$

   Under Claim 3 in the proof of Proposition 1,

$$-m(S_0,C) - m\left((N \setminus S)_0, C^{+S}\right) \leq -m(N_0,C).$$

   Thus, $\sum_{i \in N} p_i(j) \leq -m(N_0,C)$.

2. Agent j's proposal (S,t,y) is rejected. Thus,

$$p_j(j) = -c_{0j} + \sum_{i \in N^{-j}} x_j^i.$$

Under the induction hypothesis,

$$\sum_{i\in N^{-j}} p_i(j) = \sum_{i\in N^{-j}} \left(-x_j^i - \varphi_i\left(N_0^{-j}, C^{+j}\right)\right) = -\sum_{i\in N^{-j}} x_j^i - m\left(N_0^{-j}, C^{+j}\right).$$

Thus,

$$\sum_{i\in N} p_i(j) = -c_{0j} - m\left(N_0^{-j}, C^{+j}\right).$$

Applying Claim 3 in the proof of Proposition 1 to $\{j\}$,

$$\sum_{i\in N} p_i(j) \leq -m(N_0, C).$$

6.3 Proof of Theorem 2

We proceed by induction on the number of agents n. If $n = 1$, the result is trivial. Assume that for all $n \leq p$ there exists a $VSNE$ whose payoff allocation coincides with $-\varphi(N_0, C)$. We will prove it when $n = p + 1$.

Let $s = (s_i)_{i\in N}$ be the strategy profile $s = (s_i)_{i\in N}$ in $B(N_0, C)$ as in the proof of Proposition 1, with $VSNE$ instead of $SPNE$.

Given $T \subset N$, we know that if the agents in N play s, $\sum_{i\in T} u_i(s) = -\sum_{i\in T} \varphi_i(N_0, C)$.

Assume that the agents in $T \subset N$ deviate and each $j \in T$ plays s'_j instead of s_j. We will prove that

$$a = \sum_{i\in T} u_i\left((s_j)_{j\in N\setminus T}, (s'_j)_{j\in T}\right) \leq \sum_{i\in T} -\varphi_i(N_0, C).$$

We prove several claims.

Claim 1. If s'_j coincides with s_j in stages 1 and 2 for each $j \in T$, $a \leq \sum_{i\in T} -\varphi_i(N_0, C)$.

According with s, each agent $j \in T$ accepts (N, t, y) as in the definition of S. Assume that some agent in T rejects (N, t, y). Each agent $i \in N^{-\alpha}$ pays $x_\alpha^i = \varphi_i(N_0, C) - \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ to agent α, who connects to the source. The agents in $N^{-\alpha}$ play $B\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

Under the induction hypothesis, the agents in $N^{-\alpha} \cap T$ obtain, at most,

$$-\sum_{i\in N^{-\alpha}\cap T} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right).$$

We consider two cases:

1. $\alpha \notin T$. In this case,

$$a \leq -\sum_{i\in T} x_\alpha^i - \sum_{i\in T} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) = -\sum_{i\in T} \varphi_i(N_0, C).$$

2. $\alpha \in T$. In this case,

$$\begin{aligned} a &\leq -c_{0\alpha} + \sum_{i \in N^{-\alpha}} x_\alpha^i - \sum_{i \in T^{-\alpha}} x_\alpha^i - \sum_{i \in T^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &= -c_{0\alpha} + \sum_{i \in N \setminus T} x_\alpha^i - \sum_{i \in T^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &= -c_{0\alpha} + \sum_{i \in N \setminus T} \varphi_i\left(N_0, C\right) - \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &= -c_{0\alpha} + m\left(N_0, C\right) - \sum_{i \in T} \varphi_i\left(N_0, C\right) - \sum_{i \in N^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right). \end{aligned}$$

Under Claim 3 in the proof of Proposition 1 with $S = \{\alpha\}$,

$$a \leq -\sum_{i \in T} \varphi_i\left(N_0, C\right).$$

Claim 2. If s'_j coincides with s_j in stage 1 for each $j \in T$, $a \leq -\sum_{i \in T} \varphi_i\left(N_0, C\right)$.

We consider several cases.

1. $\alpha \notin T$. Under Claim 1, $a \leq \sum_{i \in T} -\varphi_i\left(N_0, C\right)$.

2. $\alpha \in T$ and α's proposal is accepted.

   The agents in $T \cap S$ obtain

$$-c\left(S_0, t, C\right) + \sum_{i \in (N \setminus T) \cap S} y_i.$$

   Since the agents in $N \setminus T$ play s and (S, t, y) is accepted, for each $i \in (N \setminus T) \cap S$

$$\begin{aligned} y_i &\leq x'^i_\alpha + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) = x_\alpha^i + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &= \varphi_i\left(N_0, C\right). \end{aligned}$$

   Thus, the agents in $T \cap S$ obtain, at most,

$$-m\left(S_0, C\right) + \sum_{i \in (N \setminus T) \cap S} \varphi_i\left(N_0, C\right).$$

   Since the agents in $N \setminus T$ play s, and under the induction hypothesis, the agents in $T \cap (N \setminus S)$ obtain, at most,

$$-\sum_{i \in T \cap (N \setminus S)} \varphi_i\left(N_0^{-S}, C^{+S}\right).$$

Hence,

$$\begin{aligned} a &\leq -m\left(S_0,C\right)+\sum_{i\in(N\setminus T)\cap S}\varphi_i\left(N_0,C\right)-\sum_{i\in T\cap(N\setminus S)}\varphi_i\left(N_0^{-S},C^{+S}\right) \\ &= -m\left(S_0,C\right)+\sum_{i\in(N\setminus T)\cap S}\varphi_i\left(N_0,C\right)-m\left(\left(N\setminus S\right)_0,C^{+S}\right) \\ &\quad +\sum_{i\in(N\setminus T)\cap(N\setminus S)}\varphi_i\left(N_0^{-S},C^{+S}\right). \end{aligned}$$

Under Claim 3 in the proof of Proposition 1,

$$a\leq -m\left(N_0,C\right)+\sum_{i\in(N\setminus T)\cap S}\varphi_i\left(N_0,C\right)+\sum_{i\in(N\setminus T)\cap(N\setminus S)}\varphi_i\left(N_0^{-S},C^{+S}\right)$$

Since φ satisfies SOL and $C^{+S}\leq C_{N\setminus S}$,

$$\begin{aligned} a &\leq -m\left(N_0,C\right)+\sum_{i\in(N\setminus T)\cap S}\varphi_i\left(N_0,C\right)+\sum_{i\in(N\setminus T)\cap(N\setminus S)}\varphi_i\left(N_0,C\right) \\ &= -m\left(N_0,C\right)+\sum_{i\in N\setminus T}\varphi_i\left(N_0,C\right)=-\sum_{i\in T}\varphi_i\left(N_0,C\right). \end{aligned}$$

3. $\alpha\in T$ and α's proposal is rejected.

   Since the agents in $N\setminus T$ play s and, s'_j coincides with s_j in Stage 1 for each $j\in T$, $x_\alpha^i=\varphi_i\left(N_0,C\right)-\varphi_i\left(N_0^{-\alpha},C^{+\alpha}\right)$ for all $i\in N$.

   Agent α obtains

$$-c_{0\alpha}+\sum_{i\in N^{-\alpha}}x_\alpha^i=-c_{0\alpha}+\sum_{i\in N^{-\alpha}}\varphi_i\left(N_0,C\right)-\sum_{i\in N^{-\alpha}}\varphi_i\left(N_0^{-\alpha},C^{+\alpha}\right).$$

   Since the agents in $N\setminus T$ play s, under the induction hypothesis the agents in $T^{-\alpha}$ obtain, at most,

$$-\sum_{i\in T^{-\alpha}}\left(x_\alpha^i+\varphi_i\left(N_0^{-\alpha},C^{+\alpha}\right)\right)=-\sum_{i\in T^{-\alpha}}\varphi_i\left(N_0,C\right).$$

   Hence,

$$\begin{aligned} a &\leq -c_{0\alpha}+\sum_{i\in N\setminus T}\varphi_i\left(N_0,C\right)-\sum_{i\in N^{-\alpha}}\varphi_i\left(N_0^{-\alpha},C^{+\alpha}\right) \\ &= -c_{0\alpha}+\sum_{i\in N\setminus T}\varphi_i\left(N_0,C\right)-m\left(N_0^{-\alpha},C^{+\alpha}\right). \end{aligned}$$

   Applying Claim 3 in the proof of Proposition 1 with $S=\{\alpha\}$,

$$a\leq -m\left(N_0,C\right)+\sum_{i\in N\setminus T}\varphi_i\left(N_0,C\right)=-\sum_{i\in T}\varphi_i\left(N_0,C\right).$$

Claim 3. If the agents in T deviate in Stage 1, $a \leq -\sum_{i\in T} \varphi_i(N_0, C)$.

Assume that the agents in T submit $\{x'^i\}_{i\in T}$ where, for each $i \in T$, $x'^i = (x'^i_j)_{j\in N^{-i}}$. We take $x' = \left((x'^i)_{i\in T}, (x^i)_{i\in N\setminus T}\right)$.

For each $j \in N \setminus T$,

$$\begin{aligned} X'(j) &= \sum_{i\in N^{-j}} x'^i_j - \sum_{i\in N^{-j}} x'^j_i = \sum_{i\in N\setminus(T\cup\{j\})} x^i_j + \sum_{i\in T} x'^i_j - \sum_{i\in N^{-j}} x^j_i \\ &= X(j) - \sum_{i\in T} x^i_j + \sum_{i\in T} x'^i_j = \sum_{i\in T} x'^i_j - \sum_{i\in T} x^i_j. \end{aligned}$$

Since α is randomly chosen out of $W(x')$, agent i's payoff, $i \in T$, is

$$\frac{1}{|W(x')|} \sum_{j\in W(x')} p_i(j).$$

We consider two cases:

(a) There exists $j \in N \setminus T$ such that $\sum_{i\in T} x'^i_j > \sum_{i\in T} x^i_j$.

In this case, $X'(j) > 0$ and for all $k \in W(x')$, $X'(k) \geq X'(j) > 0$. We now consider several subcases:

1. $\alpha \notin T$.
   Since α plays s, he proposes (N, t, y) where $y_i = x'^i_\alpha + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)$ for all $i \in N^{-\alpha}$. All the agents in $N \setminus (T \cup \{\alpha\})$ accept (N, t, y). Using arguments similar to those used in Claim 1 we can conclude that

$$\begin{aligned} \sum_{i\in T} p_i(\alpha) &\leq -\sum_{i\in T} x'^i_\alpha - \sum_{i\in T} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &< -\sum_{i\in T} x^i_\alpha - \sum_{i\in T} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &= -\sum_{i\in T} \varphi_i(N_0, C). \end{aligned}$$

2. $\alpha \in T$ and α's proposal is accepted.
   Using arguments similar to those used in case 2 of Claim 2 we can conclude that

$$\sum_{i\in T} p_i(\alpha) \leq -\sum_{i\in T} \varphi_i(N_0, C).$$

3. $\alpha \in T$ and α's proposal is rejected.

Using arguments similar to those used in case 3 of Claim 2 we can deduce that

$$\begin{aligned}\sum_{i\in T} p_i(\alpha) &\le -c_{0\alpha} + \sum_{i\in N^{-\alpha}} x_\alpha^{\prime i} - \sum_{i\in T^{-\alpha}} \left(x_\alpha^{\prime i} + \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right)\right) \\ &= -c_{0\alpha} + \sum_{i\in N\setminus T} x_\alpha^i - \sum_{i\in T^{-\alpha}} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right).\end{aligned}$$

Since $x_\alpha^{\prime i} = x_\alpha^i$ for all $i \in N \setminus T$, applying Claim 3 of Proposition 1 with $S = \{\alpha\}$ we deduce that

$$\begin{aligned}\sum_{i\in T} p_i(\alpha) &\le -m(N_0, C) + \sum_{i\in N\setminus T} x_\alpha^i + \sum_{i\in N\setminus T} \varphi_i\left(N_0^{-\alpha}, C^{+\alpha}\right) \\ &= -\sum_{i\in T} \varphi_i(N_0, C).\end{aligned}$$

Thus, if the agents in T deviate, they obtain

$$\begin{aligned}\frac{1}{|W(x')|} \sum_{j\in W(x')} \sum_{i\in T} p_i(j) &\le \frac{1}{|W(x')|} \sum_{j\in W(x')} \left(-\sum_{i\in T} \varphi_i(N_0, C)\right) \\ &= -\sum_{i\in T} \varphi_i(N_0, C).\end{aligned}$$

(b) $\sum_{i\in T} x_j^{\prime i} \le \sum_{i\in T} x_j^i$ for all $j \in N \setminus T$.

In this case, for each $j \in N \setminus T$, $X'(j) \le X(j) = 0$. Two subcases are possible:

1. There exists $j \in T$ such that $X'(j) > 0$.
   Since $\sum_{i\in N} X'(i) = 0$, we have $W(x') \subset T$. Using arguments similar to those used in case (a) when $\alpha \in T$, we can conclude that agents of T obtain, at most, $-\sum_{i\in T} \varphi_i(N_0, C)$.
2. $X'(j) \le 0$ for each $j \in T$.
   Since $\sum_{i\in N} X'(i) = 0$, we have $X'(i) = 0$ for each $i \in N$ and $W(x') = N$. Hence, $\sum_{i\in T} x_j^{\prime i} = \sum_{i\in T} x_j^i$ for each $j \in N\setminus T$. Using arguments similar to those used in case (a), we can conclude that agents of T obtain, at most, $-\sum_{i\in T} \varphi_i(N_0, C)$.

6.4 Proof of Proposition 3

We proceed by induction on the number of agents n. If $n = 1$ the result is trivial. Assume that for all $n \leq p$ there exists an $SPNE$ $s = (s_i)_{i \in N}$ in $B^4(N_0, C)$ such that $u(s) = -\varphi(N_0, C)$. We will prove it when $n = p + 1$.

Let $i_0 \in N$ be such that there exists an mt in (N_0, C) with $(0, i_0) \in t$.

We define the strategy combination s in $B^4(N_0, C)$ as follows:

Stage 1. For all $i \in N$ and $j \in N^{-i}$, $x_j^i = \varphi_i(N_0, C) - \varphi_i\left(N_0^{-j}, C^{+j}\right)$ and $i^* = i_0$.

Stage 3. By induction hypothesis, there exists an $SPNE$ $s^{+\alpha} = \left(s_i^{+\alpha}\right)_{i \in N^{-\alpha}}$ in $B^4\left(N_0^{-\alpha}, C^{+\alpha}\right)$ such that $u(s^{+\alpha}) = -\varphi\left(N_0^{-\alpha}, C^{+\alpha}\right)$. We define s as $s^{+\alpha}$ in $B^4\left(N_0^{-\alpha}, C^{+\alpha}\right)$.

Using arguments similar to those used in the proof of Proposition 1 we can prove that s is an $SPNE$ satisfying $u(s) = -\varphi(N_0, C)$.

6.5 Proof of Proposition 4

A TU game (N, v) is zero-monotonic if for all $S \subset N$ and $i \notin S$, $v(S) + v(\{i\}) \leq v(S \cup \{i\})$.

Pérez-Castrillo and Wettstein (2001) [13] proved that if (N, v) is zero-monotonic, the bidding mechanism associated with (N, v) has $SPNE$. Moreover, the payoff of each $SPNE$ coincides with the Shapley value of (N, v).

It is not difficult to prove the following claims:

Claim 1. Given an *mcstp* (N_0, C), $(N, -v_C)$ is zero-monotonic.

Claim 2. The $SPNE$ of $B^5(N_0, C)$ coincide with the $SPNE$ of the bidding mechanism applied to the TU game $(N, -v_C)$.

Thus, Proposition 4 is a consequence of both claims.

Acknowledgments. We thank David Pérez-Castrillo for helpful comments. Financial support from the Ministerio de Educación y Ciencia and FEDER through grant SEJ2005-07637-C02-01 and Xunta de Galicia through grants PGIDIT06PXIB362390PR and PGIDIT06PXIC3000184PN is gratefully aknowledged.

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