The folk solution and Boruvka's algorithm in minimum cost spanning tree problems

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Abstract

The Boruvka's algorithm, which computes the minimum cost spanning tree, is used to define a rule to share the cost among the nodes (agents). We show that this rule coincides with the folk solution, a very well-known rule of this literature.

Keywords: minimum cost spanning tree, Boruvka's algorithm, folk solution.

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1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). Consider that a group of agents, located at different geographical places, wants some particular service which can only be provided by a common supplier, called the *source*. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

In the literature on *mcstp* there are several algorithms for constructing minimum cost spanning trees (*mt*): Boruvka (1926) [6], Kruskal (1956) [13] and Prim (1957)

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[14]. In this paper we focus on Boruvka's algorithm, which has been studied later by other authors as Eppstein (1999) [11] and Cong and Wen (2007) [9]. All three are greedy algorithms that run in polynomial time. But constructing an *mt* is only a part of the problem. Another important issue is how to allocate the cost associated with the *mt* among the agents. Several authors have introduced rules in *mcstp* through some algorithms for constructing *mt*. The idea is to propose rules to divide the cost among the agents in a fair way[1].

Bird (1976) [5], Dutta and Kar (2004) [10], and Bergantiños and Vidal-Puga (2007a) [2] introduce three rules based on Prim's algorithm. Feltkamp *et al* (1994) [12] introduce a rule based on Kruskal's algorithm. The rules introduced by Bergantiños and Vidal-Puga (2007a) [2] and Feltkamp *et al* (1994) [12] coincide (Bergantiños and Lorenzo-Freire (2008) [1]). We call this rule the *folk solution*, which can be obtained in other ways (see Branzei *et al* (2004) [7], Ciftci and Tijs (2009) [8], and Bergantiños and Vidal-Puga (2007a, 2007b, 2009) [2][3][4]).

Nevertheless, as far as we know, no rule has been introduced through Boruvka's algorithm. We do it. The idea behind this algorithm is the following. Initially the network is empty and each agent is a single component. We sequentially add to the network, for each connected component, the cheapest arc joining this connected component with some agent outside it and without introducing cycles. We divide the cost of any arc selected by Boruvka's algorithm following three principles. First, each agent is assigned to the arc selected by the component he belongs to. Each agent pays, partially, the cost of the assigned arc. Second, all agents pay the same proportion of the arc assigned. Namely, each agent i pays $pc_{a(i)}$ where $c_{a(i)}$ is the cost of the arc $a(i)$ assigned to agent i. Third, the proportion paid, p, should be as large as possible.

We prove that the rule we introduce coincides with the folk solution. Our result gives more support to the folk solution as it can be obtained in several ways.

The paper is organized as follows. In Section 2 we present the notation. In Section 3 we define our rule and prove the main result.

2 The minimum cost spanning tree problem

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given $N \subset \mathcal{N}$ finite, $|N|$ denotes the number of elements in N. We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., |N|\}$. A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ *over* N represents the

[1] In this paper we refer to fairness as a general principle to achieve, and not as a well-defined mathematical object.

cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs and we denote it as $\{i, j\}$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$, we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$. We denote the set of all cost matrices over N with all the costs different as $\mathcal{D}^N$, *i.e.* $C \in \mathcal{D}^N$ if $c_{ii'} \neq c_{jj'}$ when $\{i, i'\} \neq \{j, j'\}$.

A *minimum cost spanning tree problem*, briefly *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix. A *graph* g over N_0 is a subset of $\{\{i, j\} : i, j \in N_0, i \neq j\}$. We define *tree*, *path* and *connected component* in the usual way. Given an *mcstp* (N_0, C), we denote the *mcstp* induced by C in $S \subset N$ as (S_0, C). Given a tree t, we denote the restriction of t to nodes in $S \subset N$ as t_S.

Usually, we denote a tree over N_0 as $t = \{\{i^0, i\}\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0. We denote the set of trees over N_0 as $\mathcal{T}_0^N$.

Given an *mcstp* (N_0, C) and a graph g, we define the *cost* associated with g as $c(N_0, C, g) = \sum\limits_{\{i,j\} \in g} c_{ij}$. When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree $t \in \mathcal{T}_0^N$ such that $c(t) = \min_{g \in \mathcal{T}_0^N} c(g)$. Since the number of possible trees is finite, there exists an *mt*, even though it does not need to be unique. Given an *mcstp* (N_0, C) we denote by $m(N_0, C)$ the cost associated with any *mt* t in (N_0, C).

A (*cost allocation*) *rule* is a function f such that for each *mcstp* (N_0, C), we have $f(N_0, C) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$. As usual, $f_i(N_0, C)$ represents the cost assigned to agent i.

Feltkamp *et al* (1994) [12] introduce a rule based on Kruskal's algorithm. Bergantiños and Vidal-Puga (2007a) [2] introduce a rule based on Prim's algorithm. Bergantiños and Lorenzo-Freire (2008) [1] prove that both rules coincide. We call this rule the folk solution and we denote it as φ.

3 A rule based on Boruvka's algorithm

Boruvka (1926) [6] provides an algorithm for computing an *mt*. We provide a way of sharing the cost of any arc selected by Boruvka's algorithm. We first describe Boruvka's algorithm in a formal way.

Let π be an order over the set of all possible arcs. Namely

$$\pi : \{\{i,j\} : i,j \in N_0, i \neq j\} \to \left\{1, 2, ..., \left(\begin{array}{c} |N| \\ 2 \end{array} \right) \right\}.$$

Boruvka's algorithm (associated with the order π).

Step 1: Let $g^{\pi,0} = \emptyset$. Notice that the set of connected components is $\{\{0\}, \{1\}, ..., \{|N|\}\}$. Assume we have reached Step s ($s = 1, 2, ...$) and we have defined $g^{\pi,s-1}$.

Step s: For each connected component T, $0 \notin T$,[2] let $\{i^{\pi,T}, j^{\pi,T}\} \in T \times (N_0 \backslash T)$ be the cheapest arc connecting T and $N_0 \backslash T$. In case there are more than one possible arc, we select the one with the lowest position in the order π. We then add this arc to the graph, *i.e.*

$$g^{\pi,s} = g^{\pi,s-1} \cup \left\{ \left\{ i^{\pi,T}, j^{\pi,T} \right\} : T \text{ is a connected component}, 0 \notin T \right\}.$$

Following this algorithm, $g^{\pi,s}$ is a graph with no cycles.

If the set of connected components becomes $\{N_0\}$, then $g^{\pi,s}$ is a tree and the process is over. Otherwise, we move to Step $s+1$.

The process finishes in a finite number of steps. The tree obtained by this procedure is an *mt* and we denoted it as t^π. Moreover, given an *mt* t, there exists an order π such that $t^\pi = t$. It is possible that $t^\pi = t^{\pi'}$, even if π and π' are different orders. For instance, if all the costs are different, $t^\pi = t^{\pi'}$ for all π and π'.

When no confusion arises we write g^s, i^T, ... instead of $g^{\pi,s}$, $i^{\pi,T}$, ... respectively.

We now introduce a rule in *mcstp* based on Boruvka's algorithm. At each step, each connected component selects an arc and each agent is assigned to the arc selected by the component he belongs to. Each agent pays the same proportion, say p, of the cost of the assigned arc. The proportion p must be as large as possible**.**

Let π be some order of the arcs, let (N_0, C) be a cost matrix, and let t^π (or simply t) be the arc selected following Boruvka's algorithm associated with π. We now define the rule β^π as follows:

Step 0. We define $a_i^{0,\pi} = \emptyset$ for all $i \in N$, $p^{0,\pi} = 0$, $\varrho_{ij}^{0,\pi} = 0$ for all $(i,j) \in t$, $A^{0,\pi} = t$, and $f_i^{0,\pi} = 0$ for all $i \in N$.

In general, $a_i^{s,\pi}$, or simply a_i^s, denotes the arc in t that agent i pays partially in Step s; $p^{s,\pi}$, or simply p^s, denotes the proportion of the cost of the arc that each agent pays in Step s; $\varrho_{ij}^{s,\pi}$, or simply ϱ_{ij}^s, denotes the proportion of the cost of arc $\{i,j\}$ already paid in Step s; $A^{s,\pi}$, or simply A^s, denotes the set of non-completely paid arcs in Step

[2]Usually, the condition $0 \notin T$ does not appear. We have added it in order to adapt the algorithm to our objective: to divide the cost of the *mt* among the agents. If $0 \in T$, then the agents in T do not need to be connected to more agents.

s, *i.e.* $A^s = \left\{\{i,j\} \in t : \varrho_{ij}^s < 1\right\}$; $f_i^{s,\pi}$, or simply f_i^s, denotes the cost that agent i pays in Step s, *i.e.* $f_i^s = p^s c_{a_i^s}$.

We denote $\overline{A}^s = t \backslash A^s = \left\{\{i,j\} \in t : \varrho_{ij}^s = 1\right\}$. Let $\mathcal{P}^s$ be the set of connected components of N_0 associated to $\overline{A}^s$.

Assume that we have defined Step r for all $r < s$. We now define Step s. For simplicity, we omit reference to the order π.

Given a connected component $T \in \mathcal{P}^{s-1}$, $0 \notin T$, we select the arc $\left\{i^T, j^T\right\}$ as in Boruvka's algorithm, so that $\left\{i^T, j^T\right\} \in t$. Moreover, if component T selects $\left\{i^T, j^T\right\}$ in Step $s-1$ and $\left\{i^T, j^T\right\}$ is not completely paid at the beginning of Step s, then component T also selects $\left\{i^T, j^T\right\}$ in Step s.

Given $k \in T \in \mathcal{P}^{s-1}$, we define $a_k^s = \left\{i^T, j^T\right\}$. That is, each agent will pay the cost of the arc selected by Boruvka's algorithm for the component he belongs to.

For each arc $\{i,j\} \in A^{s-1}$, let $N_{ij}^s = \{k \in N : a_k^s = \{i,j\}\}$ be the set of agents that will pay the cost of arc $\{i,j\}$. We define

$$p^s = \min\left\{\frac{1-\varrho_{ij}^{s-1}}{\left|N_{ij}^s\right|} : \{i,j\} \in A^{s-1}, N_{ij}^s \neq \emptyset\right\}.$$

Notice that, assuming that all agents must pay the same proportion of the cost for each arc, p^s is the maximum proportion that agents can pay in Step s.

For each $\{i,j\} \in A^{s-1}$, we define $\varrho_{ij}^s = \varrho_{ij}^{s-1} + \left|N_{ij}^s\right| p^s$. Thus, $\varrho_{ij}^s \leq 1$ for each $\{i,j\} \in A^{s-1}$. Moreover, there exists at least one $\{i,j\} \in A^{s-1}$ such that $\varrho_{ij}^s = 1$. Thus, $A^s \subsetneq A^{s-1}$ and $\overline{A}^{s-1} \subsetneq \overline{A}^s$. That is, there are more arcs completely paid.

This process finishes when $\overline{A}^s = t$. Since $a_i^s \in t$ for each agent i and each Step s, and $\overline{A}^{s-1} \subsetneq \overline{A}^s$, this process finishes in a finite number of steps (at most $|N|$), say γ.

Moreover, by definition the process finishes when $\sum\limits_{s=1}^{\gamma} p^s = 1$.

Definition 3.1 *Given an order π of the set of arcs and a cost matrix C, we define the Boruvka's rule induced by the order π as*

$$\beta_i^\pi (N_0, C) = \sum_{s=1}^{\gamma} f_i^s \text{ for each } i \in N.$$

We have generated an allocation for each order of the arcs following Boruvka's algorithm. Even though this allocation could depend on the order, we prove that it does not (as the rule defined in Feltkamp *et al* (1994a) [12], based on Kruskal's algorithm). Moreover we prove that this allocation coincides with the folk solution φ. All these statements are proved in the following theorem (main result):

Theorem 3.1 *For each order π, β^π coincides with φ.*

Proof. The following properties are satisfied by φ (Bergantiños and Vidal-Puga (2007a)[2]). We say that a rule f satisfies:

Separability (SEP) if for all (N_0, C) and all partition $\{S, T\}$ of N satisfying $m(N_0, C) = m(S_0, C) + m(T_0, C)$, we have $f_i(N_0, C) = f_i(S_0, C)$ if $i \in S$, and $f_i(T_0, C)$ if $i \in T$;

Equal Sharing of Extra Costs ($ESEC$) if for all (N_0, C), (N_0, C') and $c_0, c'_0 \geq 0$ such that $c_{0i} = c_0$ and $c'_{0i} = c_0$ for all $i \in N$, $c_0 < c'_0$, and $c_{ij} = c'_{ij} \leq c_0$ for all $i, j \in N$, we have $f_i(N_0, C') = f_i(N_0, C) + \frac{c'_0 - c_0}{|N|}$ for all $i \in N$;

Continuity (CON) if for all N, f is a continuous function on $\mathcal{C}^N$;

Independence of Irrelevant Trees (IIT) if for all (N_0, C) and (N_0, C') such that they both share a common mt with the same costs[3], we have $f(N_0, C) = f(N_0, C')$.

Let π be any order of the arcs in N_0, C a cost matrix, and $t^\pi = \{\{i^0, i\}\}_{i \in N}$ the mt in (N_0, C) obtained through Boruvka's algorithm. We will prove that $\beta^\pi(N_0, C) = \varphi(N_0, C)$. We proceed by induction on the number of agents. For $|N| = 1$, the result follows from the definition of rule. Assume that the result holds for less than $|N|$ agents.

We first check that it is enough to prove that the result holds for matrices in $\mathcal{D}^N$ (set of matrices where all costs are different). Notice that $\mathcal{D}^N$ is a dense subset of $\mathcal{C}^N$. For any $C \in \mathcal{C}^N \backslash \mathcal{D}^N$ and t^π the tree obtained through Boruvka's algorithm, we can find a sequence of matrices $\{C^m\}_{m=1}^\infty \subset \mathcal{D}^N$ such that (1) $C^m \to C$; (2) t^π is an mt in C^m for all m; (3) if $c_{ii'} = c_{jj'}$ and $\pi(\{i, i'\}) < \pi(\{j, j'\})$, then $c^m_{ii'} < c^m_{jj'}$ for all m. Under conditions (2) and (3), $\gamma, a^{s,\pi}$, $p^{s,\pi}$, $\varrho^{s,\pi}$ and $A^{s,\pi}$ coincide for C and any C^m. Hence, $\lim_{m \to \infty} \beta^\pi(N_0, C^m) = \beta^\pi(N_0, C)$. If $\beta^\pi = \varphi$ in $\mathcal{D}^N$, then the continuity of φ implies that the result is true in all $\mathcal{C}^N$.

Hence, we prove the result assuming that $C \in \mathcal{D}^N$. Then, $t^\pi = t^{\pi'}$ and $\beta^\pi = \beta^{\pi'}$ for any pair of orders π and π'. Thus, it is enough to prove that $\beta^\pi = \varphi$ for some order π. Let π be an order and $t = t^\pi$. Let N^0 be the set of nodes directly connected to the source in t and let (j^0, j) be the most expensive arc in t. We consider three cases:

Case 1. There are more than one agent directly connected to the source: For any of these agents, say agent $i \in N^0$, let F^i be the set of followers of agent i (agents $j \in N$ such that $\{0, i\}$ is in the unique path in t from j to 0) including agent i. Then, $\{F^i\}_{i \in N^0}$ is a partition of N satisfying that $\sum_{i \in N^0} m(F^i_0, C) = m(N_0, C)$ and $t_{F^i_0}$ is a tree in (F^i_0, C) for all $i \in N^0$.

Since φ satisfies SEP, for all $i \in N^0$ and $k \in F^i$, we have $\varphi_k(N_0, C) = \varphi_k(F^i_0, C)$. We just need to prove $\beta^\pi_k(N_0, C) = \beta^\pi_k(F^i_0, C)$ for all $i \in N^0$ and $k \in F^i$ and apply the

[3] Formally, there exists a tree t that is a mt in both (N_0, C) and (N_0, C') and, moreover, $c_{ij} = c'_{ij}$ for all $(i, j) \in t$.

induction hypothesis.

We need to prove that for each $i \in N^0$, the cost of the arcs in $t_{F_0^i}$ is paid only by the agents in F^i. Suppose not. Then, there exist $i \in N^0$ and $k \in F^i$ such that k selects in step $s+1$ an arc $a_k^{s+1} = \left\{i^T, j^T\right\} \in t \backslash t_{F_0^i}$ for some $T \in \mathcal{P}^s$ with $k \in T$. Let s be the first stage in which we can find such $i \in N^0$ and $k \in F^i$. Thus, the arcs in t_{T_0} have been paid in Step s. By definition, all the agents in T are connected through arcs in t. Thus, t_{T_0} is a tree in T_0. Since in t_{T_0} there are exactly $|T|$ arcs, the cost of the arcs in t_{T_0} is paid only by the agents in T (s is the first stage in which an agent $k \in F^i$ is paying an arc outside $t_{F_0^i}$), and each agent pays the same proportion p^r at each step r, we deduce that $\sum\limits_{r=1}^{s} p^r = 1$. This means that the procedure is already finished in Step s. Hence, there is no Step $s+1$, which is a contradiction.

Case 2. There is exactly one agent directly connected to the source, and it is not player j (hence, the most expensive arc does not connect to the source): Let F be the set of agents $i \in N$ such that arc $\{j^0, j\}$ is in the unique path in t from i to 0. Let $\overline{F} = N \setminus F$. Notice that $F \neq \emptyset$ and $\overline{F} \neq \emptyset$ because $j \in F$ and $j^0 \in \overline{F}$.

We first prove that agents in $\overline{F}$ only pay the cost of the arcs in $t_{\overline{F}_0}$. Suppose not. Then, there exists $k \in \overline{F}$ such that $a_k^{s+1} = \{j^0, j\}$ for some step s. Let s be the first stage where this happens. Let $T \in \mathcal{P}^s$ with $k \in T$. Thus, $a_i^{s+1} = \{j^0, j\}$ for all $i \in T$. Since $c_{j^0 j} > c_{ii'}$ for all $\{i, i'\} \in t_{\overline{F}_0}$ and $t_{\overline{F}_0}$ is a tree in $\overline{F}_0$, we deduce that $T = \overline{F}$ and $t_{\overline{F}_0} \subset \overline{A}^s$. Since there are exactly $\left|\overline{F}\right|$ arcs in $t_{\overline{F}_0}$, and all the agents pay the same proportion p^r at each Step r, we deduce that $\sum\limits_{r=1}^{s} p^r = 1$. This means that the procedure is already finished in Step s. Hence, there is no Step $s+1$, which is a contradiction.

Similarly, we can prove that agents in F only pay the cost of arcs in $t_{F \cup \{j^0\}}$.

Take the matrix $C' \in \mathcal{D}^N$ defined as $c'_{0j} = c_{j^0 j}$, $c'_{j^0 j} = c_{0j}$, and $c'_{il} = c_{il}$ otherwise. Following the above reasoning, $\beta^\pi (N_0, C) = \beta^\pi (N_0, C')$.

Since t is the unique mt in (N_0, C), $t' = (t \backslash \{\{j^0, j\}\}) \cup \{(0, j)\}$ is the unique mt in (N_0, C'). Thus, C' is in Case 1. Hence, $\beta^\pi (N_0, C') = \varphi (N_0, C')$.

Take now the matrix $C'' \in \mathcal{C}^N$ defined as $c''_{0j} = c_{j^0 j}$ and $c''_{il} = c_{il}$ otherwise. With this change, both t and t' are mt in C''. Since φ satisfies IIT, $\varphi (N_0, C') = \varphi (N_0, C'') = \varphi (N_0, C)$.

Case 3. Agent j is the only one directly connected to the source (hence, the most expensive arc connects to the source). Let $\{k^0, k\} \in t \backslash \{\{0, j\}\}$ be the most expensive arc in $t \backslash \{\{0, j\}\}$. Under our hypothesis, $k^0 \neq 0$.

We define a new matrix $C' \in \mathcal{C}^N$ from C by reducing the cost of the arcs in $\{(0, i)\}_{i \in N}$ to the same cost as arc (k^0, k). Namely, for each $i, l \in N$, $c'_{0i} = c_{k^0 k}$, and $c'_{il} = c_{il}$. Of course $C' \notin \mathcal{D}^N$.

We can assume wlog that the order π is such that the arcs $\{\{0,i\}\}_{i\in N}$ are the last and, among them, $\{0,j\}$ is the first one and $\{0,k\}$ is the second one. Moreover, let π' be another order such that the arcs $\{\{0,i\}\}_{i\in N}$ come first and, among them, $\{0,j\}$ is the first one and $\{0,k\}$ is the second one.

For any $i\in N$, we prove $\beta_i^{\pi}(N_0,C)=\varphi_i(N_0,C)$ following four equalities.

$$\beta_i^{\pi}(N_0,C) \overset{\text{Equality 1}}{=} \beta_i^{\pi}(N_0,C')+\frac{c_{0j}-c_{k^0k}}{|N|} \overset{\text{Equality 2}}{=} \beta_i^{\pi'}(N_0,C')+\frac{c_{0j}-c_{k^0k}}{|N|}$$
$$\overset{\text{Equality 3}}{=} \varphi_i(N_0,C')+\frac{c_{0j}-c_{k^0k}}{|N|} \overset{\text{Equality 4}}{=} \varphi_i(N_0,C).$$

We now prove the four equalities:

Equality 1: $\beta_i^{\pi}(N_0,C)=\beta_i^{\pi}(N_0,C')+\frac{c_{0j}-c_{k^0k}}{|N|}$ for all $i\in N$.

When computing $\beta_i^{\pi}(N_0,C)$ and $\beta_i^{\pi}(N_0,C')$, we realize that (a) their respective *mt* coincide with t and (b) both procedures coincide until step $\gamma-1$, where all the arcs in $t\backslash\{\{0,j\}\}$ are completely paid in both procedures and $\{0,j\}$ is not paid at all. Thus, $f_i^{s,\pi}(N_0,C)=f_i^{s,\pi}(N_0,C')$ for all $i\in N$ and all $s<\gamma$. In Step γ, all the players choose arc $\{0,j\}$ and hence, the cost of arc $\{0,j\}$ is shared equally among all agents, i.e. $p^{\gamma}=\frac{1}{|N|}$. Thus, for all $i\in N$, $f_i^{\gamma,\pi}(N_0,C)=\frac{c_{0j}}{|N|}$ and $f_i^{\gamma,\pi}(N_0,C')=\frac{c'_{0j}}{|N|}$. Hence, for all $i\in N$, $\beta_i^{\pi}(N_0,C)-\beta_i^{\pi}(N_0,C')=\frac{c_{0j}}{|N|}-\frac{c'_{0j}}{|N|}=\frac{c_{0j}-c_{k^0k}}{|N|}$.

Equality 2: $\beta^{\pi}(N_0,C')=\beta^{\pi'}(N_0,C')$.

Since the *mcstp* is the same we omit (N_0,C') from the notation. Let G be the set of agents $i\in N$ such that arc $\{k^0,k\}$ is in the unique path in t from i to 0. Let $\overline{G}=N\setminus G$. Notice that $G\neq\emptyset$ and $\overline{G}\neq\emptyset$ because $k\in G$ and $k^0\in\overline{G}$. We prove that $\beta_i^{\pi}=\beta_i^{\pi'}$ for all $i\in G$. The case $i\in\overline{G}$ can be proved in a similar way and we omit it.

We know that $t^{\pi}=t$. Because of the definition of β^{π}, there exist r^1 and r^2 such that (a) from Step 1 to Step r^1, the agents in G select arcs in t_G, and hence all arcs in t_G have been paid completely in Step r^1; (b) from Step r^1+1 to Step r^2, all the agents in G select arc $\{k^0,k\}$; and hence all the arcs in $t_G\cup\{\{k^0,k\}\}$ have been paid completely in Step r^2. Hence, $a_i^{s,\pi}=\{0,j\}$ for all $i\in N$ and $\gamma^{\pi}=r^2+1$.

Because of the definition of π', he have $t^{\pi'}=(t\backslash\{\{k^0,k\}\})\cup\{\{0,k\}\}$. Because of the definition of $\beta^{\pi'}$, (a) from Step 1 to Step r^1, the agents in G select arcs in t_G, and moreover $a_i^{s,\pi'}=a_i^{s,\pi}$ and $p^{s,\pi}=p^{s,\pi'}$ for all $s=1,...,r^1$ and all $i\in G$, and all the arcs in t_G have been paid completely in Step r^1; (b) from Step r^1+1 to Step $\gamma^{\pi'}$, all the agents in G select arc $\{0,k\}$.

Let $i \in G$. Then,

$$\beta_i^{\pi} = \sum_{s=1}^{r^1} p^{s,\pi} c'_{a_i^{s,\pi}} + \sum_{s=r^1+1}^{r^2} p^{s,\pi} c'_{a_i^{s,\pi}} + p^{\gamma^\pi,\pi} c'_{a_i^{\gamma^\pi,\pi}} = \sum_{s=1}^{r^1} p^{s,\pi} c_{a_i^{s,\pi}} + \left(1 - \sum_{s=1}^{r^1} p^{s,\pi}\right) c_{k^0 k}.$$

Moreover,

$$\beta_i^{\pi'} = \sum_{s=1}^{r^1} p^{s,\pi'} c'_{a_i^{s,\pi'}} + \sum_{s=r^1+1}^{\gamma^{\pi'}} p^{s,\pi'} c'_{a_i^{s,\pi'}} = \sum_{s=1}^{r^1} p^{s,\pi} c_{a_i^{s,\pi}} + \left(1 - \sum_{s=1}^{r^1} p^{s,\pi}\right) c_{k^0 k}.$$

Equality 3: $\beta^{\pi'}(N_0, C') = \varphi(N_0, C')$.

The proof is analogous to the proof of Equality 1 and hence we omit it.

Equality 4: $\varphi_i(N_0, C) = \varphi_i(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|}$ for all $i \in N$.

Let $C'' \in \mathcal{C}^N$ defined as $c''_{0i} = c_{0j}$ and $c''_{il} = c_{il}$ for all $i, l \in N$. Since φ satisfies $ESEC$, for all $i \in N$, $\varphi_i(N_0, C'') = \varphi_i(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|}$. Since t is an mt in (N_0, C'') and (N_0, C) and φ satisfies IIT, $\varphi(N_0, C'') = \varphi(N_0, C)$. ■

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