Clique games: a family of games with coincidence between the nucleolus and the Shapley value

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Abstract

We introduce a new family of cooperative games for which there is coincidence between the nucleolus and the Shapley value. These so-called clique games are such that agents are divided into cliques, with the value created by a coalition linearly increasing with the number of agents belonging to the same clique. Agents can belong to multiple cliques, but for a pair of cliques, at most a single agent belongs to their intersection. Finally, if two agents do not belong to the same clique, there is at most one way to link the two agents through a chain of agents, with any two non-adjacent agents in the chain belonging to disjoint sets of cliques. We examine multiple games defined on graphs that provide a fertile ground for applications of our results.

Keywords: nucleolus; Shapley value; clique; graphs

1 Introduction

The Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969) are two well known values for cooperative games. The Shapley value is an average of the contributions of an agent, while the prenucleolus is the value

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that minimizes the dissatisfaction of the worst-off coalitions. The nucleolus differs from the prenucleolus by only taking into account individually rational imputations.

Coincidence between the Shapley value and the (pre)nucleolus is uncommon and, in general, difficult to check without computing both values. Recently, Yokote et al. (2017) provide a sufficient and necessary condition for this coincidence to hold, but it requires the computation of both the Shapley value and of a parametric family of sets, for which the computation mimics that of the (pre)nucleolus.[1] This characterization can be applied in order to identify the coincidence in some particular classes of games, such as airport games (Littlechild and Owen, 1973), bidder collusion games (Graham et al., 1990) and polluted river games (Ni and Wang, 2007). Csóka and Herings (2017) also find coincidence in some three-agent games based on bankruptcy problems. As discussed by Kar et al. (2009), for general coalitional form games we have coincidence if the game only has two agents or if all agents are symmetric within the normalized game. Some other games have also been proposed (Deng and Papadimitriou, 1994; van den Nouweland et al., 1996), all having in common that the value of a coalition is equal to the sum of the values created by the pairs composing that coalition. These games are called 2-additive games. The coincidence persists in games that satisfy the so-called PS property (Kar et al., 2009). These games are such that the contributions of agent i to any coalition and its complement sum up to an agent-specific constant. A particular instance of such games is studied by Chun et al. (2016).

González-Díaz and Sánchez-Rodríguez (2014) also study the coincidence from a geometric point of view. Instead of providing classes of games where both values coincide, they study the properties that lead to this result in some already existing classes, as for example PS-games. A similar, yet different problem, is the invariance of the payoff assigned by an allocation rule to a specific player in two related games. See Béal et al. (2015) for the case of the Shapley value.

[1] Additionally, the condition also requires to check whether the sets in this parametric family are balanced.

In this paper, we present another family of games, called *clique games*, in which the Shapley value and the nucleolus coincide. The family can be described as follows: the set of agents is divided into cliques that cover it. A coalition creates value when it contains many agents belonging to the same clique, with the value increasing linearly with the number of agents in the same clique. Agents may belong to more than one clique, but the intersection of two cliques contains at most one agent. Finally, if two agents are not in the same clique, there exists at most one way to "connect" them through a chain of connected cliques.

The family of clique games has a non-empty intersection with PS-games, and that intersection consists of 2-additive games. Some clique games are not PS-games, and some PS-games are not clique games. A clique game is convex, and hence its Shapley value is the average of the extreme points in its core. We thus obtain a link between three crucial concepts of cooperative game theory: the nucleolus, the core, and the Shapley value.

Naturally, graph-induced games provide a fertile ground to apply our result. We first consider the graph-restricted cooperative games introduced by Myerson (1977). In these games, a coalitional value function is accompanied by a graph that summarizes the cooperation possibilities: a coalition S cannot fully cooperate if some of its members have no path between them that uses only the vertices of agents in S. When we consider a symmetric coalitional value function, assigning shares of the value created among agents is akin to defining centrality measures (Gomez et al., 2003). We show that when the coalitional value function increases linearly with the number of agents in a coalition, we obtain coincidence of the Shapley value (known as the Myerson value in this context) and the nucleolus for a family of graphs.

Another graph-induced game that we study is the minimum coloring game (Deng et al., 1999), in which the graph represents conflicts between pairs of agents. We wish to assign agents to facilities, but cannot assign agents that are in conflict to the same facility. As facilities all have a cost of one, we wish to minimize the number of facilities used. Okamoto (2008) noticed a coincidence between the Shapley value and the nucleolus for a particular family of graphs. We explain this coincidence by the fact that the graphs induce clique games.

Our third example is the minimum cost spanning tree (*mcst*) problem (Bird, 1976). This well-studied problem has agents connecting to a source through a network, with the cost of an edge being a fixed amount that is paid if the edge is used, regardless of the number of users of the edge. Any such problem has a non-empty core even though it may not be convex. Moreover, its Shapley value is not always in its core (Dutta and Kar, 2004). When we consider elementary *mcst* problems (in which all edges have a cost of 0 or 1), the subset of cycle-complete problems (Trudeau, 2012) generates clique games. Cycle-complete problems are such that if there exists multiple distinct free paths between a pair of agents the edge connecting them directly must also be free. Our result on clique games then applies, yielding that the nucleolus coincides with the Shapley value and the permutation-weighted average of extreme core allocations.

The paper is divided as follows: preliminary definitions are in Section 2. Section 3 describes and illustrates clique games. Section 4 contains the coincidence results. Applications to graph-induced games are discussed in Section 5.

2 Preliminaries

Let $N = \{1, \cdots, n\}$ be a set of agents. A *transferable utility game* (*TU game*, for short) is a pair (N, v) where v is a real-valued function defined on all subsets $S \subseteq N$ satisfying $v(\emptyset) = 0$. Given $T \subset N$ and $S \subseteq N \setminus T$, the contribution of coalition T to S is defined as

$$\Delta_T^v(S) = v(S \cup T) - v(S).$$

We write $\Delta_i^v(S)$ for $\Delta_{\{i\}}^v(S)$. A game is *convex* if $\Delta_i^v(S) \leq \Delta_i^v(T)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$.

A *value* is a function that associates with each TU game (N, v) an *allocation* $x \in \mathbb{R}^N$. Two well-known values are the Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

The *Shapley value* of the game (N, v) is the allocation $Sh(v)$ defined as

$$Sh_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \Delta_i^v(P_i(\pi))$$

for all $i \in N$, where Π is the set of all orderings of N and $P_i(\pi)$ is the set of predecessors of agent i in π, i.e. $P_i(\pi) = \{j : \pi(j) < \pi(i)\}$.

The *excess* of a coalition S in a TU game (N, v) with respect to an allocation x is defined as $e(S, x, v) = \sum_{i \in S} x_i - v(S)$. The vector $\theta(x)$ is constructed by rearranging the 2^n excesses in (weakly) increasing order. If $x, y \in \mathbb{R}^N$ are two allocations, then $\theta(x) >_L \theta(y)$ means that $\theta(x)$ is lexicographically larger than $\theta(y)$. As usual, we write $\theta(x) \geq_L \theta(y)$ to indicate that either $\theta(x) >_L \theta(y)$ or $x = y$.

The *nucleolus* of the game (N, v) is the set

$$Nu(v) = \{x \in X : \theta(x) \geq_L \theta(y) \forall y \in X\}$$

where $X = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \forall i \in N\right\}$ is the set of individually rational allocations. When $X \neq \emptyset$, as it is the case for the TU games we study here, it is well-known that $Nu(v)$ is a singleton, whose unique element we denote, with some abuse of notation, also as $Nu(v)$.

By contrast, the *prenucleolus* of the game (N, v) is the set

$$Pre(v) = \left\{x \in X^0 : \theta(x) \geq_L \theta(y) \forall y \in X^0\right\}$$

where $X^0 = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)\right\}$ is the set of allocations. Whenever the prenucleolus is individually rational, which will be the case in all games that we consider, it coincides with the nucleolus. Therefore, from now on, we focus exclusively on the nucleolus.

The *core* is the set of allocations such that no coalition is assigned less than its stand-alone value. Formally,

$$Core(v) = \left\{x \in X^0 : \sum_{i \in S} x_i \geq v(S) \forall S \subset N\right\}.$$

When $Core(v) \neq \emptyset$, for each $\pi \in \Pi$, let $y^{\pi} \in Core(v)$ be the allocation that lexicographically maximizes the individual allocations with respect to the order given by the permutation. The *permutation-weighted average of extreme points of the core* is the average of these allocations:

$$\bar{y}(v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^{\pi}(v).$$

If the game is convex, $\bar{y}$ is the Shapley value. This average is also closely related to the "selective value" (Vidal-Puga, 2004) and the "Alexia value" (Tijs, 2005).

3 Clique games

We say that $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ is a cover of N if $Q^k \subseteq N$ for $k = 1, ..., K$ and $\cup_{k=1}^{K} Q^k = N$. For a cover $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ and each $Q^k \in \mathcal{Q}$, the interior of Q^k, $Int\left(Q^k\right)$, is the set of agents who only belong to Q^k, i.e.

$$Int\left(Q^k\right) = \left\{i \in Q^k : i \notin Q^l \forall l \neq k\right\}.$$

A *path* between Q^k and Q^l is a sequence $P^{kl} = \left\{Q^{k_1}, \ldots, Q^{k_M}\right\}$ of different coalitions such that $Q^{k_1} = Q^k$, $Q^{k_M} = Q^l$ and $\left|Q^{k_m} \cap Q^{k_{m+1}}\right| = 1$ for all $m = 1, \ldots, M-1$. Analogously, *a path between Q^k and Q^l through agent i* is a path $P_i^{kl} = \left\{Q^{k_1}, \ldots, Q^{k_M}\right\}$ such that $Q^{k_1} \cap Q^{k_2} = \{i\}$. The set of agents connected to Q^k via a path through agent $i \in Q^k$ is denoted as

$$N_{k,i}^{P} = \left\{ j \in N : \begin{array}{l} \text{there exists a path between } Q^k \text{ and} \\ Q^l \text{ through agent } i \text{ such that } j \in Q^l \end{array} \right\}.$$

Example 1 *Let $\mathcal{Q} = \{Q^1, Q^2, Q^3\}$ with $Q^1 = \{1, 2\}$, $Q^2 = \{2, 3, 4\}$ and $Q^3 = \{4, 5, 6\}$ (see Figure 1).*

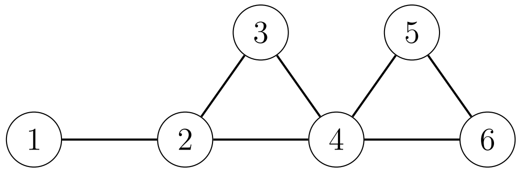

Figure 1: Example of a cover represented as cliques.

In this case, $P_2^{13} = \{Q^1, Q^2, Q^3\}$ *is a path between* Q^1 *and* Q^3 *through agent* 2*. The other paths are* $P_2^{12} = \{Q^1, Q^2\}$*,* $P_2^{21} = \{Q^2, Q^1\}$*,* $P_4^{23} = \{Q^2, Q^3\}$*,* $P_4^{32} = \{Q^3, Q^2\}$*, and* $P_4^{31} = \{Q^3, Q^2, Q^1\}$*. Moreover,* $N_{1,1}^P = \emptyset$*,* $N_{2,2}^P = \{1, 2\}$*,* $N_{1,2}^P = \{2, 3, 4, 5, 6\}$*,* $N_{2,4}^P = \{4, 5, 6\}$*, and so on.*

A game $\left(N, v^{\mathcal{Q}}\right)$ is a *clique game* if $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ covers N and there exist $\{v_i\}_{i \in N} \subset \mathbb{R}_+$ and $\{v_Q\}_{Q \in \mathcal{Q}} \subset \mathbb{R}_+$ such that:

i) for all $k, k' \in \{1, \ldots, K\}$ with $k \neq k'$, $\left|Q^k \cap Q^{k'}\right| \leq 1$

ii) for all $k \in \{1, \ldots, K\}$ and all $i, j \in Q^k$ with $i \neq j$, $N_{k,i}^P \cap N_{k,j}^P = \emptyset$ (there is at most one path between any two elements of $\mathcal{Q}$),

iii) for all $S \subseteq N$,

$$v^{\mathcal{Q}}(S) = \sum_{i \in S} v_i + \sum_{Q \in \mathcal{Q}(S)} \left(|Q \cap S| - 1\right) v_Q \tag{1}$$

with $\mathcal{Q}(S) = \{Q \in \mathcal{Q} : S \cap Q \neq \emptyset\}$.

We write $\mathcal{Q}(i)$ for $\mathcal{Q}\left(\{i\}\right)$.

Let $\mathcal{C}$ be the set of all clique games.[2]

We propose an example of a clique game.

Example 2 *(Trading goods) The agent set is* $N = \{1, 2, 3, 4, 5\}$*, with* 1 *and* 2 *being producers and* 3, 4 *and* 5 *being buyers. Producer* 1 *has a capacity to produce two units at constant marginal cost* c_1 *while producer* 2 *can produce*

[2]Alternatively, we could describe clique games using the notion of hypergraphs, a generalization of undirected graphs in which edges might consists of more than two vertices. The set of cliques would have to be represented by a linear and α-acyclic hypergraph. See Jégou and Ndiaye (2009) and Dong et al. (2018) for complete definitions.

a single unit at cost c_2. Each buyer i is interested in a single unit that she values at R_i. We suppose that these valuations are larger than the marginal cost of the producers.

Producers 1 and 2 have exclusive territories (because of vertical restraints or collusion) and buyers 3 and 4 are on the territory of producer 1 and buyer 5 on the territory of producer 2. The producers' unused capacity can be sold to external buyers at price q and buyers have the option of buying from an external supplier at price p, with $R_i > p > q > c_j$.

When a coalition forms, trades occur between buyers and sellers in the same territory, with unsatisfied demands and unsold supply resolved on the outside market. For example, coalition $\{1, 2, 3, 5\}$ can organize trades between 1 and 3 and 2 and 5, generating a surplus of $R_3 + R_5 - c_1 - c_2$. In addition, producer 1 can sell its extra unit on the outside market, generating an additional surplus of $q - c_1$.

The game can thus be represented (see Figure 2) by a clique game, with cover $\mathcal{Q} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}\}$ and $v_1 = 2q - 2c_1$, $v_2 = q - c_2$, $v_i = R_i - p$ for $i = 3, 4, 5$, $v_{\{1,2\}} = 0$ and $v_Q = r \equiv p - q$ otherwise.

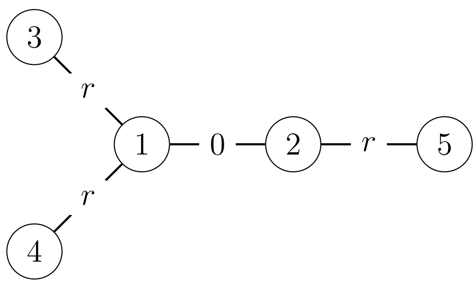

Figure 2: Clique cover of a trading goods game.

We next establish the connection between clique games and the PS-games of Kar et al. (2009).[3] We say that a game (N, v) is a *PS-game* if for all $i \in N$ and $S, S' \subseteq N \setminus \{i\}$ we have

$$\Delta_i^v(S) + \Delta_i^v\left(N \setminus (S \cup \{i\})\right) = \Delta_i^v(S') + \Delta_i^v\left(N \setminus (S' \cup \{i\})\right). \qquad (2)$$

[3]Even though Kar et al. (2009) do not explicitly mention what *PS* stands for, a plausible possibility is "Player Specific".

Let $\mathcal{PS}$ be the set of PS-games.

Clique games can be described with a similar condition. To show this, we first calculate the contributions in a clique game.

Lemma 1 *For all $(N, v^{\mathcal{Q}}) \in \mathcal{C}$, all $T \subset N$, and all $S \subseteq N \setminus T$,*

$$\Delta_T^{v^{\mathcal{Q}}}(S) = \sum_{i \in T} v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(T)} |Q \cap T| v_Q + \sum_{Q \in \mathcal{Q}(T) \setminus \mathcal{Q}(S)} (|Q \cap T| - 1) v_Q.$$

Proof. By definition of a contribution,

$$\begin{aligned}
\Delta_T^{v^{\mathcal{Q}}}(S) &= v^{\mathcal{Q}}(S \cup T) - v^{\mathcal{Q}}(S) \\
&\overset{(1)}{=} \sum_{i \in T} v_i + \sum_{Q \in \mathcal{Q}(S \cup T)} (|Q \cap (S \cup T)| - 1)\, v_Q - \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\, v_Q \\
&= \sum_{i \in T} v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(T)} [(|Q \cap (S \cup T)| - 1) - (|Q \cap S| - 1)]\, v_Q \\
&+ \sum_{Q \in \mathcal{Q}(T) \setminus \mathcal{Q}(S)} (|Q \cap (S \cup T)| - 1)\, v_Q \\
&= \sum_{i \in T} v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(T)} [|Q \cap S)| + |Q \cap T)| - 1 - (|Q \cap S| - 1)]\, v_Q \\
&+ \sum_{Q \in \mathcal{Q}(T) \setminus \mathcal{Q}(S)} (|Q \cap T| - 1)\, v_Q \\
&= \sum_{i \in T} v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(T)} |Q \cap T| v_Q + \sum_{Q \in \mathcal{Q}(T) \setminus \mathcal{Q}(S)} (|Q \cap T| - 1) v_Q.
\end{aligned}$$

■

We are now ready to show that clique games have a property related to the sum of contributions of a coalition to another coalition and its complement. The property is reminiscent of the property of PS-games.

Lemma 2 *For all $(N, v^{\mathcal{Q}}) \in \mathcal{C}$ and all $T \subset N$, if $|Q \cap T| \in \{0, |Q| - 1, |Q|\}$ for all $Q \in \mathcal{Q}$, then, for all $S, S' \subseteq N \setminus T$,*

$$\Delta_T^{v^{\mathcal{Q}}}(S) + \Delta_T^{v^{\mathcal{Q}}}(N \setminus (S \cup T)) = \Delta_T^{v^{\mathcal{Q}}}(S') + \Delta_T^{v^{\mathcal{Q}}}(N \setminus (S' \cup T)).$$

Proof. Let T be such that $|T \cap Q| \in \{0, |Q| - 1, |Q|\}$ for all $Q \in \mathcal{Q}$. Fix

$S \subseteq N \setminus T$. From Lemma 1, we have that

$$\Delta_T^{v^{\mathcal{Q}}}(S) = \sum_{i\in T} v_i + \sum_{Q\in\mathcal{Q}(S)\cap\mathcal{Q}(T)} |Q\cap T|\, v_Q + \sum_{Q\in\mathcal{Q}(T)\setminus\mathcal{Q}(S)} (|Q\cap T| - 1)\, v_Q$$

and

$$\Delta_T^{v^{\mathcal{Q}}}(N\setminus(S\cup T)) = \sum_{i\in T} v_i + \sum_{Q\in\mathcal{Q}(N\setminus(S\cup T))\cap\mathcal{Q}(T)} |Q\cap T|\, v_Q + \sum_{Q\in\mathcal{Q}(T)\setminus\mathcal{Q}(N\setminus(S\cup T))} (|Q\cap T| - 1)\, v_Q.$$

For all $i \in T$, it is obvious that the coefficient associated to v_i is 2. Fix $Q \in \mathcal{Q}$. We are interested in the coefficient associated to v_Q in $\Delta_T^{v^{\mathcal{Q}}}(S) + \Delta_T^{v^{\mathcal{Q}}}(N \setminus (S \cup T))$. We have three cases:

1. Suppose that $|Q \cap T| = 0$. Then $Q \notin \mathcal{Q}(T)$ and thus the coefficient is zero.

2. Suppose that $|Q \cap T| = |Q|$. Then $Q \cap S = Q \cap (N \setminus (S \cup T)) = \emptyset$ and $Q \cap T = Q$. Thus, the coefficient is $2\,(|Q| - 1)$.

3. Suppose that $|Q \cap T| = |Q| - 1$. Then either $|Q \cap S| = 1$ and $|Q \cap (N \setminus (S \cup T))| = 0$ or $|Q \cap S| = 0$ and $|Q \cap (N \setminus (S \cup T))| = 1$. In both cases, the coefficient associated to v_Q is $(2\,|Q| - 3)$.

Therefore, the coefficient associated to v_Q is always independent of S. Thus, $\Delta_T^{v^{\mathcal{Q}}}(S) + \Delta_T^{v^{\mathcal{Q}}}(N\setminus(S\cup T)) = \Delta_T^{v^{\mathcal{Q}}}(S') + \Delta_T^{v^{\mathcal{Q}}}(N\setminus(S'\cup T))$ for all $S, S' \subseteq N \setminus T$ as desired. ■

Notice that if $|Q| \leq 2$, the condition $|Q \cap T| \in \{0, |Q| - 1, |Q|\}$ is trivially satisfied. This is how we build a link with PS-games.

Deng and Papadimitriou (1994) define a family of games where value is created by the presence in a coalition of groups of a certain size, the *k-additive games*. Formally, for $k = 1, ..., n$, the k-additive games are defined by $(a_i)_{i\in N}$ and $(b_S)_{\substack{S\subseteq N\\|S|=k}}$ and are such that $V^{a,b}(S) = \sum_{i\in S} a_i + \sum_{\substack{T\subseteq S\\|T|=k}} b_T$. Let $\mathcal{A}^k$ be the set of k-additive games. Let $\mathcal{C}^2$ be the set of covers $\mathcal{Q}$ of N such that $|Q| = 2$ for all $Q \in \mathcal{Q}$ and such that conditions i) and ii) in the definition of clique games are satisfied. Let $\mathcal{A}^{2*}$ be the set of 2-additive games such that there exists $\mathcal{Q} \in \mathcal{C}^2$ such that $b_S > 0$ implies $S \in \mathcal{Q}$. Then, it is not difficult to check that this 2-additive game coincides with $\left(N, v^{\mathcal{Q}}\right)$ where $v_i = a_i$ and $v_Q = b_Q$ for all $Q \in \mathcal{Q}$.

We show that a clique game is not necessarily a PS-game, and vice-versa. In fact, a game is both a clique game and a PS-game if and only if it is a 2-additive game in $\mathcal{A}^{2*}$.

Theorem 1 $\mathcal{PS} \cap \mathcal{C} = \mathcal{A}^{2*}$.

Proof. "$\supseteq$" Suppose that we have a clique game $v^{\mathcal{Q}}$ such that for all $Q^k \in \mathcal{Q}$, $|Q^k| = 2$. We show that it is also a PS-game.

From Lemma 2, for all $i \in N$ and $S, S' \subseteq N \setminus \{i\}$ we have

$$\Delta_i^{v^{\mathcal{Q}}}(S) + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S \cup \{i\})) = \Delta_i^{v^{\mathcal{Q}}}(S') + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S' \cup \{i\}))$$

if $|Q \cap \{i\}| \in \{0, |Q|-1, |Q|\}$ for all $Q \in \mathcal{Q}$. Since $|Q| = 2$ for all $Q \in \mathcal{Q}$, condition (2) is trivially satisfied and $v^{\mathcal{Q}}$ is a PS-game.

"$\subseteq$" We show that in all other cases, $v^{\mathcal{Q}}$ is not a PS-game.

Suppose that there exists $Q \in \mathcal{Q}$ such that $|Q| > 2$. If $v_Q = 0$, then we can consider the alternative cover

$$\mathcal{Q}' = (\mathcal{Q} \setminus \{Q\}) \cup \{\{i\} : \mathcal{Q}(i) = Q\}.$$

It is not difficult to check that $v^{\mathcal{Q}} = v^{\mathcal{Q}'}$. Hence, we can assume that $v_Q > 0$.

Fix $i \in Q$ and let $S \subseteq N \setminus \{i\}$ be such that $|Q \cap S| = 1$ and $S' \subseteq N \setminus \{i\}$ be such that $|Q \cap S')| = |Q| - 1$.

Consider the coefficient associated to v_Q in $\Delta_i^{v^{\mathcal{Q}}}(S) + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S \cup \{i\}))$. Agent i joins a member of Q in S (since $|Q \cap S| = 1$) and at least a member of Q in $N \setminus (S \cup \{i\})$ (since $|Q| > 2$) and thus the coefficient is 2.

Consider the coefficient associated to v_Q in $\Delta_i^{v^{\mathcal{Q}}}(S') + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S' \cup \{i\}))$. Agent i joins a member of Q in S' (since $|Q \cap S'| = |Q| - 1$) but no member of Q in $N \setminus (S' \cup \{i\})$ (since $S' \cup \{i\} = Q$) and thus the coefficient is 1.

Thus, since $v_Q > 0$, $\Delta_i^{v^{\mathcal{Q}}}(S) + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S \cup \{i\})) \neq \Delta_i^{v^{\mathcal{Q}}}(S') + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S' \cup \{i\}))$ and v^Q is not a PS-game. ■

Thus, only some clique games are also PS-games. Moreover, not all PS-games are clique games, as the next example shows:

Example 3 (Example 3.12 in Kar et al. (2009)) *We consider the TU game* (N, v) *with* $N = \{1, 2, 3, 4\}$ *and such that* $v(S) = 0$ *if* $|S| = 1$, 1 *if* $|S| = 2$, $\frac{3}{2}$ *if* $|S| = 3$, *and* 3 *if* $S = N$. *This is a* PS*-game with* $\Delta_i^v(S) + \Delta_i^v(N \setminus (S \cup \{i\})) = \frac{3}{2}$ *for all* i *and* S. *However, it is not a clique game. To see this, notice that* $v(S) = 0$ *if* $|S| = 1$ *implies that* $v_i = 0$ *for all* $i \in N$. *Then,* $v(S) = 1$ *if* $|S| = 2$ *implies that any pair* i, j *belong to some clique* Q *with* $v_Q = 1$. *The no-cycle condition of clique games (condition i)) leaves us with a single candidate for the set of cliques:* $\mathcal{Q} = \{N\}$. *But then* $v(S) = |S| - 1$ *for all* S, *which is different from the* PS*-game for* $|S| = 3$.

4 Coincidence between the Shapley value and the nucleolus

We are now ready for our main result, that shows the coincidence of the Shapley value, the nucleolus and the permutation-weighted average of extreme points of the core. We also provide for it a closed-form expression.

The coincidence between the Shapley value and the permutation-weighted average of the extreme points of the core follows from the fact that clique games are convex. The closed-form expression follows from Lemma 1 and the definition of the Shapley value. The main difficulty is in showing the coincidence with the nucleolus. To do so, we evaluate excesses at the Shapley value. For each coalition $S \subset N$ that excess is a fraction of the value created by each clique containing at least a member of S. That fraction is exactly the number of missing member of the cliques over the number of members of that clique.

This allows to easily pinpoint the candidates for minimal excesses: the groups for which we have a single missing member for a clique, and either all or no members of all other cliques. This is not a coincidence, remember from Lemma 2 the special properties of the contribution of coalition T to S

and its complement when the intersection of T with cliques contain none, all or all but one agents of the clique.

We then rank cliques according to their "per-capita" value created. Starting from Q_1, the one creating the least per-capita value, we examine these coalitions that generates exactly that minimum excess of its per-capita value. By the properties of clique games, notably the "no cycle" property, there is exactly $|Q_1|$ groups generating that excess. In addition, their complements form a partition of N, with each of its elements containing a single member of Q_1 : thus, to increase the excess of one of these groups would imply reducing it for another, and that modified allocation cannot be the nucleolus. Proceeding in the same manner for cliques in increasing order of their per-capita value, we obtain that the Shapley value is also the nucleolus.

Theorem 2 *For all $v^{\mathcal{Q}} \in \mathcal{C}$ and all $i \in N$,*

$$Sh_i\left(v^{\mathcal{Q}}\right) = \bar{y}_i\left(v^{\mathcal{Q}}\right) = Nu_i\left(v^{\mathcal{Q}}\right) = v_i + \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q.$$

Proof. Fix $v^{\mathcal{Q}} \in \mathcal{C}$.

Step 1: Shapley value and permutation-weighted average of core allocations

It is obvious from Lemma 1 that v^{Q} is a convex game. Thus, the Shapley value is the average of extreme points of the core (Shapley, 1971; Ichiishi, 1981) and $Sh\left(v^{\mathcal{Q}}\right) = \bar{y}\left(v^{\mathcal{Q}}\right)$. We show that for all $i \in N$,

$$Sh_i\left(v^{\mathcal{Q}}\right) = Nu_i\left(v^{\mathcal{Q}}\right) = v_i + \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q.$$

We suppose that for all $k \in \{1, ..., K\}$, $\bigcup_{i \in Q^k} N^P_{k,i} = N \setminus Int(Q^k)$, that is, there is a (unique) path between any two elements of $\mathcal{Q}$. Without that assumption, we can partition our agents into groups unconnected by paths, and we can compute the Shapley value, the permutation-weighted average of extreme points of the core, and the nucleolus independently on each component of the partition.

We start with $Sh\left(v^{\mathcal{Q}}\right)$. Given $\pi \in \Pi$, under Lemma 1, the contribution of agent i to $P_i(\pi)$ is $v_i+\sum_{Q\in\mathcal{Q}(P_i(\pi))\cap\mathcal{Q}(i)} v_Q$. For each $Q \in \mathcal{Q}(i)$, the probability that $Q \in \mathcal{Q}\left(P_i(\pi)\right) \cap \mathcal{Q}(i)$ is the probability of agent i not being the first one to be chosen out of Q, and hence it is $\frac{|Q|-1}{|Q|}$. Summing up, we obtain the desired result.

Step 2: Computations of excesses

We now focus on $Nu\left(v^{\mathcal{Q}}\right)$. Let $x \in \mathbb{R}^N$ defined as $x_i = v_i+\sum_{Q\in\mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q$ for all $i \in N$. We have that

$$
\begin{aligned}
e\left(S, x, v^{\mathcal{Q}}\right) &= \sum_{i\in S} v_i + \sum_{i\in S}\sum_{Q\in\mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q - \sum_{i\in S} v_i - \sum_{Q\in\mathcal{Q}(S)} \left(|Q\cap S|-1\right) v_Q \\
&= \sum_{Q\in\mathcal{Q}(S)} \left(\frac{|Q\cap S|\left(|Q|-1\right)}{|Q|} - \left(|Q\cap S|-1\right)\right) v_Q \\
&= \sum_{Q\in\mathcal{Q}(S)} \frac{|Q\cap S|\left(|Q|-1\right) - \left(|Q\cap S|-1\right)|Q|}{|Q|} v_Q \\
&= \sum_{Q\in\mathcal{Q}(S)} \frac{|Q| - |Q\cap S|}{|Q|} v_Q
\end{aligned}
$$

for all $S \subset N$, $S \neq \emptyset$.

Step 3: Finding coalitions with minimal excess

Assume without loss of generality $\frac{v_{Q^1}}{|Q^1|} \leq \frac{v_{Q^2}}{|Q^2|} \leq \cdots \leq \frac{v_{Q^K}}{|Q^K|}$.

For each $i \in Q^1$, let $S_i^1 = N \setminus \left(N_{1,i}^P \cup \{i\}\right)$. Note that for all $Q \in \mathcal{Q}\setminus\{Q^1\}$, either $S_i^1 \cap Q = \emptyset$ or $S_i^1 \cap Q = Q$. In addition, $S_i^1 \cap Q^1 = Q^1 \setminus \{i\}$. Thus, $e\left(S_i^1, x, v^{\mathcal{Q}}\right) = \frac{v_{Q^1}}{|Q^1|}$. By construction, this is the lowest excess value. To see why, note that any $S \subset N$ has at least one $Q \in \mathcal{Q}\left(S\right)$ such that $|Q\cap S| < |Q|$. That generates an excess of $\frac{|Q|-|Q\cap S|}{|Q|} v_Q \geq \frac{v_Q}{|Q|} \geq \frac{v_{Q^1}}{|Q^1|}$.

Step 4: Complements of coalitions with minimal excess form a partition of N

For each $i \in Q^1$, let $T_i^1 = N_{1,i}^P \cup \{i\} = N \setminus S_i^1$. Take $\{T_i^1\}_{i\in Q^1}$. This is a partition of N. To see why, note that each T_i^1 is nonemtpy (because $i \in T_i^1$ for all $i \in Q^i$), their union is N (because all cliques are connected through a path), and they are pairwise disjoint (because of assumption i)). Thus, we have $|Q^1|$ coalitions $\{T_i^1\}_{i\in Q^1}$ whose complements $\{S_i^1\}_{i\in Q^1}$ have the minimal excess, with each agent belonging to exactly one of these coalitions in $\{T_i^1\}_{i\in Q^1}$.

We prove that, in order to increase the excess of one of the coalitions in $\{T_i^1\}_{i\in Q^1}$, we would need to decrease the excess of another coalition, and so the corresponding allocation could not be the nucleolus. Let $i \in Q^1$. First, suppose that $e(S_i^1, Nu(v^{\mathcal{Q}}), v^{\mathcal{Q}}) > e(S_i^1, x, v^{\mathcal{Q}})$. Then $e(T_i^1, Nu(v^{\mathcal{Q}}), v^{\mathcal{Q}}) < e(T_i^1, x, v^{\mathcal{Q}})$. Since $\{T_j^1\}_{i\in Q^1}$ is a partition of N, there exists $j \in Q^1$ such that $e(T_j^1, Nu(v^{\mathcal{Q}}), v^{\mathcal{Q}}) > e(T_j^1, x, v^{\mathcal{Q}})$. Then, $e(S_j^1, Nu(v^{\mathcal{Q}}), v^{\mathcal{Q}}) < e(S_j^1, x, v^{\mathcal{Q}})$. This implies that $x >_L Nu(v^{\mathcal{Q}})$. Next, suppose that $e(S_i^1, Nu(v^{\mathcal{Q}}), v^{\mathcal{Q}}) < e(S_i^1, x, v^{\mathcal{Q}})$. In this case, it immediately follows that $x >_L Nu(v^{\mathcal{Q}})$. Thus, we can conclude that, for all $i \in Q^1$, $e(S_i^1, Nu(v^{\mathcal{Q}}), v^{\mathcal{Q}}) = e(S_i^1, x, v^{\mathcal{Q}})$.

Step 5: Concluding

We repeat the process for all Q^k to obtain that

$$\sum_{j\in S_i^k} Nu_j\left(v^{\mathcal{Q}}\right) = \sum_{j\in S_i^k} x_j \tag{3}$$

for all $Q^k \in \mathcal{Q}$ and all $i \in Q^k$. In case $i \in Int(Q^k)$ for some $k \in \{1, ..., K\}$, we have $S_i^k = N \setminus \{i\}$, from where (3) and efficiency of x imply $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$.

In case $\mathcal{Q} = \{Q^1\}$, we have $N = Int\left(Q^1\right)$ and hence $Nu\left(v^{\mathcal{Q}}\right) = x$. So, we assume $|\mathcal{Q}| > 1$. From condition i) in the definition of clique games, there exist some $i \in N$ and $Q^k \in \mathcal{Q}(i)$ such that $Q = Int(Q) \cup \{i\}$ for all $Q \in \mathcal{Q}(i) \setminus \{Q^k\}$. This implies that $Nu_j\left(v^{\mathcal{Q}}\right) = x_j$ for all $j \in Q^k \in \mathcal{Q}(i) \setminus \{Q^k\}$. Under (3) and the efficiency of x, we deduce $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$. Repeating the reasoning, we can always find a new $i \in N$ and $Q^k \in \mathcal{Q}(i)$ such that $Nu_j\left(v^{\mathcal{Q}}\right) = x_j$ for all $j \in Q^k \in \mathcal{Q}(i) \setminus \{Q^k\}$, so that (3) and the efficiency of x imply $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$, and so on until we get $Nu\left(v^{\mathcal{Q}}\right) = x$. ■

5 Graph-induced games

We apply our result to three families of games that represent cooperation possibilities or impossibilities on a graph. We need the following graph theory definitions.

We now interpret N as a set of vertices, and say that a graph on N is a set of unordered pairs of distinct members of N. Let G^N be the complete graph: $G^N = \{(i, j) : i, j \in N, i \neq j\}$. A graph G is a subset of G^N. For any $S \subseteq N$, $G[S]$ is the subgraph of G induced by S.

Suppose $S \subseteq N$, $G \subseteq G^N$, $i, j \in N$. We say that i and j are connected in S by G iff there is a path in G which goes from i to j and stays within S. That is, there is some $k \geq 1$ and a sequence $i^0, i^1, ..., i^k$ such that $i^0 = i$, $i^k = j$ and $(i^{l-1}, i^l) \in G[S]$ for all $l = 1, ..., k$. A coalition T is connected in S by G if i and j are connected in S by G, for all $i, j \in T$. If S is connected by S in G, we simply say that S is connected by G. We say that a connected coalition T in S is maximal if there does not exist a coalition $T' \supset T$ that is connected in S by G.

We say that a graph G is a clique graph if there exists Q, a cover of N with $Q = \left\{Q^1, ..., Q^K\right\}$ such that for all $k = 1, ..., K$, Q^k is connected by G. In addition, a clique graph G is said to be acyclical if for all i, j such that there does not exists $Q^k \supseteq \{i, j\}$, there exists at most a single path between them: there does not exist S such that i and j are connected in both S and $N \setminus S$ by G.

We say that a clique graph is disjoint if for all $i \in N$, $|\mathcal{Q}(i)| = 1$. It is obvious that a disjoint clique graph is also acyclical.

Let $G^C = G^N \setminus G$ be the complement of graph G.

5.1 Graph-restricted cooperative game

We consider the graph-restricted cooperation game introduced by Myerson (1977), in which a coalition of agents can cooperate together only if its members are connected.

More precisely, every graph G partitions every coalition S into a set of maximal connected subcoalitions, P_S, in a natural way.

Let $V \in \mathbb{R}_+^{2^N}$ be a coalition function. A graph restricted problem is (G, V). The graph restricted game V_G is then defined by

$$V_G(S) = \sum_{T \in P_S} V(T).$$

In words, V represents the values obtained by each coalition if all of its members can cooperate. But, in practice, cooperation within a coalition might reduce to cooperation among subcoalitions, so that a coalition S fails to extract all of its potential value $V(S)$.

While any value function can be used, symmetric functions eliminate the differences coming from the game V, allowing to focus on the graph, thus defining a centrality measure (Gomez et al., 2003). We use a function that is part of the family decribed by Gonzalez-Aranguena et al. (2017): for $a > 0$, $V^a(S) = 0$ if $S = \emptyset$ and $V^a(S) = (|S|-1)a$ otherwise. Thus, in V^a, a coalition receives a value of a for each of its members, starting with the second one.

Theorem 3 *Let G be an acyclical clique graph. Then, for all $i \in S$ and all $a > 0$,*

$$Sh_i(V_G^a) = Nu_i(V_G^a) = \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} a.$$

It is not difficult to see that the combination of the acyclical clique graph and the properties of V^a gives rise to a clique game, in which $v_i = 0$ for all $i \in N$ and $v_Q = a$ for all cliques Q. The result then follows from Theorem 2. Both the Shapley value (known as the Myerson value in this context) and the nucleolus (Montero, 2013) have been proposed as centrality or power indexes. The result above allows to understand when they coincide.

5.2 Minimum coloring game

We now consider a model in which the graph represents conflict situations.

Agents have to be placed in facilities, but there is potential conflict between individuals. If two agents are in conflict, they cannot be put in the same facility. We are trying to find the minimum number of facilities needed to locate all individuals in N. We assume that facilities cost 1 unit each.

Conflicts are represented in graph G: If $(i, j) \in G$, then agents i and j are in conflict. The problem is known as the minimum coloring game, as we attempt to assign colors to all vertices (agents) in the graph, with different colors for agents in conflict, all while using the minimum number of colors (Deng et al., 1999).

For all $S \subseteq N$, let $c_G^{MC}(S)$ be the minimum number of facilities to locate all members of S, given conflict subgraph $G[S]$.

Theorem 4 *(Okamoto, 2008) If G^C is a disjoint clique graph, then, for all $i \in N$,*

$$Sh_i(c_G^{MC}) = Nu_i(c_G^{MC}) = \frac{1}{|Q_i|}$$

where Q_i is the only clique that agent i belongs to.

While Okamoto (2008) notices the coincidence between the Shapley value and the nucleolus, no explanation is provided. It is obvious that if G^C is a disjoint clique graph, we obtain (once we transform the game into a value game) a clique game in which $v_i = 0$ for all $i \in N$ and $v_Q = 1$ for all cliques Q. Therefore, we now have an explanation on why we have this coincidence. We can also be pessimistic about coincidence on a larger set of minimum coloring games, as if an agent belongs to multiple cliques, we do not obtain the linear form of value creation needed for a clique game.

5.3 Minimum cost spanning tree problems

This well-studied problem has agents connecting to a source through a network, with the cost of an edge being a fixed amount that is paid if the edge is used, regardless of the number of users of the edge. Formally, we assume that the agents in N need to be connected to a source, denoted by 0. For all $S \subseteq N$, let $S_0 = S \cup \{0\}$. We consider a subset of mcst problems where the cost of an edge can only take two values: 0 or 1. These problems are called elementary mcst problems and form a basis for all mcst problems. They are called information games in Kuipers (1993). A cost matrix $c = \{c_{ij}\}_{i,j \in N_0}$, with $c_{ij} = c_{ji}$ describe the costs for all edges.

For each coalition $S \subseteq N$, we are interested in finding the minimal cost to connect the members of S to the source, using only edges in $G[S_0]$. $C(\cdot, c)$ is the corresponding game associated to the mcst problem and its cost matrix c.

Let $G_c^0 = \{(i,j) \in N_0 \times N_0 : c_{ij} = 0\}$ be the graph containing all free edges in c. Trudeau (2012) defines a cycle complete cost matrix as one such that if we take any cycle of edges containing i and j, its most expensive edge cannot be cheaper than the direct connection on edge (i, j). It is easy to see that for elementary cost matrices, the condition reduces to G^0 being an acyclical clique graph. In other words, we obtain a clique game where cliques are such that members of a clique have free edges among themselves and they collectively have at most a single free edge to any outside agent. Thus, supposing that at most one agent has a free connection to the source, there is always a gain of 1 to adding a member of the clique, starting with the second one.[4]

Let $\mathcal{Q}^*(i) = \{Q \in \mathcal{Q} : 0 \notin Q\}$ and $\mathcal{Q}^0(i) = \{Q \in \mathcal{Q} : 0 \in Q\}$ be respectively the set of cliques distinguished by the presence or not of the source among its members. We obtain the following results:

Theorem 5 *For any elementary cost matrix c, if G_c^0 is an acyclical clique graph, then, for all $i \in N$,*

$$Sh_i(C) = Nu_i(C) = \bar{y}_i(C) = \begin{cases} 1 - \sum_{Q \in \mathcal{Q}^*(i)} \frac{|Q|-1}{|Q|} & \textit{if } \mathcal{Q}^0(i) = \emptyset \\ - \sum_{Q \in \mathcal{Q}^*(i)} \frac{|Q|-1}{|Q|} & \textit{if } \mathcal{Q}^0(i) \neq \emptyset \end{cases}.$$

Trudeau (2012) studies the Shapley value of cycle-complete cost matrices, and calls it the cycle-complete solution. Our result show that when the matrices are also elementary, we obtain nice features: the Shapley value, the nucleolus and the permutation-weighted average of extreme points of the core of the corresponding game coincide. The coincidence between the cycle-complete solution and $\bar{y}$ was shown in Trudeau and Vidal-Puga (2017).

A well-studied subset of cycle-complete matrices is the set of irreducible matrices (Bird, 1976; Bergantiños and Vidal-Puga, 2007). They are such that if we take any path from i to j, its most expensive edge cannot be cheaper than the direct connection on edge (i, j). The Shapley value of irreducible matrices is known as the folk solution (Bogomolnaia and Moulin, 2010). If c is an irreducible matrix, G_c^0 is such that if $\mathcal{Q}^0(i) \neq \emptyset$, then $\mathcal{Q}^*(i) = \emptyset$, as otherwise it would imply that there is a free path from the source to agents in $Q \in \mathcal{Q}^*(i)$. Thus, the result simplifies for irreducible matrices. For all

[4] If the source belongs to the clique then the benefit of adding a member is always zero.

$i \in N$,

$$Sh_i(C) = Nu_i(C) = \bar{y}_i(C) = \begin{cases} 1 - \sum_{Q \in \mathcal{Q}^*(i)} \frac{|Q|-1}{|Q|} \text{ if } \mathcal{Q}^0(i) = \emptyset \\ 0 \text{ if } \mathcal{Q}^0(i) \neq \emptyset \end{cases}.$$

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