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The Consistent Coalitional Value

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We describe a value for nontransferable utility games with coalition structure. This value coincides with the consistent value for trivial coalition structures, and with the Owen value for transferable utility games with coalition structure. Furthermore, we present two characterizations: the first one using properties of balanced contributions and the second one using a consistency property.

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1. Introduction. Some of the most important issues of cooperative game theory are to define "good" values, to study which properties are satisfied by these values, and to obtain axiomatic characterizations using some of these properties.

In cooperative games with transferable utility (*TU* games), Shapley [20] introduced the Shapley value. He characterized it as the only value satisfying efficiency, null player, symmetry, and additivity. Later, several authors obtained new characterizations of the Shapley value using other properties. For instance, Myerson [16] characterized the Shapley value using balanced contributions, and Hart and Mas-Colell [9] characterized it by consistency.

There are several extensions of *TU* games. The most natural one is the extension to games with nontransferable utility (*NTU* games). Another extension applies to *TU* games with coalition structure, which study situations where players are partitioned into several groups. This model was considered by Aumann and Drèze [2] and Owen [18]. Of course, a third extension is to *NTU* games with coalition structure. Several authors proposed values that are generalizations of the Shapley value in these extended models.

In *NTU* games, the Harsanyi value (Harsanyi [7]) and the Shapley *NTU* value (Shapley [21]) are generalizations of the Shapley value. Maschler and Owen [11, 12] defined the consistent value for hyperplane games and *NTU* games. The main idea behind this generalization is to maintain (as far as possible) the consistency property of the Shapley value. Later, Hart and Mas-Colell [10] developed a noncooperative game that yields the consistent value as subgame perfect equilibrium outcome. They also characterized it by means of balanced contributions.

Owen [18] introduced a generalization of the Shapley value, called the Owen value, for *TU* games with coalition structure. He characterized his value using similar axioms to those used by Shapley [20]. Later, Winter [24] characterized the Owen value using the consistency property, and Calvo et al. [4] did so using properties of balanced contributions.

In volumes 2 and 3 of the *Handbook of Game Theory with Economic Applications*, Chapters 37 ("Coalition Structures" by Greenberg [6]), 53 ("The Shapley Value" by Winter [25]), 54 ("Variations of the Shapley Value" by Monderer and Samet [15]), and 55 ("Values of Non-transferable Utility Games" by McLean [14]), it is possible to find surveys of this literature.

It is of interest to know whether the consistent value and the Owen value can be generalized in the same way. Therefore, we introduce a new value called the *consistent coalitional value*. This value is defined through a two-stage procedure. In the first stage, the utility is divided among the coalitions. In the second stage, the utility obtained in the first stage by each coalition is divided among its members.

This value can be characterized in two ways: by the consistency property and by the balanced contributions property. We must note that our characterizations generalize the results about consistency obtained by Maschler and Owen [11] for the consistent value and Winter [24] for the Owen value. We also generalize the results about balanced contributions obtained by Hart and Mas-Colell [10] for the consistent value and Calvo et al. [4] for the Owen value. We believe these characterizations make the consistent coalitional value a proper generalization of the consistent and the Owen value for *NTU* games with coalition structure.

Furthermore, Vidal-Puga [22] proposed a noncooperative game for which the consistent coalitional value arises as subgame perfect equilibrium payoff. His results are similar to those presented by Hart and Mas-Colell [10] for the consistent value.

Chae and Heidhues [5] described a value for bargaining problems with coalition structure. Bergantiños et al. [3] proved that the consistent coalitional value coincides with this value in the subclass of bargaining problems.

The consistent coalitional value cannot be obtained as an average of marginal contributions depending on equally likely permutations. The value obtained in this way is called the *random order value*. We prove that even this value satisfies some interesting properties, it satisfies neither consistency nor balanced contributions.

The paper is organized as follows. In §2 we introduce the notation and some previous results. In §3 we define the consistent coalitional value. In §4 present the axiomatic characterization with balanced contributions. In §5 we present the axiomatic characterization in hyperplane games with consistency. §6 is devoted to some concluding remarks. Finally, in Appendix A we present the proofs of the results obtained in the paper.

2. Definitions and previous results. Given a set A, $|A|$ denotes the cardinal of A. Given $x, y \in \mathbb{R}^N$, we say $y \leq x$ when $y_i \leq x_i$ for each $i \in N$ and $x * y$ is the scalar product $\sum_{i\in N} x_i y_i$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N\colon\ x_i \geq 0, \forall i\}$ and $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N\colon\ x_i > 0, \forall i\}$. We say that $x \in \mathbb{R}^N$ is *normalized* if $\sum_{i\in N} \max\{x_i, -x_i\} = 1$. Given $\lambda \in \mathbb{R}^N$ a vector orthogonal to some surface on $\mathbb{R}^N$, we say that λ is *orthonormal* if it is normalized.

A *game with nontransferable utility*, or simply an *NTU game*, is a pair (N, V) where $N = \{1, 2, \ldots, n\}$ is the set of players and V is a correspondence (*characteristic function*) which assigns to each coalition $S \subset N$ a subset $V(S) \subset \mathbb{R}^S$. This subset represents all the possible payoffs that members of S can obtain for themselves when playing cooperatively. For $S \subset N$, if there is no ambiguity, we maintain the notation V when referring to the correspondence V restricted to S as player set. We also denote $\bar{S} = N \backslash S$.

Following Maschler and Owen [12], we impose the next conditions on V:

(A1) For each $S \subset N$, the set $V(S)$ is comprehensive (i.e., if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and bounded above (i.e., for each $x \in \mathbb{R}^S$, the set $\{y \in V(S)\colon\ y \geq x\}$ is compact).

(A2) For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is smooth (on each point of the boundary there exists a unique outward orthonormal vector) and nonlevel (the outward vector on each point of $\partial V(S)$ has its coordinates positive.) We denote these orthonormal vectors as $\lambda^S = (\lambda^S_i)_{i\in S}$.

(A3) These λ^S_i are continuous functions on $\partial V(S)$.

(A4) There exists a positive number δ, such that for each $S \subset N$ and $i \in S$, $\lambda^S_i > \delta$.

(A5) For each $S \subset N$, the origin $0_S = (0, \ldots, 0) \in \mathbb{R}^S$ belongs to $V(S)$.

Property (A5) is a normalization and does not affect our results.

We denote by *NTU* the set of all *NTU* games as *NTU*.

We now introduce two particular subclasses of *NTU* games studied in this paper.

We say that (N, V) is a *game with transferable utility* (or *TU game*) if there exists a function $v\colon 2^N \to \mathbb{R}$, called the characteristic function, satisfying $v(\varnothing) = 0$ and for each $S \subset N$,

$V(S)=\{x\in\mathbb{R}^S\colon \sum_{i\in S}x_i\le v(S)\}$. Usually we represent a *TU* game as the pair (N,v). We denote by *TU* the set of all *TU* games.

We say that (N,V) is a *hyperplane game* if for each $S\subset N$ there exists $\lambda^S\in\mathbb{R}^S_{++}$ satisfying

$$V(S)=\{x\in\mathbb{R}^S\colon \lambda^S * x\le v(S)\} \tag{2.1}$$

for some $v\colon 2^N\to\mathbb{R}$ with $v(\varnothing)=0$. We denote by H the set of all hyperplane games.

Notice that each *TU* game is a hyperplane game (take $\lambda_i^S=1$ for each $S\subset N$ and $i\in S$).

A *coalition structure* C over N is a partition of the player set, i.e., $C=\{C_1,C_2,\ldots,C_m\}\subset 2^N$ where $\bigcup_{C_q\in C}C_q=N$ and $C_q\cap C_r=\varnothing$ when $q\ne r$. We denote by (N,V,C) an *NTU* game (N,V) with coalition structure C over N. We denote by *CNTU* the set of all *NTU* games with coalition structure (*CTU* for TU games, and *CH* for hyperplane games).

Given $S\subset N$, we denote by C_S the structure C restricted to the players in S, i.e., $C_S=\{C_q\cap S\}_{C_q\in C}$. Notice that this implies that C_S may have fewer or the same number of coalitions as C. For simplicity, we use C_{-i} instead of $C_{N\setminus\{i\}}$.

A *payoff configuration* for (N,V) is a set of payoffs $x=(x^S)_{S\subset N}$ with $x^S\in V(S)$ for all $S\subset N$.

Given that G is a subset of *CNTU* (or *NTU*), a *value* Γ in G is a correspondence that assigns to each $(N,V,C)\in G$ a subset $\Gamma(N,V,C)\subset V(N)$. We say that $(\Gamma^S)_{S\subset N}$ is a payoff configuration associated with Γ if $\Gamma^S\in\Gamma(S,V,C_S)$ for all $S\subset N$. When several *NTU* games or coalition structures are involved we write $\Gamma^S(V)$, $\Gamma^S(C)$, or $\Gamma^S(V,C)$ instead of Γ^S.

If $\Gamma(N,V,C)$ has a single point in $V(N)$ for all $(N,V,C)\in G$, we say that Γ is a *single value*. Of course, each single value has associated a unique payoff configuration.

We denote by ϕ^N (or $\phi^N(v)$) the *Shapley value* (Shapley [20]) of the *TU* game (N,v).

We now present the consistent value for *NTU* games following Maschler and Owen [11, 12].

Let Π be the set of all permutations over N. Given $\pi\in\Pi$, we define the set of *predecessors* of i under π as

$$P(\pi,i)=\{j\in N\colon \pi(j)<\pi(i)\}.$$

We define the *marginal contribution* of player $i\in N$ to the game V in the permutation π as

$$d_i(\pi)=\max\big\{x_i\colon ((d_j(\pi))_{j\in P(\pi,i)},x_i)\in V(P(\pi,i)\cup\{i\})\big\}.$$

Thus, $d_i(\pi)$ is the maximum that player i can obtain in $V(S)$ after his predecessors obtain their respective $d_j(\pi)$s. We denote $d(\pi)=(d_i(\pi))_{i\in N}$.

Given a hyperplane game (N,V), the *consistent value* Ψ^N (Maschler and Owen [11]) is the vector of expected marginal contributions, where each $\pi\in\Pi^N$ is equally likely, i.e.,

$$\Psi^N=\frac{1}{|\Pi|}\sum_{\pi\in\Pi}d(\pi).$$

Notice that each $d(\pi)$ is an efficient vector (it belongs to the boundary of $V(N)$). Because we are dealing with hyperplane games, this boundary is flat and the consistent value is also an efficient value.

Maschler and Owen [11] proved that, given $i\in N$,

$$\Psi_i^N=\frac{1}{|N|\lambda_i^N}\left(\sum_{j\in N\setminus\{i\}}\lambda_i^N\Psi_i^{N\setminus\{j\}}+v(N)-\sum_{j\in N\setminus\{i\}}\lambda_j^N\Psi_j^{N\setminus\{i\}}\right). \tag{2.2}$$

One procedure for extending a hyperplane solution to the general class of *NTU* games with convex $V(S)$s is to pass arbitrary hyperplanes to the various sets $V(S)$. These hyperplanes determine a hyperplane game to which we know the solution. If this solution belongs to $V(N)$, we say it is a solution for the *NTU* game (N,V). Maschler and Owen [12] adopted this procedure for extending the consistent value to the class of *NTU* games. It is known that the consistent value is not a single value in *NTU* games.

For *TU* games with coalition structure, ϕ^N (or $\phi^N(v, C)$) denotes the *Owen value* (Owen [18]), which is a generalization of the Shapley value (when $C = \{N\}$ or $C = \{\{i\}\}_{i\in N}$, the Owen value coincides with the Shapley value).

Owen [18] defined his value in a two-stage procedure. Let (N, v, C) be a *TU* game with coalition structure.

Stage 1. The total value is divided among the coalitions. Owen defines the *TU* game $(M, v^{[C]})$ where $M = \{1, \ldots, m\}$ and $v^{[C]}(Q) = v(\bigcup_{q\in Q} C_q)$ for each $Q \subset M$. This game (called the *game between coalitions*) is obtained from (N, v, C) assuming that the players are the coalitions acting as single players.

For any coalition $C_q \in C$, Owen states that the total amount obtained by players in C_q (i.e., $\sum_{i\in C_q} \phi_i^N(v, C)$) must be equal to the Shapley value of coalition C_q in the game $(M, v^{[C]})$, i.e., $\phi_q^M(v^{[C]})$.

Stage 2. The players inside each coalition $C_q \in C$ divide the amount obtained in Stage 1. For each $C_q \in C$, Owen defined the *TU* game (C_q, v^*), where $v^*(S) = \phi_q^M(v^{[C_{S\cup \bar{C}_q}]})$ for each $S \subset C_q$. Notice that $v^*(S)$ represents the amount obtained by players of S in Stage 1, assuming that players in $C_q \backslash S$ are not in the game.

The Owen value of player $i \in C_q$ is defined as

$$\phi_i^N(v, C) = \phi_i^{C_q}(v^*).$$

It is well known that the Owen value (like the Shapley value and the consistent value) can be obtained as an average of marginal contributions depending on equally likely permutations.

We say that a permutation $\pi \in \Pi$ is *admissible* with respect to C if given $i, j \in C_q \in C$ and $k \in N$, $\pi(i) < \pi(k) < \pi(j)$ implies $k \in C_q$. We denote by Π^C the set of all permutations over N admissible with respect to C. The Owen value can then be defined as

$$\phi_i^N(v, C) = \frac{1}{|\Pi^C|} \sum_{\pi\in\Pi^C} d(\pi).$$

3. The consistent coalitional value. In this section we define the consistent coalitional value for *NTU* games. We first define it in hyperplane games following the two-stage procedure used by Owen [18] in the definition of the Owen value. Second, we generalize it to *NTU* games, following the same procedure as Maschler and Owen [12] when they extended the consistent value to *NTU* games.

We end this section by proving that the consistent coalitional value cannot be obtained as an average of marginal contributions depending on equally likely permutations. The value obtained in this way is called the *random order value*.

We define the consistent coalitional value Υ in a recursive way. If $|N| = 1$, we take $\Upsilon_i^{\{i\}} = v(\{i\})/\lambda_i^{\{i\}}$.

Assume we have defined Υ^N for each hyperplane game (N, V, C) where $|N| \leq t$. When $|N| = t+1$, we define Υ^N by extending the two-stage procedure of Owen [18] to hyperplane games.

Let (N, V, C) be a hyperplane game and let $(\lambda^S)_{S\subset N}$ and v be associated with V as in (2.1).

Stage 1. The total amount is divided among the coalitions. Each coalition $C_q \in C$ receives

$$\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N = \frac{1}{|C|}\left[\sum_{C_r\in C\backslash\{C_q\}} \left(\sum_{j\in C_q} \lambda_j^N \Upsilon_j^{N\backslash C_r}\right) + v(N) - \sum_{C_r\in C\backslash\{C_q\}} \left(\sum_{j\in C_r} \lambda_j^N \Upsilon_j^{N\backslash C_q}\right)\right]. \quad (3.1)$$

Notice that (3.1) gives a recursive way to compute $\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N$.

Stage 2. The players inside each coalition $C_q \in C$ divide the amount obtained in Stage 1. For each $C_q \in C$, we consider the hyperplane game (C_q, V^*) such that for each $S \subset C_q$,

$$V^*(S) = \{x \in \mathbb{R}^S \colon \lambda^{*S} * x \le v^*(S)\},$$

where $\lambda_i^{*S} = \lambda_i^{S\cup\bar{C}_q}$ for each $i \in S$ and $v^*(S) = \sum_{j\in S} \lambda_j^{S\cup\bar{C}_q} \Upsilon_j^{S\cup\bar{C}_q}$ is the amount obtained by coalition S in Stage 1 applied to the hyperplane game $(S \cup \bar{C}_q, V, C_{S\cup\bar{C}_q})$.

The *consistent coalitional value* Υ_i^N of player $i \in C_q$ is defined as

$$\Upsilon_i^N = \Psi_i^{C_q}(V^*).$$

We now explain this definition carefully. In Stage 1 we must decide the total amount received by each coalition $C_q \in C$. In *TU* games this amount is $\sum_{j\in C_q} \Upsilon_j^N$. In hyperplane games it should be $\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N$.

In *TU* games the total amount received by coalition C_q is the Shapley value of the game $(M, v^{[C]})$ played by the coalitions. Nevertheless, in hyperplane games the game between coalitions cannot, in general, be defined in a meaningful way.

Because we are trying to generalize Owen's procedure, we will use another property of the Shapley value, which can be generalized to our setting. This property is (2.2), which provides a recursive way for the computation of the Shapley value. Notice that even though (2.2) is written in terms of the consistent value, in *TU* games the consistent value coincides with the Shapley value.

In Stage 2 the total amount received by coalition C_q in the first stage must be divided among players in C_q. In this stage, we proceed as in the second stage of Owen [18], using the consistent value instead of the Shapley value.

The next proposition shows that Υ is well-defined.

Proposition 3.1. *Υ is well defined and $\sum_{j\in N} \lambda_j^N \Upsilon_j^N = v(N)$.*

The proof is in Appendix A.

Of course, the consistent coalitional value (like the consistent value) is a single value in hyperplane games.

Remark 3.1. *Let (N, V, C) be a hyperplane game such that for each $i \in N$, λ_i^S is constant across coalitions S containing player i, i.e., there exists $\lambda = (\lambda_i)_{i\in N}$ such that $\lambda_i^S = \lambda_i$ for each $S \subset N$, $i \in S$. Let C_q and C_r be two coalitions of C. Thus, $v(C_q)$ and $v(C_r)$ represent the maximum utility that players in C_q and C_r can obtain when they compare their respective levels of utility using λ. If both coalitions join, they compare their respective levels of utility with the same vector. Then, it makes sense to compare $v(C_q \cup C_r)$ with $v(C_q)$ and $v(C_r)$.*

Under this assumption we can define in a meaningful way the game played by the coalitions. We define the TU game $(M, v^{[C]})$ where $v^{[C]}(S) = v(\bigcup_{q\in S} C_q)$.

For each coalition $C_q \in C$, the amount obtained by coalition C_q coincides with the Shapley value of the game played by the coalitions, i.e., $\sum_{j\in C_q} \lambda_j \Upsilon_j^N = \phi_q^M(v^{[C]})$. The proof of this statement is located in Appendix A.

This remark also applies to TU games because in TU games $\lambda_i^S = 1$ for all $S \subset N$ and $i \in S$. Thus, our procedure coincides with Owen's procedure in TU games.

Condition (3.1) is a generalization of condition (2.2) for the game played by the coalitions. We can also generalize condition (2.2) for the game played by the agents of each coalition $C_q \in C$. For all $i \in C_q \in C$,

$$\lambda_i^N \Upsilon_i^N = \frac{1}{|C_q|}\left(\sum_{j\in C_q\setminus\{i\}} \lambda_i^N \Upsilon_i^{N\setminus\{j\}} + \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^{N\setminus\{i\}}\right). \tag{3.2}$$

The next proposition will be very useful in the proofs of our results.

Proposition 3.2. *The consistent coalitional value is the only value in CH* (*the class of hyperplane games with coalition structure*) *satisfying conditions* (3.1) *and* (3.2).

The proof is in Appendix A.

We now extend the definition of the consistent coalitional value to *NTU* games. We follow the same approach as in Maschler and Owen [12], where they extend the consistent value from hyperplane games to *NTU* games.

For an *NTU* game with coalition structure $(N, V, \mathscr{C})$, we take for each coalition $S \subset N$ an orthonormal vector λ^S to $\partial V(S)$. Let $(N, V', \mathscr{C})$ be the resulting hyperplane game, i.e.,

$$V'(S) = \{x \in \mathbb{R}^S\colon\ \lambda^S * x \leq v'(S)\}$$

where $v'(S) = \max\{\lambda^S * x\colon\ x \in V(S)\}$, assuming this maximum is well defined. Let $x = (x^S)_{S\subset N}$ be the (unique) payoff configuration associated with $\Upsilon(N, V', \mathscr{C})$. If $x^S \in V(S)$ for all $S \subset N$, we say $(x^S)_{S\subset N}$ is a *consistent coalitional payoff configuration* for $(N, V, \mathscr{C})$.

Given $(N, V, \mathscr{C}) \in CNTU$, $y \in \Upsilon(N, V, \mathscr{C})$ if there exists a consistent coalitional payoff configuration $(x^S)_{S\subset N}$ for $(N, V, \mathscr{C})$ such that $y = x^N$.

In the next theorem we prove the existence of consistent coalitional payoff configurations.

Theorem 3.1. *Each NTU game with coalition structure has a consistent coalitional payoff configuration.*

The proof is in Appendix. A.

NTU games with coalition structure are natural generalizations of other well-known games. We can ask if Υ coincides with "nice" values in that subclass of games. The answer is yes.

When $C = \{N\}$, (3.2) coincides with (2.2) and when $C = \{\{i\}\}_{i\in N}$, (3.1) coincides with (2.2). Because Ψ is the only value in hyperplane games satisfying (2.2), by Proposition 3.2 we conclude that Υ coincides with Ψ in trivial coalition structures. Moreover, by Remark 3.1, for *TU* games with coalition structure Υ coincides with the Owen value.

Chae and Heidhues [5] and Vidal-Puga [22] introduced two values for bargaining problems with coalition structure. Bergantiños et al. [3] proved that Υ coincides with these values in bargaining problems with coalition structure.

We know that the Shapley value, the consistent value, and the Owen value are obtained as an average of marginal contributions depending on equally likely permutations. Thus, it seems reasonable to generalize these values in the same way.

Given a hyperplane game (N, V, C), the *random order coalitional value* $\digamma^N$ is defined as the vector of expected marginal contributions when all the admissible permutations with respect to C are equally likely, i.e.,

$$\digamma^N = \frac{1}{|\Pi^C|} \sum_{\pi\in\Pi^C} d(\pi).$$

In *TU* games, McLean [13] defined the random order coalitional structure values. $\digamma$ is the natural generalization to hyperplane games of McLean's values when all the admissible permutations are equally likely and the rest of the permutations have probability 0. It is remarkable that Maschler and Owen [12] even suggested the name random order value instead of consistent value.

We can extend $\digamma$ from hyperplane games to *NTU* games following the same procedure as with Υ.

We now compute Υ and $\digamma$ in a classical example that first appeared, with a different formulation, in Owen [17]. Hart [8] compared several *NTU* values in this example.

Example 3.1. Let $N = \{1, 2, 3\}$ and let V be defined as follows:

$$V(i) = \{x_i\colon\ x_i \leq 0\} \quad \text{for all } i \in N$$

$$V(\{1, 2\}) = \{(x_1, x_2)\colon\ x_1 + 2x_2 \leq 144\}$$

$$V(\{1,3\}) = \{(x_1, x_3)\colon\ x_1 + x_3 \leq 0\}$$
$$V(\{2,3\}) = \{(x_2, x_3)\colon\ x_2 + x_3 \leq 0\}$$
$$V(N) = \{(x_1, x_2, x_3)\colon\ x_1 + x_2 + x_3 \leq 144\}.$$

If $C = \{\{1,2\}, \{3\}\}$, we obtain that

$$\Upsilon^N = (63, 63, 18) \qquad \text{and} \qquad \digamma^N = (72, 54, 18).$$

This example shows that Υ and $\digamma$ are different.

Even though Υ and $\digamma$ are natural generalizations of the Owen value, we believe that Υ is a more suitable value. We will prove that Υ satisfies more interesting properties and can be characterized generalizing axiomatic characterizations of the Owen value and the consistent value.

4. An axiomatic characterization in *NTU* games. In this section we present a characterization of Υ with properties of efficiency and balanced contributions.

We say that a value Γ satisfies *efficiency (EF)* if for each $(N, V, \mathscr{C}) \in CNTU$ and each $x \in \Gamma(N, V, \mathscr{C})$, $x \in \partial V(N)$.

REMARK 4.1. *Let $(x^S)_{S\subset N}$ be a payoff configuration associated with Γ. Because V satisfies* (A2) *we can conclude that if Γ satisfies efficiency, then for each $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ such that $\lambda^S * x^S = \max\{\lambda^N * y\colon\ y \in V(S)\}$. The reciprocal statement is clearly also true.*

Hart and Mas-Colell [10, p. 367] obtained the following characterization of the consistent coalitional value. We must say that Hart and Mas-Colell present it in a slightly different way: (B1) is split into two parts.

PROPOSITION 4.1. *Let (N, V) be an NTU game and let $(x^S)_{S\subset N}$ be a payoff configuration. $(x^S)_{S\subset N}$ is a payoff configuration associated with Ψ if and only if for each $S \subset N$ there exists a vector $\lambda^S \in \mathbb{R}^S_{++}$ such that*

(B1) $\lambda^S * x^S = \max\{\lambda^S * y\colon\ y \in V(S)\}$;

(B2) *For each $i \in S$,*

$$\sum_{j\in S\setminus\{i\}} \lambda_i^S \big(x_i^S - x_i^{S\setminus\{j\}}\big) = \sum_{j\in S\setminus\{i\}} \lambda_j^S \big(x_j^S - x_j^{S\setminus\{i\}}\big).$$

Myerson [16] characterized the Shapley value as the only value in *TU* games satisfying *EF* and balanced contributions (*BC*). Note that (B1) can be considered as an efficiency property and (B2) as a generalization of *BC* to *NTU* games. Proposition 4.1 is therefore an extension of Myerson's result.

Calvo et al. [4] characterized the Owen value in *TU* games with coalition structure as the only value satisfying *EF*, balanced contributions among coalitions (*BCAC*), and balanced contributions among players (*BCAP*). Even though Calvo et al. [4] present these two balanced properties as just one, we are of the opinion that, for our paper, the formulation with two properties is more natural.

A value Γ satisfies *balanced contributions among coalitions (BCAC)* if for each $(N, v, C) \in CTU$ and $C_q, C_r \in C$ with $q \neq r$,

$$\sum_{j\in C_q} \Gamma_j^N - \sum_{j\in C_q} \Gamma_j^{N\setminus C_r} = \sum_{j\in C_r} \Gamma_j^N - \sum_{j\in C_r} \Gamma_j^{N\setminus C_q}.$$

A value Γ satisfies *balanced contributions among players in the same coalition (BCAP)* if for each $i, j \in C_q \in C$ with $i \neq j$,

$$\Gamma_i^N - \Gamma_i^{N\setminus\{j\}} = \Gamma_j^N - \Gamma_j^{N\setminus\{i\}}.$$

The result of Calvo et al. [4] for the Owen value also generalizes Myerson's [16] result.

In the next theorem we present a characterization of Υ in *NTU* games with coalition structure with properties of balanced contributions. This characterization generalizes the results of Hart and Mas-Colell [10] for the consistent value and of Calvo et al. [4] for the Owen value.

Given $S \subset N$, we denote $C'_q = C_q \cap S \in \mathscr{C}_S$ when $C_q \cap S \neq \varnothing$.

THEOREM 4.1. *Let $(N, V, \mathscr{C}) \in CNTU$ and $(x^S)_{S\subset N}$ be a payoff configuration. $(x^S)_{S\subset N}$ is a consistent coalitional payoff configuration of $(N, V, \mathscr{C})$ if and only if, for each $S \subset N$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ such that*

(B1) $\lambda^S * x^S = \max\{\lambda^S * y: y \in V(S)\}$.

(B3) *For each $C'_q \in \mathscr{C}_S$,*

$$\sum_{C'_r \in \mathscr{C}_S\setminus\{C'_q\}} \left(\sum_{j\in C'_q} \lambda^S_j \big(x^S_j - x^{S\setminus C'_r}_j\big)\right) = \sum_{C'_r \in \mathscr{C}_S\setminus\{C'_q\}} \left(\sum_{j\in C'_r} \lambda^S_j \big(x^S_j - x^{S\setminus C'_q}_j\big)\right).$$

(B4) *For each $i \in C'_q \in \mathscr{C}_S$,*

$$\sum_{j\in C'_q\setminus\{i\}} \lambda^S_i \big(x^S_i - x^{S\setminus\{j\}}_i\big) = \sum_{j\in C'_q\setminus\{i\}} \lambda^S_j \big(x^S_j - x^{S\setminus\{i\}}_j\big).$$

The proof is in Appendix A.

Note that (B3) and (B4) generalize (B2) in the same way that *BCAC* and *BCAP* generalize *BC*. (B3) and (B4), on the other hand, generalize *BCAC* and *BCAP* in the same way that (B2) generalizes *BC*.

Hence, the consistent coalitional value is the correct generalization of the Owen value and the consistent value to *NTU* games with coalition structure, if we focus on the properties of balanced contributions for both values.

We now prove that (B1), (B3), and (B4) are independent.

(B1) is independent of the rest of properties. The value $\Gamma_i = 0$ for each $(N, V, \mathscr{C}) \in CNTU$, and $i \in N$ satisfies (B3) and (B4) but fails (B1).

(B3) is independent of the rest of properties. We define the value Λ that satisfies (B1) and (B4) but fails (B3).

We first define Λ in hyperplane games. Given $(N, V, \mathscr{C}) \in CH$ we define, for each $i \in N$,

$$\Omega^N_i = \frac{v(N)}{|N|\lambda^N_i}.$$

Given $S \subset N$, let Π^S denote the set of all permutations over S. Given $\pi \in \Pi^{C_q}$, we consider $f(\pi) \in \mathbb{R}^{C_q}$ such that, for each $i \in C_q$,

$$f_i(\pi) = \max\big\{x_i: \big((\Omega^S_j)_{j\in \overline{C}_q}, (f_j(\pi))_{j\in P(\pi,i)}, x_i\big) \in V(S)\big\},$$

where $S = \overline{C}_q \cup P(\pi, i) \cup \{i\}$.

Given $i \in C_q \in \mathscr{C}$, we define Λ as follows:

$$\Lambda^N_i = \frac{1}{|\Pi^{C_q}|} \sum_{\pi\in\Pi^{C_q}} f_i(\pi).$$

It is not difficult to see that Λ is a single value in *CH*. Moreover, $\Lambda \neq \Upsilon$.

We can extend Λ to *NTU* games as we did with Υ.

LEMMA 4.1. *Λ satisfies* (B1) *and* (B4) *but fails* (B3)

The proof is in Appendix A.

(B4) is independent of the rest of properties. The random order value $\digamma$ satisfies (B1) and (B3) but fails (B4) (in §6 we prove this statement).

5. Two axiomatic characterizations in hyperplane games. In this section we present two axiomatic characterizations of the consistent coalitional value in hyperplane games. The first one uses balanced contributions and it is closely related to the characterization presented in the previous section. The second one uses the consistency property.

In Theorem 4.1 the vector $\lambda = (\lambda^S)_{S\subset N}$ must be the same in (B1), (B3), and (B4). Moreover, (B1) says that λ makes $(x^S)_{S\subset N}$ an efficient vector in $V(N)$. In hyperplane games there is only one vector λ with this property, which is the one given by (2.1).

We now present the definition of (B3) and (B4) in hyperplane games with respect to the vector λ of (2.1).

A value Γ in *CH* satisfies *average balanced contributions among coalitions (ABCAC)* if for each $(N, V, C) \in CH$ and $C_q \in C$,

$$\sum_{C_r\in C\setminus\{C_q\}}\left[\sum_{j\in C_q}\lambda_j^N\big(\Gamma_j^N-\Gamma_j^{N\setminus C_r}\big)\right]=\sum_{C_r\in C\setminus\{C_q\}}\left[\sum_{j\in C_r}\lambda_j^N\big(\Gamma_j^N-\Gamma_j^{N\setminus C_q}\big)\right].$$

A value Γ in *CH* satisfies *average balanced contributions among players in the same coalition (ABCAP)* if for each $(N, V, C) \in CH$ and $i \in C_q$,

$$\sum_{j\in C_q\setminus\{i\}}\lambda_i^N\big(\Gamma_i^N-\Gamma_i^{N\setminus\{j\}}\big)=\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\big(\Gamma_j^N-\Gamma_j^{N\setminus\{i\}}\big).$$

It is not difficult to check that *ABCAC* and *ABCAP* are the correct generalizations of *BCAC* and *BCAP* to the class of hyperplane games. We now present the characterization of Υ for hyperplane games with these properties of balanced contributions.

Theorem 5.1. *The consistent coalitional value is the only value in CH satisfying EF, ABCAC, and ABCAP.*

Proof. It is analogous to the proof of Theorem 4.1. We omit it. □

We now present the axiomatic characterization of Υ with consistency. We first introduce several properties. Some of these properties are well known in the literature of *NTU* games. Others are introduced in this paper generalizing properties of *TU* games.

We present the definitions for single values. The definition for payoff configurations associated with general values is straightforward.

Given $(N, V, C) \in CNTU$ we say that two players $i, j \in N$ are *symmetric* if two properties hold:

For each $S \subset N\setminus\{i, j\}$, if $x \in V(S\cup\{i\})$, $y_j = x_i$, and $y_k = x_k$ for each $k \in S$, we have $y \in V(S\cup\{j\})$.

For each $S \supset \{i, j\}$, if $x \in V(S)$, $y_i = x_j$, $y_j = x_i$, and $y_k = x_k$ for each $k \in S\setminus\{i, j\}$, we have $y \in V(S)$.

We say that a value Γ satisfies *individual symmetry (IS)* if for each pair of symmetric players $i, j \in C_q \in C$, $\Gamma_i^N = \Gamma_j^N$.

We now present the property of covariance in hyperplane games following Maschler and Owen [11]. Let (N, V, C) and $(N, \widetilde{V}, C)$ be two hyperplane games such that for each $S \subset N$,

$$V(S)=\{x\in\mathbb{R}^S\colon\ \lambda^S * x\le v(S)\}\qquad\text{and}\qquad \widetilde{V}(S)=\{x\in\mathbb{R}^S\colon\ \tilde{\lambda}^S * x\le\tilde{v}(S)\}.$$

We say that (N, V, C) and $(N, \widetilde{V}, C)$ are *equivalent under a linear transformation of player i's utility* if there exist two constants $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that for all $S \subset N$, $\tilde{\lambda}_i^S = \lambda_i^S/a$, $\tilde{\lambda}_j^S = \lambda_j^S$ if $j \neq i$, $\tilde{v}(S) = v(S) + b\lambda_i^S/a$ if $i \in S$, and $\tilde{v}(S) = v(S)$ if $i \notin S$. Notice that if (N, V, C) and $(N, \widetilde{V}, C)$ are equivalent under a linear transformation of player i's utility, then $\tilde{x} \in \widetilde{V}(S)$ if and only if there exists $x \in V(S)$ satisfying $\tilde{x}_i = ax_i + b$ and $\tilde{x}_j = x_j$ if $j \in S\setminus\{i\}$.

We say that a value Γ satisfies *covariance (COV)* if, given two hyperplane games (N, V, C) and $(N, \widetilde{V}, C)$ equivalent under a linear transformation of some player i's utility,

$$\Gamma_j^N(\widetilde{V}, C) = \begin{cases} a\Gamma_i^N(V, C) + b & \text{if } j = i \\ \Gamma_j^N(V, C) & \text{if } j \neq i. \end{cases}$$

Hart and Mas-Colell [9] characterized the Shapley value as the only value in *TU* games satisfying consistency and other properties. A value Γ in *TU* satisfies consistency (*CONS*) if and only if for each $(N, v) \in TU$, $S \subset N$, and $i \in S$,

$$\Gamma_i^N(v) = \Gamma_i^S(v_S)$$

where $v_S(T) = v(T \cup \bar{S}) - \sum_{j \in \bar{S}} \Gamma_j^{T \cup \bar{S}}(v)$ for each $T \subset S$.

Winter [24] extended the definition of consistency to *TU* games with coalition structure. A value Γ in *CTU* satisfies *consistency* (*CONS*) if and only if for each $(N, v, C) \in CTU$, $C_q \in C$, $S \subset C_q$, and $i \in S$,

$$\Gamma_i^N(v, C) = \Gamma_i^S(v_S, \{S\}).$$

Maschler and Owen [11] showed that if we define the property of consistency of Hart and Mas-Colell [9] in hyperplane games as in the *TU* case, there is no value satisfying consistency and other "basic" properties (for instance, efficiency). Thus, they provided a weaker definition of consistency, called l-consistency.

A value Γ in H satisfies *l-consistency* if, for each hyperplane game (N, V, C), $l \leq |N|$, and $i \in N$,

$$\sum_{S \subset N,\, i \in S,\, |S| = l} \Gamma_i^S(V_S) = \binom{|N| - 1}{l - 1} \Gamma_i^N(V)$$

where $V_S(T) = \{x \in \mathbb{R}^T \colon (x, (\Gamma_i^{T \cup \bar{S}})_{i \in \bar{S}}) \in V(T \cup \bar{S})\}$ for all $T \subset S$.

It is straightforward to prove that V_S is a hyperplane game and

$$V_S(T) = \left\{ x \in \mathbb{R}^T \colon \sum_{i \in T} \lambda_i^{T \cup \bar{S}} x_i \leq v(T \cup \bar{S}) - \sum_{i \in \bar{S}} \lambda_i^{T \cup \bar{S}} \Gamma_i^{T \cup \bar{S}} \right\}.$$

We now present a generalization of l-consistency to hyperplane games with coalition structure.

A value Γ in *CH* satisfies *l-consistency* if for each hyperplane game (N, V, C), $C_q \in C$, $l \leq |C_q|$, and $i \in C_q$,

$$\sum_{S \subset C_q,\, i \in S,\, |S| = l} \Gamma_i^S(V_S, \{S\}) = \binom{|C_q| - 1}{l - 1} \Gamma_i^N(V, C).$$

We call 2-consistency *bilateral consistency (BCONS)*.

Notice that our bilateral consistency generalizes in the natural way the consistency of Hart and Mas-Colell [9], the consistency of Winter [24], and the bilateral consistency of Maschler and Owen [11].

Hart and Mas-Colell [9] characterized the Shapley value in the class of *TU* games as the only value satisfying *EF*, *SYM*, *COV*, and *CONS*. Winter [24] and Maschler and Owen [11] extended this result in two different ways.

Winter [24] extended it to the class of *TU* games with coalition structure. He proved that the Owen value is the only value satisfying *EF*, *IS*, *COV*, *CONS*, and Game Between Coalitions Property (*GBCP*).

A value Γ satisfies *GBCP* if for each *TU* game (N, v, C) and $C_q \in C$,

$$\sum_{i \in C_q} \Gamma_i(N, v, C) = \Gamma_{C_q}\big(M, v^{[C]}, \{M\}\big).$$

This property says that the amount received by a coalition in the game played by the coalitions coincides with the sum of the amounts received by the members of this coalition in the original game. This property cannot be exported to hyperplane games. Nevertheless, it is possible to obtain the following result:

PROPOSITION 5.1. *The Owen value is the only value in CTU satisfying EF, IS, COV, CONS, and BCAC.*

PROOF. It is similar to the proof of Winter's result about the characterization of the Owen value. We omit it. □

Maschler and Owen [11] extended the result of Hart and Mas-Colell [9] to the class of hyperplane games. They proved that the consistent value is the only value satisfying *EF*, *SYM*, *COV*, and *BCONS*.

In the next theorem we generalize the results of Hart and Mas-Colell [9], Winter [24], and Maschler and Owen [11] to the class of hyperplane games with coalition structure.

THEOREM 5.2. *The consistent coalitional value is the only value in CH satisfying EF, IS, COV, BCONS, and ABCAC.*

The proof is in Appendix A.

REMARK 5.1. *The properties used in this theorem are independent* (*see Appendix A*).

The results obtained in this section about the consistent coalitional value and the relation with other values can be summarized in the following table:

	Without coalition structure		With coalition structure	
	Shapley		Owen	
TU games	*EF* *SYM* *COV* *CONS*	*EF* *BC*	*EF* *IS* *COV* *CONS* *BCAC*	*EF* *BCAP* *BCAC*
	Consistent		Consistent coalitional	
Hyperplane games	*EF* *SYM* *COV* *BCONS*	*EF* *ABC*	*EF* *IS* *COV* *BCONS* *ABCAC*	*EF* *ABCAP* *ABCAC*

Hence, the consistent coalitional value is the correct generalization of the Owen value and the consistent value if we focus on the properties of consistency and balanced contributions.

6. Concluding remarks. In this paper we present two generalizations of the Owen value and the consistent value for *NTU* games with coalition structure: the consistent coalitional value and the random order coalitional value.

We now study which of the properties introduced before are satisfied by the random order coalitional value in hyperplane games.

PROPOSITION 6.1. (i) *The random order coalitional value satisfies EF, IS, COV, and ABCAC.*

(ii) *The random order coalitional value satisfies neither BCONS nor ABCAP.*

The proof is in Appendix A.

Moreover, it is not difficult to see that in *NTU* games the random order coalitional value satisfies (B1) and (B4) but fails (B3).

The Shapley value and the consistent value have two important properties. Firstly, they have an intuitive definition (they can be computed through the vector of marginal contributions). Secondly, these values can be characterized with nice properties (consistency and balanced contributions).

The Owen value can be defined using either the two-stage procedure or the vector of marginal contributions over admissible permutations. Moreover, it can also be characterized with consistency and balanced contributions.

NTU games with coalition structure generalize the three classes of games mentioned before. In this setting, we introduce two values, Υ and $\digamma$. Whereas Υ generalizes the definition based on the two-stage procedure, $\digamma$ generalizes the definition based on marginal contributions. This fact is not surprising. There are results from *TU* games that cannot be generalized in the same way to *NTU* games. For instance, the consistent value and the Shapley *NTU* value are generalizations of the Shapley value. However, whereas the consistent value generalizes the characterizations of the Shapley value based in the properties of consistency and balanced contributions, the Shapley *NTU* value generalizes the classical axiomatization of the Shapley value (Aumann [1]). Nevertheless, only Υ can be characterized with consistency and balanced contributions ($\digamma$ satisfies neither *BCONS* nor *ABCAP*).

Recently, Pérez-Castrillo and Wettstein [19] introduced a solution to the problem of sharing the surplus among a group of agents. This solution, called Ordinal Shapley Value (*OSV*), is formulated in the preferences-endowments space, and induces an ordinal solution in *NTU* games. In this paper we generalize the Shapley value to *NTU* games with coalition structure. It is quite remarkable that in both papers the properties of consistency and balanced contributions (called fairness in the paper of Pérez-Castrillo and Wettstein) play a very important role.

NTU games with coalition structure are also studied by Winter [23], who introduces the *game coalition structure value* (Θ). This value is a generalization of the Harsanyi value (Harsanyi [7]) for *NTU* games and the Owen value for *TU* games with coalition structure. Winter characterized this value with six axioms: *EF*, *COV*, *conditional additivity*, *independence of irrelevant alternatives*, *inessential games*, and *unanimity games*.

If we compare Υ with Θ, we realize that both values generalize the Owen value. However, whereas Υ generalizes the consistent value, Θ generalizes the Harsanyi value.

The definition and axiomatic characterization of Θ are closely related to the definition and axiomatic characterization of the Harsanyi value. Nevertheless, Θ is not related with a definition or axiomatic characterization of the Owen value.

Of course, Υ, $\digamma$, and Θ are different. In Example 3.1, $\Theta^N = (60, 60, 24)$ when $C = \{\{1,2\},\{3\}\}$.

Appendix A. Proofs. We prove the results presented in the paper. Some computations are avoided.

Proof of Proposition 3.1. We first prove that Υ satisfies (3.1). Because Ψ is an efficient value, for each $C_q \in C$,

$$\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j\in C_q} \lambda_j^N \Psi_j^{C_q}(V^*) = v^*(C_q),$$

which satisfies (3.1) by definition.

Using an induction argument over the number of agents $|N|$, it is not difficult to prove that there exists a unique point Υ^N obtained through the two-stage procedure.

Because Υ satisfies (3.1), making some computations, we can prove that

$$\sum_{i\in N} \lambda_i^N \Upsilon_i^N = \sum_{C_q\in C} \left(\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N \right) = v(N). \quad \square$$

Proof of Remark 3.1. Because $(M, v^{[C]})$ is a *TU* game and $\Psi = \phi$ in *TU* games, applying (2.2) we obtain that, given $q \in M$,

$$\begin{aligned}\phi_q^M &= \frac{1}{|C|}\left(\sum_{r\in M\backslash\{q\}} \phi_q^{M\backslash\{r\}} + v^{[C]}(M) - \sum_{r\in M\backslash\{q\}} \phi_r^{M\backslash\{q\}}\right)\\ &= \frac{1}{|C|}\left(\sum_{r\in M\backslash\{q\}} \phi_q^{M\backslash\{r\}} + v(N) - \sum_{r\in M\backslash\{q\}} \phi_r^{M\backslash\{q\}}\right).\end{aligned}$$

We now prove that for all $C_q \in C$, $\phi_q^M = \sum_{j\in C_q} \lambda_j \Upsilon_j^N$. We proceed by induction on $|M|$. If $|M| = 1$, then $\phi_q^M = v(N)$ and $\sum_{j\in C_q} \lambda_j \Upsilon_j^N = v(N)$. Assume that the result holds when $|M| \leq t$. We prove it when $|M| = t+1$.

Because Υ satisfies (3.1) and $\lambda_i^S = \lambda_i$ for all $S \subset N$ such that $i \in S$,

$$\sum_{j\in C_q} \lambda_j \Upsilon_j^N = \frac{1}{|C|}\left[\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q} \lambda_j \Upsilon_j^{N\backslash C_r}\right) + v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r} \lambda_j \Upsilon_j^{N\backslash C_q}\right)\right].$$

By induction hypothesis,

$$\sum_{j\in C_q} \lambda_j \Upsilon_j^N = \frac{1}{|C|}\left[\sum_{r\in M\backslash\{q\}} \phi_q^{M\backslash\{r\}} + v(N) - \sum_{r\in M\backslash\{q\}} \phi_r^{M\backslash\{q\}}\right] = \phi_q^M. \quad \square$$

Proof of Proposition 3.2. Using an induction argument over the number of agents $|N|$, it is not difficult to prove that there exists a unique value satisfying conditions (3.1) and (3.2).

Because Γ satisfies (3.1), we only need to prove that Υ satisfies (3.2).

If we apply (2.2) to the hyperplane game (C_q, V^*) we obtain that for all $i \in C_q$,

$$\begin{aligned}\Upsilon_i^N &= \Psi_i^{C_q}(V^*)\\ &= \frac{1}{|C_q|\lambda_i^{*C_q}}\left(\sum_{j\in C_q\backslash\{i\}} \lambda_i^{*C_q}\Psi_i^{C_q\backslash\{j\}}(V^*) + v^*(C_q) - \sum_{j\in C_q\backslash\{i\}} \lambda_j^{*C_q}\Psi_j^{C_q\backslash\{i\}}(V^*)\right).\end{aligned}$$

In this expression, $\Psi_i^{C_q\backslash\{j\}}(V^*)$ represents the consistent value of player i in the hyperplane game $(C_q\backslash\{j\}, V^*)$. V^* is obtained from (N, V, C) as in Stage 2 of the definition of Υ. By definition, $\Upsilon_i^{N\backslash\{j\}} = \Psi_i^{C_q\backslash\{j\}}(V_{-j}^*)$ where V_{-j}^* is obtained from $(N\backslash\{j\}, V, C_{-j})$ as in Stage 2 of the definition of Υ. It is not difficult to see that V^* coincides with V_{-j}^* in $C_q\backslash\{j\}$. Thus,

$$\Upsilon_i^N = \frac{1}{|C_q|\lambda_i^N}\left(\sum_{j\in C_q\backslash\{i\}} \lambda_i^N \Upsilon_i^{N\backslash\{j\}} + v^*(C_q) - \sum_{j\in C_q\backslash\{i\}} \lambda_j^N \Upsilon_j^{N\backslash\{i\}}\right).$$

Because Υ satisfies (3.1), $v^*(C_q) = \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N$. Thus, Υ satisfies (3.2). $\square$

Proof of Theorem 3.1. The structure of the proof is analogous to that of Theorem 3.3 in Maschler and Owen [12], where the existence of the consistent value for general *NTU* games is proven.

We make use of induction to prove the following claim:

Given $(x^T)_{T\subsetneq N}$ with $x^T \in \mathbb{R}^T$ such that, for any $S \subsetneq N$, the collection $(x^T)_{T\subset S}$ is a consistent coalitional payoff configuration of the game (S, V, C_S), there exists $x^N \in \partial V(N)$ such that $(x^T)_{T\subset N}$ is a consistent coalitional payoff configuration of (N, V, C).

For $n = 1$, the claim is trivially true, the collection being the empty set.

Assume now that the claim holds true for fewer than n players. Hence, there exists a collection $(x^T)_{T \subsetneq N}$ such that, for any $S \subsetneq N$, $(x^T)_{T \subset S}$ is a consistent coalitional payoff configuration of the game (S, V, C_S).

Assume that $z \in \partial V(N)$. For each $T \subsetneq N$, let $\lambda^T = (\lambda_i^T)_{i \in T}$ be the orthonormal vector outwards x^T. Moreover, $(\lambda_i^N)_{i \in N}$ is the orthonormal vector outwards z.

Consider the hyperplane game (N, V^z, C) such that, for each $S \subset N$,

$$V^z(S) = \{y \in \mathbb{R}^S \colon \lambda^S * y \leq v(S)\}$$

where $v(S) = \lambda^S * x^S$ when $S \neq N$ and $v(N) = \lambda^N * z$.

Let $(\Upsilon^S(z))_{S \subset N}$ be the (unique) consistent coalitional payoff configuration for the hyperplane game (N, V^z, C). Under the definition of V^z, $\Upsilon^S(z) = x^S$ for all $S \subsetneq N$, independently of the chosen z.

We will now show that there exists a point $x^N \in \partial V(N)$ such that the collection $(x^T)_{T \subset N}$ is a consistent coalitional payoff configuration for (N, V, C). Notice that it is enough to prove that $\Upsilon^N(x^N) = x^N$. For this purpose we will use a fixed-point theorem. Because Υ satisfies (3.1) and (3.2) and the λ_i^S's are strictly positive and continuous functions, $\Upsilon^N(z)$ is also a continuous function of z.

We define $M = \max\{|x_i^T|/\delta \colon i \in T \subsetneq N\}$, where δ is given by (A4).

Given $C_q \in C$, by (3.1),

$$|C| \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(z) = \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N x_j^{N \setminus C_r} \right) + v(N) - \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N x_j^{N \setminus C_q} \right).$$

By (A5), $v(N) \geq 0$, and because the λ_j^Ns are normalized, it is easy to prove that

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(z) \geq -M\delta \quad \text{for all } C_q \in C.$$

Given $i \in C_q \in C$, because Υ satisfies (3.2), $\lambda_i^N > \delta$, λ^N is normalized, and $\delta < 1$, making some computations we can prove that

$$\Upsilon_i^N(z) > -2M.$$

The remainder of the proof is analogous to Maschler and Owen's proof [12]. Below (Figure 1) we provide a geometric description for the case $n = 2$.

We define $D = \{x \in \mathbb{R}^N \colon x_i \geq -2M \text{ for all } i \in N\}$. Given a vector z in $\partial V(N) \cap D$ (the thick line in Figure 1), and given that we have proved that $\Upsilon^N(z) \in D$; then the point $F(z)$

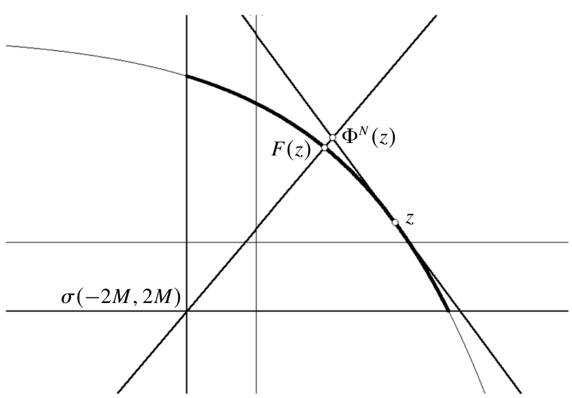

FIGURE 1. Geometric description when $n = 2$.

obtained by applying a projection centered in $\sigma = (-2M, \ldots, -2M) \in \mathbb{R}^N$ also belongs to $\partial V(N) \cap D$ (Figure 1). By applying a standard fixed-point theorem over the (continuous) function F, we can obtain the desired x^N. □

Proof of Theorem 4.1. We first prove that if $(x^S)_{S\subset N}$ is a consistent coalitional payoff configuration of $(N, V, \mathscr{C})$ then for each $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying (B1), (B3), and (B4).

Because $(x^S)_{S\subset N}$ is associated with Υ, there exists $(N, V', \mathscr{C}) \in CH$ such that $x^S = \Upsilon(S, V', \mathscr{C}_S)$. Let $(\lambda^S)_{S\subset N}$ and v': $2^N \to \mathbb{R}$ be associated with V'.

For each $S \subset N$, $\lambda^S * x^S = v'(S) = \max\{\lambda^S * y\colon y \in V(S)\}$ because $x^S = \Upsilon(S, V', \mathscr{C}_S)$. Thus, x^S satisfies (B1).

We now prove that $(\Upsilon(S, V', \mathscr{C}_S))_{S\subset N}$ satisfies (B3). In order to simplify the notation, we assume that $S = N$ and we denote $\Upsilon^N = \Upsilon(N, V', \mathscr{C})$. By (B1) and Remark 4.1, $v'(N) = \sum_{C_r\in C}(\sum_{j\in C_r} \lambda_j^N \Upsilon_j^N)$.

Applying this to (3.1) and making some computations, we obtain that for all $C_q \in C$,

$$\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N = \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q} \lambda_j^N\left(\Upsilon_j^{N\setminus C_r} - \Upsilon_j^N\right)\right) + \sum_{j\in C_q}\lambda_j^N\Upsilon_j^N + \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\Upsilon_j^N - \Upsilon_j^{N\setminus C_q}\right)\right),$$

which means that $(x^S)_{S\subset N}$ satisfies (B3).

We now prove that $(\Upsilon(S, V', \mathscr{C}_S))_{S\subset N}$ satisfies (B4). Again, we assume that $S = N$. Take $i \in C_q \in C$, because Υ satisfies (3.2) making some computations we can prove that

$$|C_q|\lambda_i^N\Upsilon_i^N = \sum_{j\in C_q\setminus\{i\}} \lambda_i^N\left(\Upsilon_i^{N\setminus\{j\}} - \Upsilon_i^N\right) + |C_q|\lambda_i^N\Upsilon_i^N + \sum_{j\in C_q\setminus\{i\}}\lambda_j^N\left(\Upsilon_j^N - \Upsilon_j^{N\setminus\{i\}}\right),$$

which means that $(x^S)_{S\subset N}$ satisfies (B4).

We shall now prove the reciprocal theorem. Assume that $(x^S)_{S\subset N}$ is a payoff configuration of $(N, V, \mathscr{C})$ such that, for each $S \subset N$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying (B1), (B3), and (B4). We shall prove that $x^S \in \Upsilon(S, V, \mathscr{C}_S)$ for each $S \subset N$.

We proceed by induction for the number of players. The result is trivially true for $n = 1$. Assume the result holds for each $n \le t$. It is proved when $n = t + 1$.

By (B1), for each $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying

$$\lambda^S * x^S = \max\{\lambda^S * y\colon y \in V(S)\}.$$

Let $v'(S) = \lambda^S * x^S$ for all $S \subset N$. Let $(N, V', \mathscr{C})$ be the hyperplane game where for each $S \subset N$,

$$V'(S) = \{x \in \mathbb{R}^S\colon \lambda^S * x \le v'(S)\}.$$

It is sufficient to prove that $x^S = \Upsilon^S(V')$ for each $S \subset N$. Under the induction hypothesis, for each $S \subsetneq N$, $x^S = \Upsilon^S(V')$. For simplicity, we denote $\Upsilon^S = \Upsilon^S(V')$ for all $S \subset N$. We now prove that $x^N = \Upsilon^N(V')$.

Assume that $i \in C_q \in \mathscr{C}$. We first prove that $\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j\in C_q} \lambda_j^N x_j^N$.

Because $v'(N) = \sum_{C_r\in\mathscr{C}}(\sum_{j\in C_r} \lambda_j^N x_j^N)$ and Υ satisfies (3.1), making some computations we can prove that

$$|\mathscr{C}|\left(\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N - \sum_{j\in C_q}\lambda_j^N x_j^N\right) = \sum_{C_r\in\mathscr{C}\setminus\{C_q\}}\left(\sum_{j\in C_r}\lambda_j^N\left(x_j^N - x_j^{N\setminus C_q}\right)\right) - \sum_{C_r\in\mathscr{C}\setminus\{C_q\}}\left(\sum_{j\in C_q}\lambda_j^N\left(x_j^N - x_j^{N\setminus C_r}\right)\right).$$

Because $(x^S)_{S\subset N}$ satisfies (B3), the last expression equals 0. Thus,

$$\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j\in C_q} \lambda_j^N x_j^N.$$

Because Υ satisfies (3.2), the induction hypothesis, and the previous result, making some computations we can prove that

$$|C_q|\lambda_i^N(\Upsilon_i^N - x_i^N) = \sum_{j\in C_q\setminus\{i\}} \lambda_j^N\big(x_j^N - x_j^{N\setminus\{i\}}\big) - \sum_{j\in C_q\setminus\{i\}} \lambda_i^N\big(x_i^N - x_i^{N\setminus\{j\}}\big).$$

Because $(x^S)_{S\subset N}$ satisfies (B4), the last expression equals 0. Thus, $x_i^N = \Upsilon_i^N$. $\square$

Proof of Lemma 4.1. We prove two claims.

Claim 1. Λ satisfies (B1) and (B4) in CH.

Let $(N, V, \mathscr{C}) \in CH$.

Given $i \in C_q \in \mathscr{C}$, $\pi \in \Pi^{C_q}$, and $S = \overline{C}_q \cup P(\pi, i) \cup \{i\}$,

$$\lambda_i^S f_i(\pi) = v(S) - \sum_{j\in\overline{C}_q} \lambda_j^S \Omega_j^S - \sum_{j\in P(\pi,i)} \lambda_j^S f_j(\pi).$$

Thus, for each $C_q \in \mathscr{C}$ and $\pi \in \Pi^{C_q}$,

$$\sum_{j\in C_q} \lambda_j^N f_j(\pi) = v(N) - \sum_{j\in\overline{C}_q} \lambda_j^N \Omega_j^N = \sum_{j\in C_q} \lambda_j^N \Omega_j^N.$$

Hence, $\sum_{j\in C_q} \lambda_j^N \Lambda_j^N = \sum_{j\in C_q} \lambda_j^N \Omega_j^N$ and

$$\sum_{j\in N} \lambda_j^N \Lambda_j^N = \sum_{C_q\in\mathscr{C}} \left(\sum_{j\in C_q} \lambda_j^N \Lambda_j^N\right) = \sum_{C_q\in\mathscr{C}} \left(\sum_{j\in C_q} \lambda_j^N \Omega_j^N\right) = v(N),$$

which means that Λ satisfies (B1).

We now prove that Λ satisfies (B4). For each $j \in C_q$, we denote the set of permutations of Π^{C_q} where j is the last player as $\Pi^{C_q}(j)$. If $j \neq i$, then player i's expected marginal contribution conditioned to j being last is the same as in the game $(N\setminus\{j\}, V, C_{-j})$, which is $\Lambda_i^{N\setminus\{j\}}$, i.e.,

$$\frac{1}{|\Pi^{C_q}(j)|} \sum_{\pi\in\Pi^{C_q}(j)} f_i(\pi) = \frac{1}{|\Pi^{C_q\setminus\{i\}}|} \sum_{\pi\in\Pi^{C_q\setminus\{i\}}} f_i(\pi) = \Lambda_i^{N\setminus\{j\}}.$$

Given $\pi \in \Pi^{C_q}(i)$, it is not difficult to prove that

$$\lambda_i^N f_i(\pi) = \sum_{j\in C_q} \lambda_j^N \Lambda_j^N - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N f_j(\pi), \quad \text{and}$$

$$\frac{1}{|\Pi^{C_q}(i)|} \sum_{\pi\in\Pi^{C_q}(i)} \lambda_i^N f_i(\pi) = \sum_{j\in C_q} \lambda_j^N \Lambda_j^N - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Lambda_j^{N\setminus\{i\}}.$$

Take $i \in C_q \in \mathscr{C}$. Because $|\Pi^{C_q}| = |C_q||\Pi^{C_q}(j)|$ for each $j \in C_q$,

$$\begin{aligned}\Lambda_i^N &= \frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus\{i\}} \frac{1}{|\Pi^{C_q}(j)|} \sum_{\pi\in\Pi^{C_q}(j)} f_i(\pi) + \frac{1}{|\Pi^{C_q}(i)|} \sum_{\pi\in\Pi^{C_q}(i)} f_i(\pi)\right] \\ &= \frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus\{i\}} \Lambda_i^{N\setminus\{j\}} + \frac{1}{\lambda_i^N}\left(\sum_{j\in C_q} \lambda_j^N \Lambda_j^N - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Lambda_j^{N\setminus\{i\}}\right)\right].\end{aligned}$$

Making some computations we can prove that Λ satisfies (B4).

CLAIM 2. Λ satisfies (B1) and (B4) in *CNTU*.

We prove that if $(x^S)_{S\subset N}$ is associated with Λ, then for each $S\subset N$ there exists $\lambda^S\in\mathbb{R}^S_{++}$ satisfying (B1) and (B4).

There exists $(N, V', \mathscr{C})\in CH$ such that $x^S=\Lambda(S, V', \mathscr{C}_S)$. Let $(\lambda^S)_{S\subset N}$ and $v'\colon 2^N\to\mathbb{R}$ be associated with V'. For each $S\subset N$, $\lambda^S * x^S = v'(S)=\max\{\lambda^S * y\colon\, y\in V(S)\}$ because $x^S=\Lambda(S, V', \mathscr{C}_S)$ and Λ satisfies (B1) in *CH*.

Applying Claim 1 to the hyperplane game $(S, V', \mathscr{C}_S)$ and agent $i\in C'_q\in\mathscr{C}_S$ we obtain that Λ satisfies (B4). $\square$

PROOF OF THEOREM 5.2. The following result appears in Maschler and Owen [11].

LEMMA A.1. *Given a hyperplane game (N, V, C) and $i\in S\subset C_q\in C$,*

$$(S\backslash\{i\}, V_S, \{S\backslash\{i\}\}) = (S\backslash\{i\}, V_{S\backslash\{i\}}, \{S\backslash\{i\}\}).$$

Maschler and Owen [11] proved that Ψ satisfies l-consistency for all $l=1,\ldots,n$. In the next lemma we obtain a similar result for Υ.

LEMMA A.2. *The consistent coalitional value satisfies l-consistency for each $l = 1,\ldots,n$.*

PROOF. We proceed by induction on l. The result is trivially true for $l=1$. Assume it is true when $l\le t$. We prove it when $l=t+1$.

If we apply the induction hypothesis to the game $(N\backslash\{j\}, V, C_{-j})$ with $j\in C_q\backslash\{i\}$ (if $C_q=\{i\}$, the result is trivially true for C_q), then

$$\sum_{T\subset C_q\backslash\{j\}\colon i\in T,\ |T|=l-1}\Upsilon_i^T(V_T)=\binom{|C_q|-2}{l-2}\Upsilon_i^{N\backslash\{j\}}(V). \tag{A.1}$$

We want to prove that, for each $C_q\in C$ and $i\in C_q$,

$$l\lambda_i^N\sum_{S\subset C_q\colon i\in S,\ |S|=l}\Upsilon_i^S(V_S)=l\lambda_i^N\binom{|C_q|-1}{l-1}\Upsilon_i^N(V). \tag{A.2}$$

To do so, we analyze the left side of this expression. Assume that $i\in S\subset C_q$ and $|S|=l$. Applying (3.2) to $(S, V_S, \{S\})$, which is also a hyperplane game, we obtain

$$l\lambda_i^N\Upsilon_i^S(V_S)=\sum_{j\in S\backslash\{i\}}\lambda_i^N\Upsilon_i^{S\backslash\{j\}}(V_S)+\sum_{j\in S}\lambda_j^N\Upsilon_j^S(V_S)-\sum_{j\in S\backslash\{i\}}\lambda_j^N\Upsilon_j^{S\backslash\{i\}}(V_S).$$

Making some computations, we can prove that

$$\begin{aligned}
l\lambda_i^N\sum_{S\subset C_q\colon i\in S,\ |S|=l}\Upsilon_i^S(V_S) ={}& \sum_{j\in C_q\backslash\{i\}}\left(\sum_{S\subset C_q\colon i,\, j\in S,\ |S|=l}\lambda_i^N\Upsilon_i^{S\backslash\{j\}}(V_S)\right)+\binom{|C_q|-1}{l-1}v(N)\\
&-\sum_{j\in N\backslash\{i\}}\left(\sum_{S\subset C_q\colon i\in S,\, j\notin S,\ |S|=l}\lambda_j^N\Upsilon_j^N(V)\right)\\
&-\sum_{j\in C_q\backslash\{i\}}\left(\sum_{S\subset C_q\colon i,\, j\in S,\ |S|=l}\lambda_j^N\Upsilon_j^{S\backslash\{i\}}(V_S)\right).
\end{aligned}$$

We now analyze the four terms separately. By Lemma A.1 and (A.1), the first term is equal to

$$\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\left(\sum_{T\subset C_q\backslash\{j\}\colon i\in T,\ |T|=l-1}\Upsilon_i^T(V_T)\right).$$

It is not difficult to prove that the third term is equal to

$$-\binom{|C_q|-2}{l-1}\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\Upsilon_j^N(V)-\binom{|C_q|-1}{l-1}\sum_{j\in N\setminus C_q}\lambda_j^N\Upsilon_j^N(V).$$

By Lemma A.1 and (A.1), the fourth term is equal to

$$-\binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\Upsilon_j^{N\setminus\{i\}}(V).$$

Because Υ satisfies (B4), making some computations we can prove that

$$l\lambda_i^N\sum_{S\subset C_q:\, i\in S,\,|S|=l}\Upsilon_i^S(V_S)=l\binom{|C_q|-1}{l-1}\lambda_i^N\Upsilon_i^N(V),$$

which is the right side of (A.2). $\square$

We now prove that Υ satisfies these five properties. By Theorem 3, Υ satisfies *EF* and *ABCAC*. By Lemma A.2, Υ satisfies *BCONS*. It is straightforward to prove that Υ satisfies *IS*.

We now prove that Υ satisfies *COV*. Given $i\in C_q\in C$, let $(N,\widetilde{V},C)$ be obtained from (N,V,C) by a change in player i's utility. Let a and b be the corresponding constants. We proceed by induction for the number of coalitions in C.

Assuming C has only one coalition ($C=\{N\}$), because Ψ satisfies *COV*, Υ also satisfies *COV*.

Assume the result holds when $|C|$ has at most $m-1$ coalitions. We prove it when $|C|=m$.

Because Υ satisfies (3.1), under the induction hypothesis, making some computations we can prove that

$$|C|\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^N(\widetilde{V})=|C|\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N(V)+|C|\frac{b\lambda_i^N}{a}.$$

Given $k\in C_q$, by (3.2),

$$|C_q|\tilde{\lambda}_k^N\Upsilon_k^N(\widetilde{V})=\sum_{j\in C_q\setminus\{k\}}\tilde{\lambda}_k^N\Upsilon_k^{N\setminus\{j\}}(\widetilde{V})+\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^N(\widetilde{V})-\sum_{j\in C_q\setminus\{k\}}\tilde{\lambda}_j^N\Upsilon_j^{N\setminus\{k\}}(\widetilde{V}).$$

Because Υ satisfies (3.2), the induction hypothesis, and the previous result, making some computations we can prove that

$$|C_q|\tilde{\lambda}_i^N\Upsilon_i^N(\widetilde{V})=|C_q|\lambda_i^N\Upsilon_i^N(V)+\frac{b\lambda_i^N}{a}|C_q|.$$

Therefore,

$$\Upsilon_i^N(\widetilde{V})=a\Upsilon_i^N(V)+b.$$

Using similar arguments to those used before, we can prove that $\Upsilon_k^N(\widetilde{V})=\Upsilon_k^N(V)$ when $k\neq i$. Thus, Υ satisfies *COV*.

We now prove the uniqueness. Let $\widetilde{\Upsilon}$ be a value satisfying these five properties. We will show that $\widetilde{\Upsilon}=\Upsilon$. We proceed by induction on the number of players. If there is only one player, then, by *EF*, $\widetilde{\Upsilon}_i=\max\{x\colon x\in V(\{i\})\}=\Upsilon_i$.

When $|N|=2$, by *COV* it is not difficult to prove that

$$\widetilde{\Upsilon}_i^N=\Upsilon_i^N=\frac{v(N)+\lambda_i^N v(\{i\})-\lambda_j^N v(\{j\})}{2\lambda_i^N}.$$

Hence,

$$\lambda_i^N \Upsilon_i^N - \lambda_j^N \Upsilon_j^N = \lambda_i^N \widetilde{\Upsilon}_i^N - \lambda_j^N \widetilde{\Upsilon}_j^N = \lambda_i^N v(\{i\}) - \lambda_j^N v(\{j\}). \tag{A.3}$$

Assume that $\widetilde{\Upsilon} = \Upsilon$ when $2 \leq n \leq t$. We prove $\widetilde{\Upsilon} = \Upsilon$ when $n = t+1$.

We first prove that, for each $C_q \in C$,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N. \tag{A.4}$$

Because Υ satisfies (3.1) and *EF*, under the induction hypothesis, making some computations we can prove that

$$|C| \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N = \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N (\widetilde{\Upsilon}_j^{N \setminus C_r} - \widetilde{\Upsilon}_j^N) \right) + |C| \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N$$
$$+ \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N (\widetilde{\Upsilon}_j^N - \widetilde{\Upsilon}_j^{N \setminus C_q}) \right).$$

As $\widetilde{\Upsilon}$ satisfies *ABCAC*, we conclude that $\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N$.

We now prove that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$ for each $i \in C_q \subset N$. We denote by V_S and $\widetilde{V}_S$ the reduced games associated with Υ and $\widetilde{\Upsilon}$, respectively.

By (A.4), if $C_q = \{i\}$, we conclude that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$.

Assume that $C_q \neq \{i\}$. For each $j \in C_q \setminus \{i\}$, we consider $S = \{i, j\}$. We know that V_S and $\widetilde{V}_S$ are hyperplane games. We denote by v_S and $\tilde{v}_S$ the characteristic functions associated with V_S and $\widetilde{V}_S$, respectively.

Because Υ and $\widetilde{\Upsilon}$ satisfies *EF* and *COV*, (A.3) and (A.4), making some computations we can prove that

$$\lambda_i^N \widetilde{\Upsilon}_i^N + \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S(\widetilde{V}_S) = \lambda_i^N \Upsilon_i^N + \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \Upsilon_j^S(V_S) \tag{A.5}$$

and

$$(|C_q| - 1)\lambda_i^N \widetilde{\Upsilon}_i^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S(\widetilde{V}_S) = (|C_q| - 1)\lambda_i^N \Upsilon_i^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \Upsilon_j^S(V_S). \tag{A.6}$$

Adding (A.5) and (A.6) we obtain that $|C_q| \lambda_i^N \Upsilon_i^N = |C_q| \lambda_i^N \widetilde{\Upsilon}_i^N$, which means that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$. □

Proof of Remark 5.1. *ABCAC* is independent of the rest of properties because the consistent value satisfies *EF*, *IS*, *COV*, and *BCONS* but not *ABCAC*.

Using arguments similar to those used by Winter [24], we can conclude that the rest of properties are independent. □

Proof of Proposition 6.1. We first prove the following lemma.

Lemma A.3. *F satisfies* (3.1) *in the class of hyperplane games.*

Proof. Let $F = (F^S)_{S \subset N}$ be the random order coalitional payoff configuration for (N, V, C). By definition, F_j^N is the expected marginal contribution of player j over all the $|\Pi^C|$ admissible permutations of players with respect to C. We classify these permutations in $|C|$ groups with respect to the last coalition C_r in such permutations.

Let $\Pi^C(C_r)$ be the set of admissible permutations with respect to C in which players of coalition C_r are in the last position. Notice that $|\Pi^C| = |C||\Pi^C(C_r)|$ for each $C_r \in C$.

If $C_r \neq C_p$, then the expected marginal contribution for each player $j \in C_p$ in the permutations of $\Pi^C(C_r)$ coincides with the expected marginal contribution of player j in the game $(N\backslash C_r, V, C\backslash C_r)$, which is $F_j^{N\backslash C_r}$, i.e.

$$\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi) = \frac{1}{|\Pi^{C\backslash C_r}|}\sum_{\pi\in\Pi^{C\backslash C_r}} d_j(\pi) = F_j^{N\backslash C_r}. \tag{A.7}$$

It is not difficult to prove that

$$\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left[\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right] = v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N F_j^{N\backslash C_q}\right). \tag{A.8}$$

By (A.7) and (A.8), making some computations we can prove that

$$\sum_{j\in C_q}\lambda_j^N F_j^N = \frac{1}{|C|}\left[\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N F_j^{N\backslash C_r}\right) + v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N F_j^{N\backslash C_q}\right)\right]$$

which is precisely the statement of this lemma. □

We now prove Proposition 6.1.

(i) It is trivial to see that F satisfies *EF* and *IS*.

Maschler and Owen [11] showed that, for any $\pi \in \Pi$, the vector $d(\pi)$ satisfies *COV*. Because F is the average of some of these $d(\pi)$'s, we conclude that F also satisfies *COV*.

By Lemma A.3, F satisfies (3.1). Now, using arguments similar to those used for Υ, we can conclude that F also satisfies *ABCAC*.

(ii) It is a consequence of Theorems 5.1 and 5.2. □

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