A sequential bargaining protocol for land rental arrangements*

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Abstract

We consider land rental between a single tenant and several lessors. The tenant should negotiate sequentially with each lessor for the available land. In each stage, we apply the Nash bargaining solution. Our results imply that, when all land is necessary, a fixed price per unit is more favourable for the tenant than a lessor-dependent price. Furthermore, a lessor is better off with a lessor-dependent price only when negotiating first. For the tenant, lessors' merging is relevant with lessor-dependent price but not with fixed price.

Keywords: Bargaining; non-cooperative game; Nash solution; land rental

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1 Introduction

Assume that there exists a single tenant that needs to negotiate with several lessors for the use of their land. Examples may arise in natural resource exploitation situations as well as in the construction of public or private facilities, and urbanization in populated areas.

For example, Sosa (2011) reports the following conflict between the mining industry and indigenous communities in Peru. The Southern Copper Corporation (SCC) is a company that has the right to exploit the underground resources in the southern Peruvian region. However, this land is customary used by local residents, who have the right to use the surface.

A possible approach to solve the conflict is from a centralized point of view. A planner, for example the government, determines a fair compensation for the use of land. This is the idea of the market design approach applied to land problems. See Sen (2007) for a survey. Two recent different approaches in this direction are proposed by Valencia-Toledo and Vidal-Puga (2017) and Sarkar (2017).

However, the Peruvian government did not try this option and, instead, encouraged both sides to reach an agreement by themselves. Even though the protocol of the negotiation is undetermined, we can still try to figure out how this negotiation may take place. We study this situation from a non-cooperative game perspective.

This is the approach taken by Bergantiños and Lorenzo (2004), who modeled the negotiation process that took place between villagers and the authorities in order to construct pipelines to connect individual houses to a water dam.

Following a similar idea, we analyze a situation that took place in Galicia, Spain. A military base[1] was established in 1968 over the land owned by three land communities (named Salcedo, Vilaboa and Figueirido, respectively). In October 2008, due to the government decision to create a wide security

[1]The Airborne Light Infantry Brigade, or *Brigada Ligera AeroTranspotada* (BRILAT).

perimeter around the base, these land communities engaged in a lawsuit claiming to be the rightful owners of the land. In November 2012, the Spanish Court of Justice settled in favour of the local communities.

As local communities are the rightful owners of the land, the government was legally compelled to reach an agreement with them. We can identify the government as a single tenant and Salcedo, Vilaboa and Figueirido communities as lessors who negotiated on a price per unit of land and a quantity of land through a negotiation process. This negotiation process, as reported by the local media, had the following features:

- One of the first announces made by the government was that they would pay the same price per hectare to all communities.

- A neighbourly community, named Marin, offered their land to the government in order to move part of the settlement. Hence, the land communities become four, instead of the original three.

- The main land community, Salcedo, offered the others to join forces and negotiate with the government as one voice. The others refused.

- After a *cheap talk* stage, that lasted almost one year, the government made a final sequential negotiation round with each community separately.

- The assented protocol stated that, if a community agrees on a price, and, later on, the price increases for another community, then it is automatically increased for the former, without repeating the negotiation.

- Any agreement between the government and one community was conditional on reaching a minimum amount of land at the end of the process.

- At the end, the tenant rented 121 Ha of land which was less than total available (around 216 Ha).

In this paper, we model the above protocol assuming that lessors are agents that want to increase the price, and that the single tenant is an agent who wants to decrease it.

This situation has many similarities with the negotiation problem between SCC and the indigenous communities in Peru: there are one tenant and several lessors; the identities of the lessors are loose (land customary in Peru, and the possible presence of more communities in Spain); the tenant does not necessarily need all the land; and, even though negotiation protocol is undetermined, the tenant is the one who takes the initiative.

In our model, the tenant sequentially negotiates with each lessor, following a pre-established ordering. In each stage, there exists a bargaining problem between the tenant and the lessor. We do not formalize the non-cooperative game in each stage. Instead, we assume that the agents reach an agreement given by the Nash solution (Nash, 1950). Even though the Nash solution has a solid axiomatic justification, the reason to use it here is non-cooperative. In fact, the Nash solution arises in equilibrium in many natural two-player negotiation processes, such as those modeled by Nash (1953), Rubinstein (1982), Binmore et al. (1986), Van Damme (1986) or Papatya and Trockel (2016). There are also protocols with more than two players, but the results are not so satisfactory, as they need refinements in the equilibrium concept, such as stationary strategies (Hart and Mas-Colell, 1996, 2010; Trockel, 2002). Non-cooperative foundations of other bargaining solutions, such as the Kalai-Smorodinsky and the discrete Raiffa solutions, use some counter-intuitive features, such as bids on probabilities (Moulin, 1984) or the existence of a finite pre-predetermined number of stages (Ståhl, 1972).

Matsushima and Shinohara (2015) also use the Nash solution as a way to identify the agreements in indeterminate non-cooperative settings. As opposed to this paper, they use the Nash solution in bargaining problems with more than two players.

In this paper, we study two cases: In the first case, we assume that the tenant has the freedom to negotiate different prices per unit of land for each lessor (i.e. prices are lessor-dependent). We call this case as the *variable price* case. In the second case, we assume that all the prices are updated, at the end of the process, to the highest price agreed upon. We call this case as the *fixed price* case. One of our results is that the fixed price case if more favourable for the tenant than the variable price case.

The remainder of this paper is structured as follows. In Section 2 we introduce the notation and present the model. In Section 3 we illustrate the non-cooperative protocol with an example with two lessors. In Section 4 we present the sequential bargaining protocol. In Section 5 we present the results for the unanimity case. In Section 6 we present the results for the non-unanimity case. In Section 7 we presents some concluding remarks.

2 The model

We denote the set of non-negative real numbers as $\mathbb{R}_+$, and the set of positive real numbers as $\mathbb{R}_{++}$. Let $N = \{1, \dots, n\}$ be a generic finite set. For any $S \subseteq N$, $\mathbb{R}^S$ is the Euclidean space of dimension $|S|$ whose coordinates are indexed by the elements of S. When $|S| = 2$ and there is no ambiguity, we write $\mathbb{R}^2$ instead of $\mathbb{R}^S$. Given $y \in \mathbb{R}^S$, we write $y(S) = \sum_{i \in S} y_i$. We denote by $\mathbf{0_N}$ and $\mathbf{1_N} \in \mathbb{R}^N$, the vectors whose coordinates are all 0 and 1, respectively. Given $x, y \in \mathbb{R}^N$ we write $x \leq y$ when $x_i \leq y_i$ for all $i \in N$. Given $S, T \subset N$, $S \cap T = \emptyset$, $x \in \mathbb{R}^S$ and $y \in \mathbb{R}^T$, we define $z = (x, y) \in \mathbb{R}^{S \cup T}$ as the combination of x and y, i.e. $z_i = x_i$ for all $i \in S$ and $z_i = y_i$ for all $i \in T$. For simplicity, given $i \in N$, $T \subseteq N \setminus \{i\}$, $x \in \mathbb{R}^T$, and $h \in \mathbb{R}$, we write $(h, x) \in \mathbb{R}^{\{i\} \cup T}$ instead of $((h), x)$, and so on. Given $A, B \subset \mathbb{R}^S$, we define $A + B = \{a + b : a \in A, b \in B\} \subset \mathbb{R}^S$.

A *two-person bargaining problem* is a pair (D, d) where $d \in D \subset \mathbb{R}^2$, D is closed and bounded above (i.e. for all $x \in D$ the set $\{y \in D : y \geq x\}$ is compact). D is the set of feasible utility pairs, and d is the disagreement point.

The *Pareto frontier* of D is the set $\partial D := \{x \in D : (\{x\} + \mathbb{R}^2_+) \cap D = \{x\}\}$.

We denote as $\mathcal{B}_2$ the set of two-person bargaining problems. Given $\hat{\mathcal{B}}_2 \subseteq \mathcal{B}_2$, a *bargaining solution* on $\hat{\mathcal{B}}_2$ is a map $\psi : \hat{\mathcal{B}}_2 \rightarrow \mathbb{R}^2$ such that for each $(D,d) \in \hat{\mathcal{B}}_2$, $\psi(D,d) \in D$ and $\psi(D,d) \geq d$, where $\psi(D,d)$ represents the expected final payoff for the players when facing the bargaining problem (D,d). The most well-known solution on bargaining problems is the *Nash solution* (Nash, 1950). Let $g^d(x) = (x_1 - d_1)(x_2 - d_2)$ for each $x \in \mathbb{R}^2$. Let $\mathcal{B}_2^N \subset \mathcal{B}_2$ be the set of all two-person bargaining problems (D,d) such that g^d reaches a unique maximizer over D. This unique maximizer determines the Nash solution.

We define a *land rental problem* as a tuple (N_0, c, r, K, E) with $N_0 = \{0\} \cup N$ where 0 is a single tenant and $N = \{1, \ldots, n\}$ is a finite set of lessors, $c \in \mathbb{R}_+^N$ represents the amount of available land for each lessor, $r \in \mathbb{R}_+$ represents the reservation price per unit of land for each lessor, and $K > 0$ is the utility that the tenant can obtain by having at least $E > 0$ units of land. We assume that $E \leq c(N)$, and there exists $y \in \mathbb{R}_+^N$, $y \leq c$, $y(N) = E$, such that $K \geq rE$.

The set of admissible agreements is given by:

$$A = \left\{(p,x) : p, x \in \mathbb{R}_+^N, x \leq c\right\}.$$

Given $(p,x) \in A$, the utility for the tenant and each lessor $i \in N$ are, respectively,

$$u_0(p,x) = \begin{cases} K - \sum_{i \in N} p_i x_i, & \text{if } E \leq x(N) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_i(p,x) = \begin{cases} (p_i - r)x_i, & \text{if } E \leq x(N) \\ 0, & \text{otherwise.} \end{cases}$$

3 An illustration with two lessors

In order to illustrate the sequential bargaining protocol, we consider the land rental problem (N_0, c, r, K, E) with $N = \{1, 2\}$. For each $i \in N$ we use the notation $(p_i, x_i) \in \mathbb{R}_+ \times [0, c_i]$ to represent the agreement between the tenant and lessor i, so that p_i is the price per unit of land and x_i is the amount of rented land.

Since there are two lessors, the protocol has two stages:

Stage 1: The tenant and the first lessor in the order (lessor 1) agree on a pair $(p_1, x_1) \in \mathbb{R}_+ \times [0, c_1]$. In case of disagreement, we assume $p_1 = 0$ and $x_1 = 0$.

Stage 2. The tenant and the second lessor in the order (lessor 2) agree on a pair $(p_2, x_2) \in \mathbb{R}_+ \times [0, c_2]$. In case of disagreement, we assume $p_2 = 0$ and $x_2 = 0$.

After Stage 2, we have $(p, x) \in A$. In the variable price case, this is the final agreement, so that the final payoff allocation is $(u_0(p, x), u_1(p, x), u_2(p, x))$. For the fixed price case, the final agreement is $(\hat{p}, x) \in A$ with $\hat{p}_i = \max\{p_1, p_2\}$ for all $i \in N$, and the final payoff allocation is $(u_0(\hat{p}, x), u_1(\hat{p}, x), u_2(\hat{p}, x))$.

Notice that, in this land rental problem, the tenant needs to reach an agreement with both lessor 1 (because $c_2 < E$) and lessor 2 (because $c_1 < E$).

Example 3.1 *Let (N_0, c, r, K, E) be a land rental problem with $N = \{1, 2\}$, $c = (9, 9)$, $r = 2$, $K = 250$ and $E = 10$. Assume that, in Stage 1, the tenant and lessor 1 agree on $p_1 = 12$ and $x_1 = 4$. Assume also that, in Stage 2, the tenant and lessor 2 agree on $p_2 = 10$ and $x_2 = 6$. Then, in the variable price case, the final payoffs are $u_0((12, 10), (4, 6)) = 250 - (12 \cdot 4 + 10 \cdot 6) = 142$ for tenant, $u_1((12, 10), (4, 6)) = (12 - 2)4 = 40$ for lessor 1, and $u_2((12, 10), (4, 6)) = (10 - 2)6 = 48$ for lessor 2. Thus, the final payoff allocation is $(142, 40, 48)$. In the fixed price case, the final payoffs are $u_0((12, 12), (4, 6)) = 250 - 12 \cdot (4 + 6) = 130$ for tenant, $u_1((12, 12), (4, 6)) = (12 - 2)4 = 40$ for lessor 1, and $u_2((12, 10), (4, 6)) = (12 - 2)6 = 60$ for lessor 2. Thus, the final payoff allocation is $(130, 40, 60)$.*

Assume now that the tenant and lessor 1 reach the same agreement as before but, in Stage 2, the tenant and lessor 2 do not reach an agreement. Then, $p_2 = x_2 = 0$ and the final payoff allocation is $(0, 0, 0)$ in both cases.

Intuitively, it seems natural that reaching a Pareto efficient agreement in the last stage will imply an efficient final payoff allocation, i.e. $x_1 + x_2 = E$ should hold. This is the case for the protocol with variable price, but not always for the protocol with fixed price, as we can see in Example 3.2.

Example 3.2 *Consider the land rental problem in Example 3.1. Assume that, in Stage* 1*, the tenant and lessor* 1 *agree on* $p_1 = 2$ *and* $x_1 = 9$ *and, in Stage* 2*, the tenant and lessor* 2 *agree on* $p_2 = 20$*. The final agreement is efficient iff* $x_2 = 1$*. However,* $x_2 = 1$ *yields a Pareto inefficient payoff allocation in Stage 2 for the tenant and lessor* 2*. When* $x_2 = 1$*, their final payoffs are, respectively,* 50 *and* 18*. If they agreed instead on* $p_2' = 15$ *and* $x_2' = 2$*, then their final payoffs would be, respectively,* 85 *and* 26*. The tenant and lessor* 2 *are both better off but there is an inefficiency because* $x_1 + x_2' > E$*.*

We now describe the corresponding two-person bargaining problem in each stage according to the protocol with variable price and fixed price respectively.

Protocol with variable price

In Stage 1, the tenant and lessor 1 agree on either a pair $(p_1, x_1) \in \mathbb{R}_+ \times [0, c_1]$, or, in case of disagreement, $p_1 = 0$ and $x_1 = 0$. In order to know the final payoff, they need to anticipate what will happen in Stage 2, where the tenant and lessor 2 bargain. In order to guarantee that the agreement can be possible in Stage 2, it is necessary that

$$x_1 \geq E - c_2 \tag{1}$$

(otherwise, the tenant would not be able to get E), and

$$(p_1 - r)x_1 \leq K - rE \tag{2}$$

(otherwise, the tenant and lessor 2 would not have benefit of cooperation). Henceforth, (1) and (2) are strategic constraints.

In Stage 2, the tenant and lessor 2, knowing (p_1, x_1), agree on a pair $(p_2, x_2) \in \mathbb{R}_+ \times [0, c_2]$. In case of disagreement, $p_2 = 0$ and $x_2 = 0$. We consider the land rental problem in Example 3.1. Then, for each possible choice

(p_1, x_1), both the tenant and lessor 2 face a bargaining problem (D^2, d^2) with

$$D^2 = D^2(p_1, x_1) = \{(u_0(p,x), u_2(p,x)) : p_2 \geq 0, x_2 \in [0,9]\}$$

and $d^2 = (0,0)$. Recall that p_1 and x_1 are fixed (they come from Stage 1). A Pareto efficient agreement is obtained if and only if $x_2 = 10 - x_1$. Then, we have that

$$\begin{aligned}\partial D^2 &= \{(250 - p_1x_1 - p_2(10 - x_1), (p_2 - 2)(10 - x_1)) : p_2 \geq 0\} \\ &= \{(u_0, u_2) : u_0 + u_2 \leq 230 - (p_1 - 2)\, x_1, u_0 \leq 250 - p_1x_1\}.\end{aligned}$$

The Nash bargaining solution gives both agents the same utility $u_0 = u_2 = \frac{230-(p_1-2)x_1}{2}$, which are uniquely determined by $p_2^* = p_2^*(p_1, x_1) = \frac{250-p_1x_1}{2(10-x_1)} + 1$ and $x_2^* = x_2^*(p_1, x_1) = 10 - x_1$. We can see illustrations of three possible choices of (p_1, x_1) and its corresponding D^2 in Figure 1.

Given this, we now can go back to Stage 1 when the tenant and lessor 1 can anticipate the final payoff for each possible choice of (p_1, x_1). The bargaining problem (D^1, d^1) is given by

$$D^1 = \{(u_0(p,x), u_1(p,x)) : p_2 = p_2^*, x_2 = x_2^*, p_1 \geq 0, x_1 \in [0,9]\}$$

and $d^1 = (0,0)$.

Assuming (1) and (2), we have $u_0(p,x) = 250 - p_1x_1 - p_2^* \cdot (10 - x_1) = \frac{230-(p_1-2)x_1}{2}$ and $u_1(p,x) = (p_1 - 2)x_1$. Hence, we can rewrite D^1 as:

$$D^{*1} = \left\{ \left(\frac{230 - (p_1 - 2)x_1}{2}, (p_1 - 2)x_1 \right) : (p_1, x_1) \in \Delta^1 \cup \{(0,0)\} \right\}$$

where

$$\Delta^1 = \{(p_1, x_1) \in \mathbb{R}_+ \times [0,9] : x_1 \geq 1, (p_1 - 2)x_1 \leq 230\}$$

is the set of partial agreements that satisfy (1) and (2). Clearly $\partial D^{*1} = \partial D^1$ and hence the Nash solution coincides in both (D^1, d^1) and (D^{*1}, d^1). In particular, the Nash solution maximizes $\left(\frac{230-(p_1-2)x_1}{2}\right)(p_1 - 2)x_1$. This maximum is reached when $(p_1 - 2)x_1 = 115$. See Figure 2 for the illustration of ∂D^1 and the Nash solution for $(p_1 - 2)x_1 = 115$. Then, the final payoff allocation is $(57.5, 115, 57.5)$.

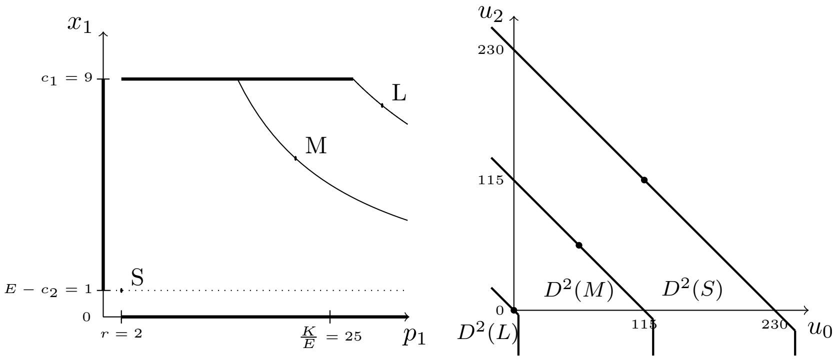

Figure 1: The bargaining problem with variable price that arises in Stage 2. Left: Three possible choices of (p_1, x_1) in Stage 1 subject to feasible ($p_1 \geq 0$ and $0 \leq x_1 \leq c_1$) and strategic ($p_1 \geq r$, $x_1 \geq E - c_2$ and $(p_1 - r)\, x_1 \leq K - rE$) constraints. Right: Examples of D^2, with their respective Nash solutions, for each choice (p_1, x_1): $D^2(S)$ when $(p_1 - r)x_1 = 0$, $D^2(M)$ when $(p_1 - r)x_1 = \frac{K-rE}{2}$, and $D^2(L)$ when $(p_1 - r)x_1 = K - rE$. Equation $u_0 + u_2 = K - rE - (p_1 - r)x_1$ determines the Pareto frontier of each D^2. Hence, the smaller $(p_1 - r)x_1$, the better the tenant and lessor 2 are.

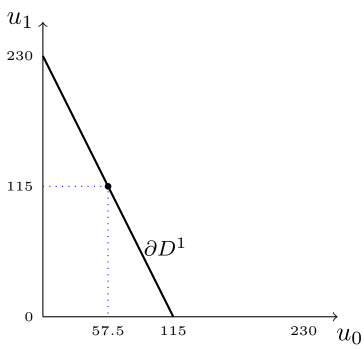

Figure 2: ∂D^1 with its Nash solution for the bargaining problem with variable price that arises in Stage 1.

Protocol with fixed price

In Stage 1, the tenant and lessor 1 agree on either a pair $(p_1, x_1) \in \mathbb{R}_+ \times [0, c_1]$, or, in case of disagreement, $p_1 = 0$ and $x_1 = 0$. In order to know the final payoff, they need to anticipate what will happen in Stage 2, where the tenant and lessor 2 bargain. In order to guarantee that the agreement can be possible in Stage 2, it is necessary that (1) holds (otherwise, the tenant would not be able to get E), and

$$p_1 \leq \frac{K}{E} \tag{3}$$

(otherwise, the tenant and lessor 2 would not have benefit of cooperation). Henceforth, (1) and (3) are strategic constraints.

In Stage 2, the tenant and lessor 2, knowing (p_1, x_1), agree on a pair $(p_2, x_2) \in \mathbb{R}_+ \times [0, c_2]$. In case of disagreement, $p_2 = 0$ and $x_2 = 0$. We also consider the same land rental problem in Example 3.1. Then, for each possible (p_1, x_1), both the tenant and lessor 2 face a bargaining problem $(\hat{D}^2, \hat{d}^2)$ with

$$\hat{D}^2 = \hat{D}^2(p_1, x_1) = \{(u_0\,(\hat{p}, x)\,, u_2\,(\hat{p}, x)) : p_2 \geq 0, x_2 \in [0, 9]\}$$

and $\hat{d}^2 = (0, 0)$. As opposed to the protocol with variable price, inefficiency can arise when $x_2 = 10 - x_1$, as we can see in Example 3.2.

It is straightforward to check that $\partial\hat{D}^2$ can be written as:

$$\partial\hat{D}^2 = \left\{(u_0\,(p_2, x_2)\,, g_2\,(u_0\,(p_2, x_2))) : (p_2, x_2) \in \hat{\Delta}^2\right\}$$

where

$$\hat{\Delta}^2 = [p_1, \infty[\times [10 - x_1, 9]$$

and

$$g_2(u_0) = \max\left\{\left(\frac{250 - u_0}{x_1 + x_2} - r\right) x_2 : x_2 \in \left[10 - x_1, \min\left\{9, \frac{250 - u_0}{p_1} - x_1\right\}\right]\right\},$$

for all $u_0 \in]-\infty, 10p_1]$. By maximizing $\left(\frac{250-u_0}{x_1+x_2} - r\right) x_2$, we obtain $x_2^* = \sqrt{\frac{x_1}{2}(250 - u_0)} - x_1$ and $p_2^* = \sqrt{\frac{2(250-u_0)}{x_1}}$. From this, we distinguish the following cases: $x_2^* < 10 - x_1$, $10 - x_1 \leq x_2^* < q$, and $q \leq x_2^*$, where

$$q = \min\left\{9, \frac{250 - u_0}{p_1}\right\}.$$

These cases can be simplified as follows:[2]

- First case (small p_1x_1): $p_1x_1 < 20$.
- Second case (medium p_1x_1): $20 \leq p_1x_1 < (x_1 + 9) \cdot 2$.
- Third case (large p_1x_1): $p_1x_1 \geq (x_1 + 9) \cdot 2$.

We can see illustrations of three possible choices of (p_1, x_1) and its corresponding $\hat{D}^2$ in Figure 3.

In Stage 1, the tenant and lessor 1 face a bargaining problem $(\hat{D}^1, \hat{d}^1)$ with $\hat{d}^1 = (0, 0)$ given as

$$\hat{D}^1 = \{(u_0(\hat{p}, x), u_1(\hat{p}, x)) : p_2 = p_2^*, x_2 = x_2^*, p_2 \geq p_1, x_1 \in [0, 9]\}.$$

Assuming (1) and (3), we have $u_0(\hat{p}, x) = 250 - (x_1 + x_2^*)p_2^*$ and $u_1(\hat{p}, x) = (p_2^* - r)x_1$. Hence, we can rewrite $\hat{D}^1$ as:

$$\hat{D}^{*1} = \left\{(250 - (x_1 + x_2^*)p_2^*, (p_2^* - 2)x_1) : (p_1, x_1) \in \hat{\Delta}^1 \cup \{(0, 0)\}\right\}.$$

where

$$\hat{\Delta}^1 = [0, 25] \times [1, 9]$$

is the set of partial agreements that satisfy (1) and (3). Clearly $\partial\hat{D}^{*1} = \partial\hat{D}^1$, and hence the Nash solution coincide in both $\left(\hat{D}^1, \hat{d}^1\right)$ and $\left(\hat{D}^{*1}, \hat{d}^1\right)$. In particular, the Nash solution maximizes $u_0^*(p_1, x_1)u_1^*(p_1, x_1)$.The final payoff allocation is $(107, 53.5, 53.5)$. This payoff allocation is inefficient, i.e. $u_0 + u_1 + u_2 < K - rE$. We can see the illustration of $\partial\hat{D}^1$ in Figure 3. Notice that the $\hat{D}^1$ is not convex.

[2]General computations are given in the proof of Theorem 6.2 (Section 6).

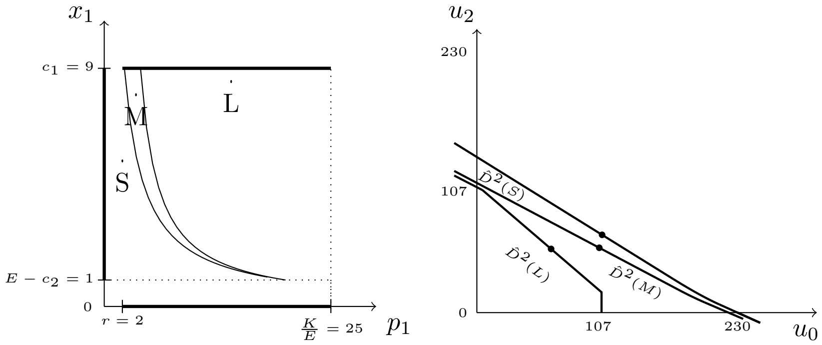

Figure 3: The bargaining problem with fixed price that arises in Stage 2. Left: Three Possible choices of (p_1, x_1) in Stage 1 subject to feasible ($p_1 > 0$ and $0 \leq x_1 \leq c_1$) and strategic ($x_1 \geq E - c_2$ and $r \leq p_1 \leq \frac{K}{E}$) constraints, e.g. S with $(p_1, x_1) = (2, 5.5)$, M with $(p_1 x_1) = (3.5, 8)$ and L with $(p_1 x_1) = (14, 8.5)$. Right: Examples of $\hat{D}^2$, with their respective Nash solutions, for each possible choice (p_1, x_1): $\hat{D}^2(S)$ with $p_1 x_1 < rE$, $\hat{D}^2(M)$ with $rE \leq p_1 x_1 < (x_1 + c_2)r$, and $\hat{D}^2(L)$ with $p_1 x_1 \geq (x_1 + c_2)r$.

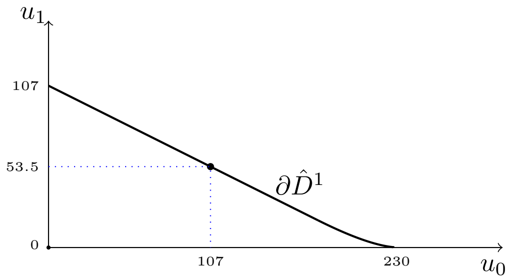

Figure 4: $\partial \hat{D}^1$ with Nash solution for the bargaining problem with fixed price that arises in Stage 1.

4 The sequential bargaining protocol

We formalize the general sequential bargaining protocol. Given a land rental problem (N_0, c, r, K, E), the tenant bargains sequentially with each lessor. Negotiations in each stage are bilateral between the tenant and the lessor in that stage. Without loss of generality, we consider that the ordering is

$1, 2, \ldots, n$. For each $i \in N$, we use the notation $a_i = (p_i, x_i) \in \mathbb{R}_+ \times [0, c_i]$ to represent the agreement reached by the tenant and lessor i, so that p_i is the price per unit of land and x_i is the amount of rented land.

The tenant and lessor s bargain in stage s. For each lessor $s \in N$,

$$A^s = \{((p_i, x_i))_{i=1}^{s} : p_i \geq 0, x_i \in [0, c_i] \ \forall \ i\}$$

is the set of feasible agreements in which the tenant and lessors $1, 2, \ldots, s$ can agree, and

$$\hat{A} = \left\{((p_i, x_i))_{i \in N} \in A : p_i = \hat{p} \in \mathbb{R}_+ \ \forall \ i \in N\right\}$$

is the set of feasible agreements in which the price is equal for all lessors.

In particular, $A^n = A$. For notational convenience we denote $A^0 = \{\emptyset\}$.

We begin by characterizing the final payoff allocation for the simplest case in which there exists an unique lessor (hereinafter called 1-*lessor land renting problem*). In this case, we assume w.l.o.g. $N = \{1\}$. Clearly, protocols with fixed price and variable price coincide when there is a single lessor. Proposition 4.1 shows the result for both protocols:

Proposition 4.1 *Given a* 1*-lessor land renting problem* (N_0, c, r, K, E)*, the final payoff allocation is* $\left(\frac{K-rE}{2}, \frac{K-rE}{2}\right)$.

Proof. The tenant and lessor 1 face a bargaining problem (D, d) with $d = (0, 0)$ and

$$\begin{aligned} D &= \{(u_0(a), u_1(a)) : a \in A\} \\ &= \{(K - p_1 x_1, (p_1 - r)x_1) : p_1 \geq 0, x_1 \in [E, c_1]\} \cup \{(0, 0)\}. \end{aligned}$$

Since $E \leq c_1$, it is straightforward to check that the Pareto frontier of D is:

$$\begin{aligned} \partial D &= \{(K - p_1 E, (p_1 - r)E) : p_1 \geq 0\} \\ &= \{(u_0, u_1) : u_1 + u_0 = K - rE, u_0 \leq K\}. \end{aligned}$$

The Nash solution is $\left(\frac{K-rE}{2}, \frac{K-rE}{2}\right)$. ∎

For more than one lessor, the final payoff allocation depends on the protocol.

Protocol with variable price

We describe the negotiation process of the non-cooperative protocol with variable price inductively as follows: Stage 1: The tenant and lessor 1 agree on a pair $a_1 \in \mathbb{R}_+ \times [0, c_1]$. In case of disagreement, we assume $a_1 = (0, 0)$. Stage s: Knowing $(a_i)_{i=1}^{s-1} \in A^{s-1}$, the tenant and lessor s agree on a pair $a_s \in \mathbb{R}_+ \times [0, c_s]$. In case of disagreement, we assume $a_s = (0, 0)$.

After finishing Stage s, we obtain an element in A^s. After finishing Stage n, we obtain an element in A.

We now describe the corresponding two-person bargaining problem in each stage. We proceed backwards from Stage n to Stage 1.

Stage n: $(a_i)_{i=1}^{n-1} \in A^{n-1}$ represents the agreements reached in the previous stages. The tenant and lessor n should agree on a pair $a_n = (p_n, x_n)$, so that the final agreement is $a^n = \left((a_i)_{i=1}^{n-1}, a_n\right) \in A$. Hence, the tenant and lessor n face the two-person bargaining problem (D^n, d^n) with

$$D^n = \{(u_0(a^n), u_n(a^n)) : p_n \geq 0, x_n \in [0, c_n]\}$$

and

$$d^n = \left(u_0(a^{n,0}), u_n(a^{n,0})\right)$$

where $a^{n,0} = \left((a_i)_{i=1}^{n-1}, a_n^0\right) \in A$ with $a_n^0 = (0, 0)$. It is clear that D^n is nonempty, closed and bounded above, and $d^n \in D^n$.

Let $a_n^* = (p_n^*, x_n^*)$ be a pair that determines a Nash solution in (D^n, d^n). We define $\alpha^{*n} : A^{n-1} \to A$ as $\alpha^{*n}\left((a_i)_{i=1}^{n-1}\right) = \left((a_i)_{i=1}^{n-1}, a_n^*\right)$.

Assume that we have defined $\alpha^{*s+1} : A^s \to A$ for any $s < n$.

Stage s: $(a_i)_{i=1}^{s-1} \in A^{s-1}$ represents the agreements reached at the previous stages. The tenant and lessor s should agree on a pair $a_s = (p_s, x_s)$, so that the final agreement is $a^s = \alpha^{*s+1}\left((a_i)_{i=1}^{s-1}, a_s\right) \in A$. Hence, they face the two-person bargaining problem (D^s, d^s) with

$$D^s = \{(u_0(a^s), u_s(a^s)) : p_s \geq 0, x_s \in [0, c_s]\}$$

and

$$d^s = \left(u_0(a^{s,0}), u_s(a^{s,0})\right)$$

where $a^{s,0} = \alpha^{*s+1}\left((a_i)_{i=1}^{s-1}, a_s^0\right) \in A$ with $a_s^0 = (0,0)$.

Let $a_s^* = (p_s^*, x_s^*)$ be a pair that determines a Nash solution in (D^s, d^s). We define $\alpha^{*s} : A^{s-1} \to A$ as

$$\alpha^{*s}\left((a_i)_{i=1}^{s-1}\right) = \alpha^{*s+1}\left((a_i)_{i=1}^{s-1}, a_s^*\right).$$

The final agreement is given by $\alpha^{*1}(\emptyset) \in A$.

Protocol with fixed price

The negotiation process of the non-cooperative protocol with fixed price is analogous to the process with variable price. The only difference is that the final price is updated after the last stage, at the highest price reached upon, i.e. $p^{\max} = \max_{i \in N}\{p_i\}$ for all lessors.

We now describe the corresponding two-person bargaining problem in each stage. We proceed backwards from Stage n to Stage 1.

Stage n: $(a_i)_{i=1}^{n-1} \in A^{n-1}$ represents the agreements reached in the previous stages. The tenant and lessor n should agree on a pair $a_n = (p_n, x_n)$. For such a_n, we define $\hat{a}_i^n = (\hat{p}_i^n, \hat{x}_i^n)$ with $\hat{p}_i^n = p^{max}$ and $\hat{x}_i^n = x_i$ for all $i \in N$. We define the updated agreement of $(a_i)_{i=1}^n$ as $\hat{a}^n = (\hat{a}_i^n)_{i=1}^n \in \hat{A}$. Hence, the tenant and lessor n face the two-person bargaining problem $(\hat{D}^n, \hat{d}^n)$ with

$$\hat{D}^n = \{(u_0(\hat{a}^n), u_n(\hat{a}^n)) : p_n \geq 0, x_n \in [0, c_n]\}$$

and

$$\hat{d}^n = \left(u_0(\hat{a}^{n,0}), u_n(\hat{a}^{n,0})\right)$$

where $\hat{a}^{n,0} \in \hat{A}$ is the updated agreement of $\left((a_i)_{i=1}^{n-1}, a_n^0\right)$ with $a_n^0 = (0,0)$.

Let $a_n^* = (p_n^*, x_n^*)$ be a pair that determines a Nash solution in $(\hat{D}^n, \hat{d}^n)$. We define $\hat{\alpha}^{*n} : A^{n-1} \to \hat{A}$ as $\hat{\alpha}^{*n}\left((a_i)_{i=1}^{n-1}\right) = \hat{a}^{*n}$ where $\hat{a}^{*n}$ is the updated agreement of $\left((a_i^n)_{i=1}^{n-1}, a_n^*\right)$.

Assume that we have defined $\hat{\alpha}^{*s+1} : A^s \to \hat{A}$ for $s < n$.

Stage s: $(a_i)_{i=1}^{s-1} \in A^{s-1}$ represents the agreements reached in the previous stages. The tenant and lessor s should agree on a pair $a_s = (p_s, x_s)$. The updated agreement of $(a_i)_{i=1}^{s}$ is $\hat{a}^s = \hat{\alpha}^{*s+1}\left((a_i)_{i=1}^{s-1}, a_s\right) \in \hat{A}$. Hence, the tenant and lessor s face the two-person bargaining problem $(\hat{D}^s, \hat{d}^s)$ with

$$\hat{D}^s = \{(u_0(\hat{a}^s), u_s(\hat{a}^s)) : p_s \geq 0, x_s \in [0, c_s]\}$$

and

$$\hat{d}^s = \left(u_0(\hat{a}^{s,0}), u_s(\hat{a}^{s,0})\right)$$

where $\hat{a}^{s,0} \in \hat{A}$ is the updated agreement of $\left((a_i)_{i=1}^{s-1}, a_s^0\right)$ with $a_s^0 = (0, 0)$.

Let $a_s^* = (p_s^*, x_s^*)$ be a pair that determines a Nash solution in $(\hat{D}^s, \hat{d}^s)$. We define $\hat{\alpha}^{*s} : A^{s-1} \to \hat{A}$ as

$$\hat{\alpha}^{*s}\left((a_i)_{i=1}^{s-1}\right) = \hat{\alpha}^{*s+1}\left((a_i)_{i=1}^{s-1}, a_s^*\right).$$

The final agreement is given by $\hat{\alpha}^{*1}(\emptyset) \in \hat{A}$.

5 The unanimity case

Let (N_0, c, r, K, E) be a land rental problem. We consider that the tenant needs all available land, i.e. $E = c(N)$, which implies that all lessors are necessary. In this context, an efficient agreement implies $x_i = c_i$ for all $i \in N$.

Theorem 5.1 presents a result in case we consider a variable price.

Theorem 5.1 *Given a land rental problem (N_0, c, r, K, E) with $E = c(N)$, the final payoff allocation according to the protocol with variable price is:*

$$\left(\frac{K - rE}{2^n}, \frac{K - rE}{2}, \frac{K - rE}{2^2}, \dots, \frac{K - rE}{2^n}\right).$$

Proof. We prove the following (stronger) result:

> Given $(a_i)_{i=1}^{s-1} \in A^{s-1}$ where $a_i = (p_i, x_i)$ for all $i < s$ in stage s, and $\beta^s = K - \sum_{i<s} p_i c_i - r \sum_{i \geq s} c_i$, the final payoff allocation is
>
> $$\left(\frac{\beta^s}{2^{n-s+1}}, (p_1 - r)c_1, \dots, (p_{s-1} - r)c_{s-1}, \frac{\beta^s}{2}, \frac{\beta^s}{2^2}, \dots, \frac{\beta^s}{2^{n-s+1}}\right)$$

if $\beta^s > 0$ and $x_i = c_i$ for all $i < s$, and $(0, \ldots, 0)$ if $\beta^s < 0$ or $x_i < c_i$ for some $i < s$.

We proceed by backward induction on s.

Assume $s = n$. Let $(a_i)_{i=1}^{n-1} \in A^{n-1}$ with $a_i = (x_i, p_i)$ for all $i < n$. In case $\sum_{i<n} x_i < E - c_n$, then there is not enough land left and the final payoff is zero for everyone. Since $E = c(N)$, $\sum_{i<n} x_i < E - c_n$ implies $x_i < c_i$ for some $i < n$. For the same reason, $\sum_{i<n} x_i \geq E - c_n$ implies $x_i = c_i$ for all $i < n$. Hence, $\sum_{i<n} x_i \geq E - c_n$ implies $\sum_{i<n} x_i = E - c_n$.

If $\beta^n < 0$, then the prices are too high and agreement is not possible, so the final payoff is zero for everyone.

Assume now $\beta^n > 0$ and $x_i = c_i$ for all $i < n$. Then, the tenant and lessor n face the bargaining problem (D^n, d^n) with $d^n = (0, 0)$.

An efficient agreement implies $x_n = c_n$. So, the Pareto frontier of D^n is:

$$\partial D^n = \left\{ \left(u_0\left(a^n\right), u_n\left(a^n\right)\right) \in \mathbb{R}^2 : p_n \geq 0, x_n = c_n \right\}.$$

The Nash solution is obtained by the maximization problem

$$\begin{aligned}\max\left\{u_0 u_n : (u_0, u_n) \in \partial D^n\right\} &= \max\left\{\left(K - \sum_{i\in N} p_i c_i\right)(p_n - r)c_n : p_n \geq 0\right\} \\ &= \max\left\{\left(\beta^n + r c_n - p_n c_n\right)(p_n - r)c_n : p_n \geq 0\right\}\end{aligned}$$

where the unique maximum is reached at $p_n^* = \frac{\beta^n + 2rc_n}{2c_n}$. Given this, it is straightforward to check that the final payoff allocation is:

$$\left(\frac{\beta^n}{2}, (p_1 - r)c_1, \ldots, (p_{n-1} - r)c_{n-1}, \frac{\beta^n}{2}\right).$$

So, the hypothesis is satisfied for stage $s = n$.

We now consider stage s.

Assume that the result is true in stage $s+1$ for $s < n$. Let $(a_i)_{i=1}^{s-1} \in A^{s-1}$ with $a_i = (x_i, p_i)$ for all $i < s$. By analogous reasoning as in stage n, we deduce that in case $\sum_{i<s} x_i < E - \sum_{i\geq s} c_i$, there is not enough land left and the final payoff is zero for everyone, and $\sum_{i<s} x_i \geq E - \sum_{i\geq s} c_i$ implies $\sum_{i<s} x_i = E - \sum_{i\geq s} c_i$.

If $\beta^s < 0$, then the prices are too high and agreement is not possible, so the final payoff is zero for everyone.

Assume now $\beta^s > 0$ and $x_i = c_i$ for all $i < s$. The tenant and lessor s face the bargaining problem (D^s, d^s) with $d^s = (0, 0)$.

An efficient agreement implies $x_s = c_s$. So, the Pareto frontier of D^s is:

$$\partial D^s = \left\{ \left(u_0\left(a^s\right), u_s\left(a^s\right)\right) \in \mathbb{R}^2 : p_s \geq 0, x_s = c_s \right\}.$$

The Nash solution is obtained by the maximization problem, which is given as follows. By induction hypothesis, given that the payoff for the tenant in stage $s+1$ is $\frac{\beta^{s+1}}{2^{n-s}}$, $\max\{u_0 u_s : (u_0, u_s) \in \partial D^s\}$ is equal to

$$\max \left\{ \frac{\beta^{s+1}}{2^{n-s}}(p_s - r)c_s : p_s \geq 0 \right\}$$

where the unique maximum is reached at $p_s^* = \frac{\beta^s + 2rc_s}{2c_s}$. Given this, the final payoff allocation is:

$$\left(\frac{\beta^s}{2^{n-s+1}}, (p_1 - r)c_1, \ldots, (p_{s-1} - r)c_{s-1}, \frac{\beta^s}{2}, \frac{\beta^s}{2^2}, \ldots, \frac{\beta^s}{2^{n-s+1}} \right).$$

■

Theorem 5.2 presents a result in case we consider a fixed price.

Theorem 5.2 *Given a land rental problem (N_0, c, r, K, E) with $E = c(N)$, the final payoff allocation according to the protocol with fixed price is:*

$$\left(\frac{K - rE}{2}, \frac{K - rE}{2E}c_1, \ldots, \frac{K - rE}{2E}c_n \right).$$

Proof. We define $\pi^* : \mathbb{R}_+ \to \mathbb{R}_+$ as $\pi^*(p) = \max\left\{p, \frac{K+rE}{2E}\right\}$. We prove the following (stronger) result:

Given $(a_i)_{i=1}^{s-1} \in A^{s-1}$ with $a_i = (p_i, x_i)$ for all $i < s$ at stage s, and $p'_{s-1} = \max_{i<s}\{p_i\}$ (for notational convenience, we assume $p'_0 = 0$ when $s = 1$), the final price is $\pi^*\left(p'_{s-1}\right)$ and the final payoff allocation is:

$$\left(\frac{K - rE}{2}, \frac{K - rE}{2E}c_1, \ldots, \frac{K - rE}{2E}c_n \right) \tag{4}$$

if $p'_{s-1} < \frac{K+rE}{2E} < \frac{K}{E}$ and $x_i = c_i$ for all $i < s$,

$$\left(K - p'_{s-1}E, \left(p'_{s-1} - r\right) c_1, \ldots, \left(p'_{s-1} - r\right) c_n\right) \tag{5}$$

if $\frac{K+rE}{2E} \leq p'_{s-1} \leq \frac{K}{E}$ and $x_i = c_i$ for all $i < s$, and

$$(0, \ldots, 0) \tag{6}$$

if $p'_{s-1} > \frac{K}{E}$ or $x_i < c_i$ for some $i < s$.

For any stage $s \in N$, in case $p'_{s-1} > \frac{K}{E}$ or $x_i < c_i$ for any $i < s$, there is no possible agreement, so the final payoff is zero for everyone, as stated in (6). Hence, from now on, we assume $p'_{s-1} \leq \frac{K}{E}$ and $x_i = c_i$ for all $i < s$.

We proceed by backward induction on s.

Assume $s = n$. Let $(a_i)_{i=1}^{n-1} \in A^{n-1}$ with $a_i = (p_i, x_i)$ for all $i < n$. The tenant and lessor n face the bargaining problem $(\hat{D}^n, \hat{d}^n)$ with $\hat{d}^n = (0, 0)$. An efficient agreement is only possible when $x_n = c_n$ and $p_n \leq \frac{K}{E}$, so the Pareto frontier of $\hat{D}^n$ is:

$$\begin{aligned}
\partial\hat{D}^n &= \left\{\left(u_0\left(\hat{a}^n\right), u_0\left(\hat{a}^n\right)\right) \in \mathbb{R}^2 : p_n \leq \frac{K}{E}, x_n = c_n\right\} \\
&= \left\{(K - p'_n E, (p'_n - r)x_n) : p_n \leq \frac{K}{E}, x_n = c_n\right\} \\
&= \left\{(K - p_n E, (p_n - r)c_n) : p'_{n-1} \leq p_n \leq \frac{K}{E}\right\}.
\end{aligned}$$

The Nash solution is obtained by the maximization problem:

$$\max\left\{u_0 u_n : (u_0, u_n) \in \partial\hat{D}^n\right\} = \max\left\{(K - p_n E)(p_n - r)c_n : p'_{n-1} \leq p_n \leq \frac{K}{E}\right\}.$$

The product $u_0 u_n$ determines a concave parabola whose vertex is at $\frac{K+rE}{2E}$.

We have the following three cases:

First Case: If $p'_{n-1} < \frac{K+rE}{2E} < \frac{K}{E}$, the maximum is reached at $p^*_n = \frac{K+rE}{2E}$. By definition, $\pi^*(p'_{n-1}) = \max\{p'_{n-1}, \frac{K+rE}{2E}\} = \frac{K+rE}{2E}$. Then, the final price is $\pi^*(p'_{n-1}) = \frac{K+rE}{2E}$. From this, it is straightforward to check that the final payoff allocation is given as in (4).

Second Case: If $\frac{K+rE}{2E} \leq p'_{n-1} \leq \frac{K}{E}$, the maximum is reached at $p_n^* = p'_{n-1}$. The final price is $\pi^*(p'_{n-1}) = p'_{n-1}$. From this, it is straightforward to check that the final payoff allocation is given as in (5).

Third Case: If $\frac{K+rE}{2E} \geq \frac{K}{E}$, we deduce that $rE \geq K$ which is a contradiction because of the condition $rE < K$. Therefore, this case is not possible.

We now consider stage s.

Assume that the result is true in stage $s+1$ for $s < n$. Let $(a_i)_{i=1}^{s-1} \in A^{s-1}$ with $a_i = (p_i, x_i)$ for all $i < s$. The tenant and lessor s face the bargaining problem $(\hat{D}^s, \hat{d}^s)$ with $\hat{d}^s = (0,0)$. An efficient agreement is only possible when $x_s = c_s$ and $p_s \leq \frac{K}{E}$. By induction hypothesis, the price is $\pi^*(p'_s)$. So the Pareto frontier of $\hat{D}^s$ is:

$$\begin{aligned}\partial\hat{D}^s &= \left\{(u_0(\hat{a}^s), u_s(\hat{a}^s)) \in \mathbb{R}^2 : p_s \leq \frac{K}{E}, x_s = c_s\right\}\\ &= \left\{(K - \pi^*(p'_s)E, (\pi^*(p'_s) - r)\, x_s) : p_s \leq \frac{K}{E}, x_s = c_s\right\}\\ &= \left\{(K - \pi^*(p_s)E, (\pi^*(p_s) - r)\, c_s) : p'_{s-1} \leq p_s \leq \frac{K}{E}\right\}.\end{aligned}$$

The Nash solution is obtained by the maximization problem $\max\{u_0 u_s : (u_0, u_s) \in \partial\hat{D}^s\}$, or equivalently,

$$\max\left\{(K - \pi^*(p_s)E)(\pi^*(p_s) - r)c_s : p'_{s-1} \leq p_s \leq \frac{K}{E}\right\}.$$

Since $\pi^*(p_s)$ is constant for $p_s \leq \frac{K+rE}{2E}$, the maximization problem can be rewritten as

$$\max\left\{(K - p_s E)(p_s - r)c_s : \pi^*\left(p'_{s-1}\right) \leq p_s \leq \frac{K}{E}\right\}.$$

The product $(K - p_s E)(p_s - r)c_s$ determines a concave parabola whose vertex is at $\frac{K+rE}{2E}$.

Now, we have the following three cases:

First Case: If $p'_{s-1} < \frac{K+rE}{2E} < \frac{K}{E}$, since $\pi^*(p'_{s-1}) = \max\{p'_{s-1}, \frac{K+rE}{2E}\}$, we deduce $\pi^*(p'_{s-1}) = \frac{K+rE}{2E}$. Hence, $\pi^*(p'_{s-1}) \leq \frac{K+rE}{2E} \leq \frac{K}{E}$, and thus the maximum is reached at $p_s^* = \frac{K+rE}{2E}$. Then, the final price is $\frac{K+rE}{2E} = \pi^*(p'_{s-1})$.

From this, it is straightforward to check that the final payoff allocation is given as in (4).

Second Case: If $\frac{K+rE}{2E} \leq p'_{s-1} \leq \frac{K}{E}$, since $\pi^*(p'_{s-1}) = \max\{p'_{s-1}, \frac{K+rE}{2E}\}$, we deduce $\pi^*(p'_{s-1}) = p'_{s-1}$. Hence, $\frac{K+rE}{2E} \leq \pi^*(p'_{s-1}) \leq \frac{K}{E}$, and thus the maximum is reached at $p^*_s = \pi^*(p'_{s-1})$. Then, the final price is $p'_{s-1} = \pi^*(p'_{s-1})$. From this, it is straightforward to check that the final payoff allocation is given as in (5).

Third Case: If $\frac{K+rE}{2E} \geq \frac{K}{E}$, we deduce that $rE \geq K$ which is a contradiction, because of the hypothesis condition $rE < K$. Therefore, this case is not possible. ■

We can now give a first answer to the question of what is the best protocol for the tenant. When there are more than one lessor and all the land is necessary ($E = c(N)$), then the tenant strictly prefers a protocol with fixed price, as stated in in Corollary 5.1.

Corollary 5.1 *Given a land rental problem* (N_0, c, r, K, E) *with* $n > 1$ *and* $E = c(N)$*, the utility of the tenant with fixed price is higher than with variable price.*

Proof. By Theorem 5.2 according to the protocol with fixed price the utility of the tenant is $\frac{K-rE}{2}$, and by Theorem 5.1 according to the protocol with variable price the utility of the tenant is $\frac{K-rE}{2^n}$. Since $K > rE$ and $n > 1$, we deduce $\frac{K-rE}{2} > \frac{K-rE}{2^n}$. ■

We can also give an answer to the question of whether it is profitable for the lessors to merge. Merging of the lessors is unprofitable under protocol with variable price. However, merging is irrelevant under protocol with fixed price, as stated in Corollary 5.2.

Corollary 5.2 *Given a land rental problem* (N_0, c, r, K, E) *with* $n > 1$ *and* $E = c(N)$*, under the protocol with variable price, the utility of the lessors when they merge is less than their aggregate utility. Under the protocol with fixed price, merging of lessors is irrelevant.*

Proof. Under the protocol with variable price: by Theorem 5.1, the sum of the utilities of the lessors can be computed as $K-rE-u_0 = K-rE-\frac{K-rE}{2^n} = \frac{(2^n-1)(K-rE)}{2^n}$. By Proposition 4.1, the utility of the unique lessor is $\frac{K-rE}{2}$. Since $n > 1$, it is straightforward to check that $\frac{K-rE}{2} < \frac{(2^n-1)(K-rE)}{2^n}$.

Under the protocol with fixed price: by Theorem 5.2, the sum of utilities of the lessors is $\frac{K-rE}{2}$, which coincide with the utility when they merge (by Proposition 4.1). Hence, merging of the lessors is irrelevant. ■

6 The general two-lessors case

In Section 5, we consider that all available land is necessary. In this section, we study the general case, focusing on the case of a land rental problem with two lessors (hereinafter called 2-*lessor land renting problem*). In this case, we assume w.l.o.g. $N = \{1, 2\}$. In general, both protocols (variable price and fixed price) yield the same final payoff allocation when at least one of the lessors $i \in N$ has enough land to cover the needs of the the tenant (i.e. $E \leq c_i$). However, results critically change when both lessors are necessary.

Theorem 6.1 characterizes the final payoff allocation in the protocol with variable price for 2-lessor land rental problems.

Theorem 6.1 *In a* 2*-lessor land renting problem* (N_0, c, r, K, E)*, the final payoff allocation of the protocol with variable price is given by:*

$$u^* = \begin{cases} \left(3\frac{K-rE}{4}, \frac{K-rE}{4}, 0\right) & \text{if } E \leq c_1, c_2 \\ \left(\frac{K-rE}{2}, \frac{k-rE}{2}, 0\right) & \text{if } c_2 < E \leq c_1 \\ \left(\frac{K-rE}{2}, 0, \frac{K-rE}{2}\right) & \text{if } c_1 < E \leq c_2 \\ \left(\frac{K-rE}{4}, \frac{K-rE}{2}, \frac{K-rE}{4}\right) & \text{if } E > c_1, c_2. \end{cases}$$

Proof. Assume first $E \leq c_1, c_2$. The tenant, in case she does not reach an agreement with lessor 1 in stage 1, will face a bargaining problem (D^2, d^2) in stage 2 with lessor 2 with $d^2 = (0, 0)$ and $D^2 = \{(u_0, u_2) \in \mathbb{R}^2 : u_0 + u_2 \leq K - rE\}$. The Nash solution is $\left(\frac{K-rE}{2}, \frac{K-rE}{2}\right)$. Hence, the expected final payoff for the tenant, provided there is disagreement in stage 1, is $\frac{K-rE}{2}$. From this, in stage 1, the tenant and lessor 1 face a bargaining problem (D^1, d^1) with $d^1 = \left(\frac{K-rE}{2}, 0\right)$ and

$$D^1 = \left\{(K - p_1 x_1, (p_1 - r)\, x_1) : (p_1, x_1) \in \Delta^0\right\}$$

where

$$\Delta^0 = \{(p_1, x_1) \in [r, \infty) \times [E, c_1] : p_1 x_1 \leq K\}.$$

Now, any agreement $(p_1, x_1) \in \Delta^0$ with $x_1 > E$ is Pareto dominated by $(p_1', x_1') = \left(\frac{p_1 x_1}{E}, E\right)$. Hence, we can rewrite D^1 as

$$D^1 = \left\{(K - p_1 E, (p_1 - r)E) : p_1 \in \left[r, \frac{K}{E}\right]\right\}.$$

The Nash solution for (D^1, d^1) maximizes $\left(K - p_1 E - \frac{K-rE}{2}\right)(p_1 - r)E$. This maximum is at $p_1^* = \frac{3r}{4} + \frac{K}{4E}$, so that the final payoff is $\left(3\frac{K-rE}{4}, \frac{K-rE}{4}, 0\right)$.

Assume now $c_2 < E \leq c_1$ or $c_2 < E \leq c_1$. The smallest lessor is a dummy, so the tenant and the other lessor will equally share $K - rE$, and the dummy receives zero.

Finally, assume $E > c_1, c_2$. Assume that the tenant and lessor 1 agree on some $(p_1, x_1) \in [0, \infty) \times [0, c_1]$ in stage 1. In order for agrement to be possible in stage 2, suppose $x_1 \geq E - c_2$ and $(p_1 - r)\, x_1 \leq K - rE$. Now, in stage 2, any individually rational choice (p_2, x_2) with $x_2 > E - x_1$ is Pareto dominated by (p_2', x_2') with $p_2' = \frac{x_2}{E - x_1} p_2$ and $x_2' = E - x_1$. Hence, the tenant and lessor 2 face a bargaining problem (D^2, d^2) with $d^2 = (0, 0)$ and

$$D^2 = \left\{(K - p_1 x_1 - (E - x_1)\, p_2, (p_2 - r)\,(E - x_1)) : p_2 \in \Delta^2\right\}$$

where $\Delta^2 = \left[r, \frac{K - p_1 x_1}{E - x_1}\right]$. Taking $\alpha = (E - x_1)\, p_2$, we can rewrite D^2 as

$$D^2 \quad = \{(K - p_1 x_1 - \alpha, \alpha - (E - x_1)\, r) : \alpha \in [(E - x_1)\, r, K - p_1 x_1]\}$$

equivalently (see Figure 1 - Right),

$$D^2 \quad = \{(u_0, u_2) \in \mathbb{R}^2 : u_0 + u_2 \leq K - rE - (p_1 - r)x_1\}.$$

By symmetry and efficiency, the Nash solution gives both players the same utility $\frac{K-rE-(p_1-r)x_1}{2}$ and it is uniquely determined with $x_2 = E - x_1$ and $p_2 = \frac{K-p_1x_1}{2(E-x_1)} + \frac{r}{2}$.

Given this, the bargaining problem in stage 1 is (D^1, d^1) given by $d^1 = (0,0)$ and

$$D^1 = \left\{ \left(\frac{K - rE - (p_1 - r)x_1}{2}, (p_1 - r)\, x_1 \right) : (p_1, x_1) \in \Delta^1 \right\}$$

where

$$\Delta^1 = \{(p_1, x_2) \in [0, \infty) \times [E - c_2, c_1] : (p_1 - r)\, x_1 \leq K - rE\}.$$

Taking $\beta = (p_1 - r)\, x_1$, we can rewrite D^1 as

$$D^1 \ = \left\{ \left(\tfrac{K-rE-\beta}{2}, \beta \right) : \beta \in [-rc_1, K - rE] \right\}$$

equivalently (Figure 2),

$$D^1 \ = \{(u_0, u_2) : 2u_0 + u_2 \leq K - rE\}.$$

The Nash solution maximizes $(K - rE - \beta)\, \beta$. This maximium is at $\beta^o = \frac{K-rE}{2}$. Hence the final payoff allocation is

$$\left(\frac{K - rE - \beta^o}{2}, \beta^o, \frac{K - rE - \beta^o}{2} \right) = \left(\frac{K - rE}{4}, \frac{K - rE}{2}, \frac{K - rE}{4} \right).$$

■

Theorem 6.2 shows the final payoff allocation in the protocol with fixed price for 2-lessor land rental problem.

Theorem 6.2 *In a 2-lessor land renting problem (N_0, c, r, K, E) the final payoff allocation of the bargaining protocol with fixed price is given by:*

$$u^* = \begin{cases} \left(3\frac{K-rE}{4}, \frac{K-rE}{4}, 0\right) & \text{if } E \leq c_1, c_2 \\ \left(\frac{K-rE}{2}, \frac{k-rE}{2}, 0\right) & \text{if } c_2 < E \leq c_1 \\ \left(\frac{K-rE}{2}, 0, \frac{K-rE}{2}\right) & \text{if } c_1 < E \leq c_2, \end{cases}$$

and when $c_1, c_2 < E$ and K is large enough ($K > \frac{E+c_2}{E-c_2} rE$), the final payoff allocation is given by:

$$\left(\frac{K-(c_1+c_2)\,r}{2}, \frac{c_1 K}{2\,(c_1+c_2)} - \frac{c_1 r}{2}, \frac{c_2 K}{2\,(c_1+c_2)} - \frac{c_2 r}{2}\right).$$

Proof. We focus on the case $c_1, c_2 < E$. The proof for the other cases is analogous to the proof of the Theorem 6.1.

Assume that the tenant and lessor 1 agree on some $(p_1, x_1) \in [0, \infty) \times [0, c_1]$ in stage 1. In order for agrement to be possible in stage 2, we also assume $x_1 \geq E - c_2$ and $p_1 \leq \frac{K}{E}$. Given this, the tenant and lessor 2 face the bargaining problem $\left(\hat{D}^2, \hat{d}^2\right)$ with $\hat{d}^2 = (0, 0)$ and

$$\hat{D}^2 = \left\{(K - (x_1 + x_2)\,p_2, (p_2 - r)\,x_2) : (p_2, x_2) \in \Delta^2\right\}$$

where

$$\Delta^2 = [p_1, \infty) \times [E - x_1, c_2]\,.$$

The individually rational Pareto frontier of $\hat{D}^2$ are the points $(u_0, g_2(u_0))$, where $u_0 \in [0, K - p_1 E]$ and $g_2(u_0)$ is a function defined as follows: $g_2(u_0)$ the maximum of $(p_2 - r)\,x_2$ subject to $p_2 \geq p_1$, $p_2 \geq r$, $E - x_1 \leq x_2 \leq c_2$ and $K - (x_1 + x_2)\,p_2 = u_0$.

Assume first $r = 0$, this means that $g_2(u_0)$ reaches its maximum at $x_2 = c_2$. Thus, we have that $g_2(u_0) = \frac{K-u_0}{x_1+c_2} c_2$.

Assume now $r > 0$. The maximization is equivalent to maximizing the function

$$f(x_2) := \left(\frac{K - u_0}{x_1 + x_2} - r\right) x_2$$

on the interval $x_2 \in \left[E - x_1, \min\left\{c_2, \frac{K-u_0}{\max\{r,p_1\}} - x_1\right\}\right]$.

Since $\max\{r, p_1\} > 0$, and for any admissible u_0, the unique global non-negative maximum of f is at

$$x_2^o := \sqrt{\frac{x_1}{r}\,(K - u_0)} - x_1$$

associated to the price $p_2^o = \sqrt{\frac{(K-u_0)r}{x_1}}$.

We distinguish three cases:

Case 1 If $x_2^o < E - x_1$ or, equivalently, $u_0 > K - \frac{rE^2}{x_1}$, the unique maximum is at $x_2 = E - x_1$ (with $p_2 = \frac{K-u_0}{E}$), and it gives

$$g_2(u_0) = f(E - x_1) = \left(\frac{K - u_0}{E} - r\right)(E - x_1) = \frac{E - x_1}{E}(K - rE - u_0)$$

which implies that, for $u_0 > K - \frac{rE^2}{x_1}$, the frontier of $\hat{D}^2$ is a line with slope $-\frac{E-x_1}{E}$.

Case 2 If $x_2^o \geq \min\left\{c_2, \frac{K-u_0}{\max\{r,p_1\}} - x_1\right\}$, we have two subcases:

Case 2a If $c_2 \leq \frac{K-u_0}{\max\{r,p_1\}} - x_1$ and $x_2^o \geq c_2$, or, equivalently, $u_0 \leq K - (x_1 + c_2)\max\left\{p_1, \frac{x_1+c_2}{x_1}r\right\}$, the maximum is at $x_2 = c_2$ (with $p_2 = \frac{K-u_0}{x_1+c_2}$), and it gives

$$g_2(u_0) = f(c_2) = \frac{c_2}{x_1 + c_2}(K - (x_1 + c_2)r - u_0)$$

which implies that, for $u_0 \leq K - (x_1 + c_2)\max\left\{p_1, \frac{x_1+c_2}{x_1}r\right\}$, the frontier of $\hat{D}^2$ is a line with slope $-\frac{c_2}{x_1+c_2}$.

Case 2b If $c_2 \geq \frac{K-u_0}{\max\{r,p_1\}} - x_1$ and $x_2^o \geq \frac{K-u_0}{\max\{r,p_1\}} - x_1$, or, equivalently, $u_0 \geq K - (x_1 + c_2)\min\left\{p_1, \frac{x_1}{(x_1+c_2)r}\max\{r,p_1\}^2\right\}$, the maximum is at $x_2 = \frac{K-u_0}{\max\{r,p_1\}} - x_1$ (with $p_2 = \max\{r,p_1\}$), and it gives

$$\begin{aligned} g_2(u_0) &= f\left(\frac{K - u_0}{\max\{r,p_1\}} - x_1\right) \\ &= \frac{\max\{r,p_1\} - r}{\max\{r,p_1\}}(K - \max\{r,p_1\}x_1 - u_0) \end{aligned}$$

which implies that, for

$$u_0 \geq K - (x_1 + c_2)\min\left\{\max\{r,p_1\}, \frac{x_1}{(x_1 + c_2)r}\max\{r,p_1\}^2\right\},$$

the frontier of $\hat{D}^2$ is a line with slope $-\frac{\max\{r,p_1\}-r}{\max\{r,p_1\}}$.

Case 3 In any other case, the maximum is at $x_2 = x_2^o$ (with $p_2 = p_2^o$), and it is

$$g_2(u_0) = f\left(\sqrt{\frac{x_1}{r}(K - u_0)} - x_1\right) = \left(\sqrt{\frac{x_1}{r}(K - u_0)} - x_1\right)^2 \frac{r}{x_1}$$

which implies that, in some cases, the frontier of D^2 is a convex function.

From these cases allow us to define $g_2(u_0)$, for any $u_0 \in [0, K - \max\{p_1, r\} E]$, as follows:

$$g_2(u_0) = \begin{cases} \frac{E-x_1}{E}(K - rE - u_0) & \text{if } u_0 > K - \frac{rE^2}{x_1} \\ \frac{c_2}{x_1+c_2}(K - (x_1 + c_2) r - u_0) & \text{if } u_0 \le K - (x_1 + c_2) \max\left\{\max\{r, p_1\}, \frac{x_1+c_2}{x_1} r\right\} \\ \frac{\max\{r,p_1\}-r}{\max\{r,p_1\}}(K - \max\{r, p_1\}x_1 - u_0) & \text{if } u_0 \ge K - \frac{\max\{r,p_1\}x_1}{r} \min\left\{\frac{x_1+c_2}{x_1} r, \max\{r, p_1\}\right\} \\ \left(\sqrt{K - u_0} - \sqrt{rx_1}\right)^2 & \text{otherwise} \end{cases}$$

for any $u_0 \in [0, K - \max\{p_1, r\} E]$. Notice that Case 1 only applies when $K - \frac{rE^2}{x_1} < K - p_1 E$ or, equivalently, $p_1 x_1 < rE$. In that case, and since $E \le x_1 + c_2$, we have $\max\{r, p_1\} < \frac{x_1+c_2}{x_1} r$ and hence case 2b reduces to $u_0 \ge K - \frac{\max\{r,p_1\}^2 x_1}{r}$ which implies $u_0 \ge K - \max\{r, p_1\}E$, which is impossible. Hence, case 1 and case 2b cannot happen simultaneously, and so g_2 is well-defined on the interval $[0, K - \max\{p_1, r\} E]$.

Since $E \le x_1 + c_2$, we have $\frac{rE}{x_1} \le \frac{x_1+c_2}{x_1} r$. There are three possibilities depending on $p_1 x_1$:

Small $p_1 x_1$ If $p_1 x_1 < rE$, then case 2b vanishes:

$$g_2(u_0) = \begin{cases} \frac{E-x_1}{E}(K - rE - u_0) & \text{if } u_0 \ge K - \frac{rE^2}{x_1} \\ \frac{c_2}{x_1+c_2}(K - (x_1 + c_2) r - u_0) & \text{if } u_0 \le K - \frac{(x_1+c_2)^2}{x_1} r \\ \left(\sqrt{K - u_0} - \sqrt{rx_1}\right)^2 & \text{if } u_0 \in \left[K - \frac{(x_1+c_2)^2}{x_1} r, K - \frac{rE^2}{x_1}\right]. \end{cases}$$

In this case, g_2 is convex. There are three candidates for the Nash solution:

Case 1 $u_0^{s1} = \frac{K-rE}{2}$ only if $u_0^{s1} \ge K - \frac{rE^2}{x_1}$, but this case is not possible when K is large enough ($K > K^{s1} := \frac{E+c_2}{E-c_2} rE$);

Case 2a $u_0^{s2} = \frac{K-(x_1+c_2)r}{2}$ only if $u_0^{s2} \leq K - \frac{(x_1+c_2)^2}{x_1}r$, which always holds for K large enough ($K > K^{s2} := \frac{E+c_2}{E-c_2}(c_1+c_2)r$); and

Case 3 $u_0^{s3} = \frac{K-rx_1}{8}\left(1-\sqrt{1-\frac{16K}{K-rx_1}}\right)$ only if $K - rx_1 \geq 16K$, which is not possible.

Hence, for $K = \min\{K^{s1}, K^{s2}\} = K^{s1}$ large enough, and for each pair (x_1, p_1) with $x_1p_1 < rE$, the tenant and lessor 2 will agree on some (p_2^*, x_2^*) such that the tenant's final payoff is u_0^{2a} (case 2a). This implies $p_2^* = r + \frac{K}{(x_1+c_2)}$ and $x_2^* = c_2$.

Medium p_1x_1 If $p_1x_1 \in [rE, (x_1+c_2)r]$, then case 1 vanishes:

$$g_2(u_0) = \begin{cases} \frac{c_2}{x_1+c_2}(K-(x_1+c_2)r-u_0) & \text{if } u_0 \leq K - \frac{(x_1+c_2)^2}{x_1}r \\ \frac{p_1-r}{p_1}(K-p_1x_1-u_0) & \text{if } u_0 \geq K - \frac{x_1}{r}p_1^2 \\ \left(\sqrt{K-u_0}-\sqrt{rx_1}\right)^2 & \text{if } u_0 \in \left[K - \frac{(x_1+c_2)^2}{x_1}r, K - \frac{x_1}{r}p_1^2\right]. \end{cases}$$

In this case, g_2 is again convex and there are three candidates for the Nash solution :

Case 2a $u_0^{2a} = \frac{K-(x_1+c_2)r}{2}$ only if $u_0^{2a} \leq K - \frac{(x_1+c_2)^2}{x_1}r$, which always holds for K large enough ($K > K^{s3} = \frac{E+c_2}{E-c_2}(c_1+c_2)r$);

Case 2b $u_0^{2b} = \frac{K-p_1x_1}{2}$ only if $u_0^{2b} \geq K - \frac{x_1}{r}p_1^2$, which does not hold for K large enough ($K > K^{s4} = (c_1+2c_2)\frac{rE}{E-c_2}$); and

Case 3 $u_0^3 = \frac{K-rx_1}{8}\left(1-\sqrt{1-\frac{16K}{K-rx_1}}\right)$ only if $K - rx_1 \geq 16K$, which is not possible.

Hence, for $K = \min\{K^{s3}, K^{s4}\} = K^{s4}$ large enough, and for each pair (p_1, x_1) with $p_1x_1 \in [rE, (x_1+c_2)r]$, the tenant and lessor 2 will agree on some (p_2^*, x_2^*) such that the tenant's final payoff is u_0^{2a} (case 2a). This implies again that $p_2^* = r + \frac{K}{(x_1+c_2)}$ and $x_2^* = c_2$.

Large x_1p_1 If $x_1p_1 \geq (x_1+c_2)r$, then cases 1 and 3 vanish

$$g_2(u_0) = \begin{cases} \frac{c_2}{x_1+c_2}(K-(x_1+c_2)r-u_0) & \text{if } u_0 \leq K-(x_1+c_2)p_1 \\ \frac{p_1-r}{p_1}(K-p_1x_1-u_0) & \text{if } u_0 \geq K-(x_1+c_2)p_1. \end{cases}$$

In this case, g_2 in convex. The (unique) Nash solution if given by:

$$u_0^* = \begin{cases} u_0^{2a} & \text{if } u_0^{2a} \leq K - (x_1 + c_2)\, p_1 \\ u_0^{2b} & \text{if } u_0^{2b} \geq K - (x_1 + c_2)\, p_1 \\ K - (c_1 + c_2)\, p_1 & \text{otherwise} \end{cases}$$

which is equivalent to:

$$u_0^* = \begin{cases} \frac{K - (x_1 + c_2) r}{2} & \text{if } p_1 \leq \frac{K}{2(x_1 + c_2)} + \frac{r}{2} \\ \frac{K - p_1 x_1}{2} & \text{if } p_1 \geq \frac{K}{x_1 + 2c_2} \\ K - (c_1 + c_2)\, p_1 & \text{if } p_1 \in \left[\frac{K}{2(x_1 + c_2)} + \frac{r}{2}, \frac{K}{x_1 + 2c_2}\right]. \end{cases}$$

For K large enough ($K > K^{s5} = (c_1 + c_2)\left(1 + 2\frac{c_2}{E - c_1}\right) r$), case $p_1 \leq \frac{K}{2(x_1+c_2)} + \frac{r}{2}$ is compatible with $x_1 p_1 \geq (x_1 + c_2)\, r$ and hence the three cases are nondegenerate. Hence, for each pair (p_1, x_1) with $p_1 x_1 \geq (x_1 + c_2)\, r$, the tenant and lessor 2 will agree on some (p_2^*, x_2^*) such that the tenant's final payoff is u_0^* given as before.

See Figure 3 for three examples of $(\hat{D}^2, \hat{d}^2)$ for three possible choices of (p_1, x_1).

Assume now we are in stage 1 and K is large enough ($K > K^{s6} = \frac{(c_1+2c_2)(c_1+c_2)r}{E-c_2}$). For each possible agreement (p_1, x_1), the above cases allow us to anticipate the agreement $\left(p_2^{(p_1,x_1)}, x_2^{(p_1,x_1)}\right)$ in stage 2:

$$\left(p_2^{(p_1,x_1)}, x_2^{(p_1,x_1)}\right) = \begin{cases} \left(\frac{K}{2(x_1+c_2)} + \frac{r}{2}, c_2\right) & \text{if } p_1 \leq \frac{K}{2(x_1+c_2)} + \frac{r}{2} \\ \left(p_1, \frac{K}{2p_1} - \frac{x_1}{2}\right) & \text{if } p_1 \geq \frac{K}{x_1+2c_2} \\ (p_1, c_2) & \text{if } p_1 \in \left[\frac{K}{2(x_1+c_2)} + \frac{r}{2}, \frac{K}{x_1+2c_2}\right]. \end{cases} \quad (7)$$

Therefore, we have a bargaining problem $\left(\hat{D}^1, \hat{d}^1\right)$ with $\hat{d}^1 = (0, 0)$ and

$$\hat{D}^1 = \left\{\left(K - \left(x_1 + x_2^{(p_1,x_1)}\right) p_2^{(p_1,x_1)}, \left(p_2^{(p_1,x_1)} - r\right) x_1\right) : (p_1, x_1) \in \hat{\Delta}^1\right\}$$

where $\hat{\Delta}^1 = \left[0, \frac{K}{E}\right] \times [E - c_2, c_1]$.

In particular, given an agreement $(p_1, x_1) \in \hat{\Delta}^1$, the final payoff for the tenant and lessor 1 is given by $(u_0^*, u_1^*) =$

$$\begin{cases} \left(\frac{K-(x_1+c_2)r}{2}, \frac{K-(x_1+c_2)r}{2(x_1+c_2)}\right) & \text{if } p_1 \leq \frac{K}{2(x_1+c_2)} + \frac{r}{2} \\ \left(\frac{K-p_1 x_1}{2}, (p_1 - r)\, x_1\right) & \text{if } p_1 \geq \frac{K}{x_1+2c_2} \\ (K - (x_1 + c_2)\, p_1, (p_1 - r)\, x_1) & \text{if } p_1 \in \left[\frac{K}{2(x_1+c_2)} + \frac{r}{2}, \frac{K}{x_1+2c_2}\right]. \end{cases} \tag{8}$$

The Nash solution is given by a pair (p_1, x_1) that maximizes $u_0^* u_1^*$. Let $u_0 \in [0, K - rE]$ be the utility that the tenant can get. The Pareto frontier of $\hat{D}^1$ is determined by a function $g_1(u_0)$ which gives the maximum that lessor 1 can get when the tenant gets u_0, i.e.

$$\partial \hat{D}^1 = \left\{(u_0, g_1(u_0)) \in \mathbb{R}^2 : u_0 \in [0, K - rE]\right\}$$

with $g_1(u_0)$ the maximum of u_1^* subject to $u_0^* = u_0$. From (8), we have three cases depending on p_1. If $p_1 \leq \frac{K}{2(x_1+c_2)} + \frac{r}{2}$, we maximize $\frac{K-(x_1+c_2)r}{2(x_1+c_2)} x_1$ subject to $\frac{K-(x_1+c_2)r}{2} = u_0$, which is equivalent to maximize $\frac{r u_0 x_1}{K-2u_0}$. Since it is increasing on x_1, we deduce that the optimal x_1 satisfies $p_1 \geq \frac{K}{2(x_1+c_2)} + \frac{r}{2}$. If $p_1 \geq \frac{K}{x_1+2c_2}$, we maximize $(p_1 - r)\, x_1$ subject to $\frac{K-p_1 x_1}{2} = u_0$, which is equivalent to maximize $K - 2u_0 - r x_1$. Since it is decreasing on x_1, we deduce that the optimal x_1 satisfies $p_1 \leq \frac{K}{x_1+2c_2}$.

We then maximize $(p_1 - r)\, x_1$ subject to a $K - (x_1 + c_2)\, p_1 = u_0$, which is equivalent to maximize $\left(\frac{K-u_0}{x_1+c_2} - r\right) x_1$ whose maximum is at

$$x_1^o = \sqrt{\frac{(K - u_0)\, c_2}{r}} - c_2$$

Hence, we have three cases depending on wherever $x_1^o \geq c_1$ (equivalently, $u_0 \leq K - \frac{(c_1+c_2)^2 r}{c_2}$), $x_1^o \in [E - c_2, c_1]$ (equivalently, $u_0 \in \left[K - \frac{(c_1+c_2)^2 r}{c_2}, K - \frac{rE^2}{c_2}\right]$), or $x_1^o \leq E - c_2$ (equivalently, $u_0 \geq K - \frac{rE^2}{c_2}$). The maximum is obtained, respectively, with $(p_1, x_1) = \left(\frac{K-u_0}{c_1+c_2}, c_1\right)$, $(p_1, x_1) = \left(\sqrt{\frac{(K-u_0)r}{c_2}}, x_1^0\right)$, and

$(p_1, x_1) = \left(\frac{K-u_0}{E}, E - c_2\right)$. From this, we have

$$g_1(u_0) = \begin{cases} \frac{c_1}{c_1+c_2}\left(K - (c_1+c_2)\, r - u_0\right) & \text{if } u_0 \leq K - \frac{(c_1+c_2)^2 r}{c_2} \\ \left(\sqrt{K-u_0} - \sqrt{rc_2}\right)^2 & \text{if } u_0 \in \left[K - \frac{(c_1+c_2)^2 r}{c_2}, K - \frac{rE^2}{c_2}\right] \\ \frac{1}{(E-c_2)E}\left(K - rE - u_0\right) & \text{if } u_0 \geq K - \frac{rE^2}{c_2} \end{cases}$$

which determines the bargaining problem in stage 1 (see Figure 4). For K large enough ($K > K^{s7} = \frac{2c_1+c_2}{c_2}\left(c_1+c_2\right) r$), the Nash solution determines $u_0 = \frac{K-(c_1+c_2)r}{2}$ as final payoff for the tenant, with $(p_1, x_1) = \left(\frac{K}{2(c_1+c_2)} + \frac{r}{2}, c_1\right)$. From (7), we deduce that, at stage 2, the final agreement is $(p_2^*, x_2^*) = \left(\frac{K}{2(c_1+c_2)} + \frac{r}{2}, c_2\right)$ and so the final payoff for each lessor $i \in \{1,2\}$ is $u_i = (p_2 - r)x_i = \frac{c_i K}{2(c_1+c_2)} - \frac{c_i r}{2}$. ■

In general this result does not hold for $K < \frac{E+c_2}{E-c_2} rE$ when $c_1, c_2 < E$. In Example 3.1 for protocol with fixed price, we obtain an inefficient payoff allocation because condition $K > \frac{E+c_2}{E-c_2} rE$ is not satisfied.

The tenant prefers to negotiate according to the protocol with fixed price, as stated in Corollary 6.1.

Corollary 6.1 *Let (N_0, c, r, K, E) be a 2-lessor land renting problem with K large enough $\left(K > \frac{E+c_2}{E-c_2} rE\right)$. Then, the tenant is indifferent to negotiate according to protocol with variable price or to protocol with fixed price, unless $E > c_1, c_2$, where the tenant strictly prefers to negotiate according to protocol with fixed price.*

Proof. By Theorem 6.1 and Theorem 6.2, the utility of the tenant is equal in both protocols with variable and fixed price, unless $E > c_1, c_2$.

We now prove that when $E > c_1, c_2$ and $K > \frac{E+c_2}{E-c_2} rE$, the utility of the tenant according to protocol with fixed price ($\frac{K-(c_1+c_2)r}{2}$ by Theorem 6.2) is strictly higher than with the protocol with variable price ($\frac{K-rE}{4}$ by Theorem 6.1), i.e. $\frac{K-(c_1+c_2)r}{2} > \frac{K-rE}{4}$, or, equivalently, $K > (2(c_1+c_2) - E)r$. Since $K > \frac{E+c_2}{E-c_2} rE$, it is enough to check that $\frac{E+c_2}{E-c_2} rE > (2(c_1+c_2) - E)r$. This is equivalent to

$$E^2 > (E - c_2)(c_1 + c_2).$$

Since $E > c_1, c_2 > 0$ and $c_1 + c_2 \geq E$,

$$E^2 > E^2 - c_2^2 = (E - c_2)(E + c_2) > (E - c_2)(c_1 + c_2).$$

■

In this case, we cannot give a definitive answer to the question of whether it is profitable for the lessors to merge, since it would depend on both the protocol and the amount of available land of each lessor, as next Corollary shows.

Corollary 6.2 *Let* (N_0, c, r, K, E) *be a 2-lessor land renting problem. Under the protocol with fixed price with* K *large enough* $\left(K > \frac{E+c_2}{E-c_2} rE\right)$, *the utility of the lessors (weakly) increases when they merge. Under the protocol with variable price, the utility of the lessors also (weakly) increases when they merge, except in case* $E > c_1, c_2$ *where merging is unfavorable for lessors.*

Proof. We prove first the case with variable price. When $c_2 < E \leq c_1$ or $c_1 < E \leq c_2$, by Theorem 6.1, the sum of utilities of the lessors is $\frac{K-rE}{2}$, and by Proposition 4.1, the utility for the unique lessor (when they merge) is $\frac{K-rE}{2}$. Then, the utilities are equal in both cases. When $E \leq c_1, c_2$, by Theorem 6.1, the sum of utilities of the lessors is $\frac{K-rE}{4}$, and by Proposition 4.1, the utility of the unique lessor is $\frac{K-rE}{2}$. Then, merging is profitable for lessors. On the contrary, when $E > c_1, c_2$, by Theorem 6.1, the sum of utilities of the lessors is $\frac{3(K-rE)}{4}$, and by Proposition 4.1, the utility of the unique lessor is $\frac{K-rE}{2}$. Then, merging is unprofitable for lessors.

We prove now the case with fixed price with $K > \frac{E+c_2}{E-c_2} rE$. When $c_2 < E \leq c_1$, $c_1 < E \leq c_2$, or $E \leq c_1, c_2$, the proof is analogous to the protocol with variable price using Theorem 6.2 instead of Theorem 6.1. Now, we focus on the case $E > c_1, c_2$. By Theorem 6.2, the sum of utilities of the lessors is $\frac{K-(c_1+c_2)r}{2}$, and by Proposition 4.1, the utility of the unique lessor is $\frac{K-rE}{2}$. Then, since $E < c_1 + c_2$, it is straightforward to check that $\frac{K-(c_1+c_2)r}{2} < \frac{k-rE}{2}$.

■

7 Concluding remarks

We propose two sequential bargaining protocols between a single tenant and several lessors and study their equilibria. These protocols differ on the price per unit of land (lessor-depended or not). They mimic the observed actual protocol followed by the negotiation process for the setting of a military base in Pontevedra, Spain, and cover other potential situations such as negotiations between the mining industry and indigenous communities.

Our model is simple in the sense that it makes very strong simplifications, such as a common reservation price for each lessor and the perfect substitutability of land. Yet quite relevant results are obtained.

First, and quite counter-intuitively, a lessor-independent price is much favorable for the tenant than when she has freedom to agree on a different price for each lessor. This is due to the fact that, when the tenant negotiates with the first lessors, she can claim a lower price arguing that, otherwise, she would have to pay that same price to everyone and that would be unfeasible.

Another relevant feature is that, despite the perfect information setting and von Neumann-Morgenstern utility functions, there arise bargaining problems with non-convex sets of feasible agreements. Failure of convexity compromises the uniqueness of the Nash bargaining solution. Yet, the Nash bargaining solution is unique in all the subclasses and examples of games that we have studied, and we suspect this could be a general result.

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