# A bargaining approach to the Owen value and the Nash solution with coalition structure[1]

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**Summary.** The mechanism by Hart and Mas-Colell (1996) for non-transferable utility (NTU) games is generalized so that a coalition structure among players is taken into account. The new mechanism yields the Owen value for transferable utility (TU) games with coalition structure as well as the consistent value (Maschler and Owen 1989, 1992) for NTU games with trivial coalition structure. Furthermore, we obtain a solution for pure bargaining problems with coalition structure which generalizes the Nash (1950) bargaining solution.

**Keywords and Phrases:** Bargaining, NTU game, Coalition structure, Owen value, Nash solution.

**JEL Classification Numbers:** C71, C78.

## 1 Introduction

Hart and Mas-Colell (1996) (from now on, H&M) developed a bargaining mechanism which yields the consistent value (Maschler and Owen 1989, 1992) for non-transferable utility (NTU) games. First, a player is randomly chosen in order to propose a payoff. If this proposal is not accepted by all the other players, the mechanism is played again under the same conditions with probability $\rho \in [0, 1)$. With probability $1 - \rho$, the proposer leaves the game and the mechanism is repeated with the rest of the players. H&M consider that the consistent value is a very appropriate generalization for the Shapley (1953) value (used in transferable utility (TU) games) to NTU games.

Other non-cooperative mechanisms which implement the Shapley value may be found, for example, in Gul (1989), Hart and Moore (1990), Winter (1994), Evans (1996), Dasgupta and Chiu (1998), Pérez-Castrillo and Wettstein (2001) and Mutuswami, Pérez-Castrillo and Wettstein (2002). Navarro and Perea

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(2001) designed a mechanism which implements the Myerson (1977) value, which is an extension of the Shapley value to graph-restricted games.

Sometimes, however, players are associated in *a priori* coalitions. Owen (1977) studied them in TU games. He proposed a value, known as the Owen value, which generalizes the Shapley value for games with a coalition structure. Later, Winter (1991) proposed a value, called the Game Coalition Structure value, which is a generalization of the Harsanyi (1963) value for NTU games and the Owen value for TU games with a coalition structure.

A non-cooperative mechanism which implements the Owen value in the TU case is given by Vidal-Puga and Bergantiños (2003).

In this paper, we develop a non-cooperative mechanism that takes into account the coalition structure and implements both the consistent value for NTU games and the Owen value for TU games.

The mechanism is as follows: First, a player is randomly chosen out of each coalition and proposes a payoff. Then, each proposal is voted by the rest of the members of its own coalition. If one of them rejects the proposed payoff, the mechanism is either played again under the same conditions (probability $\rho$), or the proposer leaves the game and the mechanism is repeated with the rest of the players (probability $1-\rho$). If there is no rejection, the proposal of one of the coalitions is randomly chosen. If this proposal is not accepted by all other coalitions, the mechanism is played again under the same conditions (probability $\rho$), or the entire proposing coalition leaves the game and the mechanism is repeated with the rest of the players (probability $1-\rho$).

Thus, this mechanism has two stages. First, agreements are negotiated within coalitions and then through delegates between coalitions. This structure appears in many economic and political situations, where negotiations are carried out by agents who are the representatives, or delegates, of a larger group of agents or players. For example, international affairs are negotiated by a minister of foreign affairs and not by the whole government. Negotiations between a syndicate and a company are carried out by the main spokesperson for each side. When a scandal involving a government or company breaks, a single spokesperson states the official position on it. In any case, this delegate does not act independently, but follows a strategy previously agreed on by the agents he is representing.

When the coalition structure is trivial (i.e., either there is a single grand coalition or all the coalitions are singletons), the mechanism coincides with that of H&M. Thus, the consistent value arises in equilibrium. Furthermore, when the mechanism is applied to a TU game with coalition structure, the Owen value is implemented.

When the mechanism is applied to pure bargaining problems, we obtain a coalitional solution that coincides with the Nash bargaining solution (Nash, 1950) when the coalition structure is trivial.

As for general NTU games with coalition structure, the arising equilibrium payoff is a recently studied solution concept: the consistent coalitional value (Bergantiños and Vidal-Puga, 2002).

The structure of this paper is as follows: In Section 2 we give the definitions and results used in the paper. In Section 3 we describe the coalitional mechanism and give the main results: Proposition 9 deals with the existence of equilibria. Theorem 10 proves the result for TU games and the Owen value. Theorem 12 gives the result for pure bargaining problems. In Section 4, we present some concluding remarks. Finally, the proofs are located in the Appendix.

## 2 Definitions and previous results

Mainly, we follow the same notation as in H&M. Let $N = \{1, 2, ..., n\}$ and $2^N = \{S : S \subset N\}$. Given $x, y \in \mathbb{R}^N$, we say $y \leq x$ when $y^i \leq x^i$ for every $i \in N$. We denote by $x \cdot y$ the scalar product $\sum_{i \in N} x^i y^i$. We denote $\mathbb{R}_+ := \{x \in \mathbb{R}^N : x \geq 0\}$, and $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N : x^i > 0, \forall i\}$.

A *non-transferable utility game*, or *NTU game*, is a pair $(N, V)$ where $N$ is the set of *players* and $V$ is a correspondence which assigns to each *coalition* $S \subset N$, $S \neq \emptyset$ a subset $V(S) \subset \mathbb{R}^S$ representing all the possible payoffs that the members of $S$ can obtain for themselves when playing cooperatively. For $S \subset N$, we maintain the notation $V$ when refering to the application $V$ restricted to $S$ as player set. For simplicity, we denote $V(i)$ instead of $V(\{i\})$, $S \cup i$ instead of $S \cup \{i\}$ and $N \backslash i$ instead of $N \backslash \{i\}$.

We impose the next conditions on the function $V$:

**(A.1)** For each $S \subset N$, the set $V(S)$ is closed, convex, *comprehensive* (i.e., if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and *upper bounded* (i.e., for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is bounded).

**(A.2)** For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is *smooth* (this means that on each point of the boundary there exists a unique outward orthonormal vector) and *nonlevel* (this means that the outward vector on each point of $\partial V(S)$ has positive coordinates).

**(A.3)** *Monotonicity*: For each $T \subset S$, $V(T) \times \{0^{S \backslash T}\} \subset V(S)$.

**(A.4)** *Normalization*: For each $S \subset N$, $0^S$ belongs to $V(S)$.

We denote by $TU(N)$ the set of TU games over $N$.

For each $i \in N$, let $r^i := \max\{x : x \in V(i)\}$ (notice that, by (A.4), $r^i \geq 0$).

When

$$V(S) = \left\{ x \in \mathbb{R}^S : \sum_{i \in S} x^i \leq v(S) \right\}$$

for some $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, we say that $(N, V)$ is a *transferable utility game* (or *TU game*) and it is represented by $(N, v)$.

When

$$V(S) = \left\{ x \in \mathbb{R}^S : \lambda_S \cdot x \leq v(S) \right\} \quad (1)$$

for some $\lambda_S \in \mathbb{R}^S_{++}$ and $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$, we say that $(N, V)$ is a *hyperplane game*.

Notice that every TU game is a hyperplane game with $\lambda^i_S = 1$ for all $i \in S \subset N$.

If $r^S \in \partial V(S)$ for all $S \subsetneq N$ and $r^N \in V(N)$, we say that $(N, V)$ is a *pure bargaining problem*.

We say that an NTU game is *zero-monotonic* if $V(i) \times V(S \backslash i) \subset V(S)$ for all $i \in S \subset N$.

Given $N$, we call *coalition structure* over $N$ a partition of the player set, i.e., $\mathcal{C} = \{C_1, C_2, ..., C_p\} \subset 2^N$ is a coalition structure if it satisfies $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. A coalition structure $\mathcal{C}$ over $N$ is *trivial* if either $\mathcal{C} = \{\{1\}, \{2\}, ..., \{n\}\}$ or $\mathcal{C} = \{N\}$.

We denote by $(N, V, \mathcal{C})$ an NTU game $(N, V)$ with coalition structure $\mathcal{C}$ over $N$. We denote by $CNTU(N)$ the set of NTU games with coalition structure over $N$. For coalitions $S \subset N$, we denote by $\mathcal{C}_S$ the restriction of $\mathcal{C}$ to the players in $S$ (notice that this implies that $\mathcal{C}_S$ may have less or the same number of coalitions as $\mathcal{C}$). We also denote $N^{-q}$ instead of $N \backslash C_q$, $\mathcal{C}_{-i}$ instead of $\mathcal{C}_{N \backslash i}$, and $\mathcal{C}^{-q}$ instead of $\mathcal{C} \backslash C_q$.

Given $G$ is a subset of $NTU(N)$ or $CNTU(N)$, a *value* in $G$ is a correspondence which assigns to each element in $G$ a subset of $\mathbb{R}^N$. When these subsets are singletons we call the value a *single value*. A well known single value for TU games is the *Shapley value* (Shapley, 1953). We denote by $\varphi_N \in \mathbb{R}^N$ the Shapley value of the TU game $(N, v)$. For TU games with coalition structure, Owen (1977) proposed a single value based on Shapley's which takes into account the coalition structure $\mathcal{C}$. We call this value the *Owen coalitional value*, or simply the *Owen value*. We denote by $\phi_N \in \mathbb{R}^N$ the Owen value of the TU game with coalition structure $(N, v, \mathcal{C})$.

The *consistent value* for NTU games is introduced by Maschler and Owen (1989, 1992). Let $(N, V)$ be a hyperplane game defined as in (1). Given $i \in N$, the consistent value $\Psi$ is defined recursively as follows:

$$\Psi^i_{\{i\}} = r^i.$$

Assume we know $\Psi^j_S$ for all $S \subsetneq N$ and $j \in S$. Then,

$$\Psi^i_N = \frac{1}{|N| \lambda^i_N} \left( \sum_{j \in N \backslash i} \lambda^i_N \Psi^i_{N \backslash j} - \sum_{j \in N \backslash i} \lambda^j_N \Psi^j_{N \backslash i} + v(N) \right). \tag{2}$$

For a general NTU game $(N, V)$, Maschler and Owen (1992) take for each coalition $S \subset N$ a vector $\lambda_S$ normal to the boundary of $V(S)$. Let $(N, V')$ be the resulting hyperplane game, i.e. $V'(S) = \left\{ x \in \mathbb{R}^S : \lambda_S \cdot x \leq v(S, \lambda_S) \right\} \supset V(S)$, with

$$v(S, \lambda_S) := \max \{ \lambda_S \cdot x : x \in V(S) \}.$$

Let $\Psi = (\Psi_S)_{S\subset N}$ with $\Psi_S$ the (only) consistent value for $(S, V')$. If $\Psi$ is a feasible payoff in $(N, V)$ (i.e., $\Psi_S \in V(S), \forall S \subset N$) then $\Psi_N$ is a *consistent value* for $V$.

The consistent value coincides with the Shapley value for TU games. Maschler and Owen (1992) also show that the consistent value exists (it is not always unique though) for any NTU game.

Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure. Bergantiños and Vidal-Puga (2002) define recursively the *consistent coalitional value* as follows: Given $i \in C_q \in \mathcal{C}$,

$$\Phi^i_{\{i\}} = r^i.$$

Assume we know $\Phi^j_S$ for all $S \subsetneq N$ and $j \in S$. Then,

$$\left.\begin{aligned}\Phi^i_N = \quad & \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N}\left(\sum_{C_r\in\mathcal{C}^{-q}}\left(\sum_{j\in C_q}\lambda^j_N\Phi^j_{N^{-r}} - \sum_{j\in C_r}\lambda^j_N\Phi^j_{N^{-q}}\right)\right)\\ & +\frac{1}{|C_q|\,\lambda^i_N}\left(\sum_{j\in C_q\backslash i}\lambda^i_N\Phi^i_{N\backslash j} - \sum_{j\in C_q\backslash i}\lambda^j_N\Phi^j_{N\backslash i}\right)\\ & +\frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N}v\,(N)\,.\end{aligned}\right\} \quad (3)$$

It is straightforward to check that (3) coincides with (2) when $\mathcal{C}$ is a trivial coalition structure.

Following the usual practice, we consider a *payoff configuration* as a set of payoffs $x = (x_S)_{S\subset N}$ with $x_S \in \mathbb{R}^S$ for all $S \subset N$.

The generalization of $\Phi$ to NTU games (not necessarily hyperplane games) is done analogously to the consistent value. For an NTU game with coalition structure $(N, V, \mathcal{C})$, we take for each coalition $S \subset N$ a normal vector $\lambda_S$ to the boundary of $V(S)$. Let $(N, V', \mathcal{C})$ be the resulting hyperplane game. Let $\Phi := (\Phi_S)_{S\subset N}$ for all $S \subset N$ be the (unique) consistent coalitional payoff configuration for $(N, V', \mathcal{C})$. If $\Phi$ is a feasible payoff configuration for $(N, V, \mathcal{C})$, then $\Phi$ is a *consistent coalitional payoff configuration* for $(N, V, \mathcal{C})$.

Bergantiños and Vidal-Puga (2002) proved that the consistent coalitional value exists for any NTU game (although it is not necessarily unique) and gave the following characterization: Given $S \subset N$ player set, we denote by $C'_q := C_q \cap S$ (when different from $\emptyset$) the restriction of $C_q$ in $\mathcal{C}_S$. We also denote $S^{-q} := S\backslash C'_q$ and $\mathcal{C}^{-q}_S := \mathcal{C}_S\backslash C'_q$. The set $\Phi = (\Phi_S)_{S\subset N}$ is a consistent coalitional payoff configuration for $(N, V, \mathcal{C})$ if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$, orthogonal to $V(S)$, such that:

**(B.1)** $\Phi_S \in \partial V(S)$;

**(B.2)** for every $C'_q \in \mathcal{C}_S$

$$\sum_{C'_r\in\mathcal{C}^{-q}_S}\left[\sum_{i\in C'_q}\lambda^i_S(\Phi^i_S - \Phi^i_{S^{-r}})\right] = \sum_{C'_r\in\mathcal{C}^{-q}_S}\left[\sum_{i\in C'_r}\lambda^i_S(\Phi^i_S - \Phi^i_{S^{-q}})\right];$$

**(B.3)** for every $i \in C'_q \in \mathcal{C}_S$

$$\sum_{j \in C'_q \backslash i} \lambda_S^i \left( \Phi_S^i - \Phi_{S \backslash j}^i \right) = \sum_{j \in C'_q \backslash i} \lambda_S^j \left( \Phi_S^j - \Phi_{S \backslash i}^j \right).$$

Thus, (B.1), (B.2), and (B.3) generalize the characterization by means of Pareto optimality and balanced contributions of the Owen value (Theorem 2 in Calvo, Lasaga and Winter, 1996) and the consistent value (Proposition 4 in H&M). Calvo, Lasaga and Winter (1996) characterized by Pareto optimality and balanced contributions the levels structure value (Winter, 1989) which is a generalization of the Owen value.

# 3 The coalitional mechanism

In this section we describe the coalitional mechanism. This mechanism is a modification of the bargaining mechanism presented by H&M.

Even though the model is defined for NTU games, we focus on TU games and on pure bargaining problems. We show that there exists at least an equilibrium. Moreover, the final payoff in any equilibrium coincides with the Owen value in zero-monotonic TU games. For pure bargaining problems, we obtain a generalization of the Nash bargaining solution.

For each $S \subset N$, we denote by $\Gamma_S$ the set of applications $\gamma : \mathcal{C}_S \to S$ satisfying $\gamma_q \in C'_q$ for each $C'_q \in \mathcal{C}_S$. For simplicity, we denote $\Gamma := \Gamma_N$ and $\gamma_q := \gamma\left(C'_q\right)$.

The coalitional bargaining mechanism associated to $(N, V, \mathcal{C})$ and $\rho \in [0, 1)$ is defined as follows:

> In each round there is a set $S \subset N$ of active players. In the first round, $S = N$. Each round has one or two stages. In the first stage, a *proposer* is randomly chosen from each coalition. Namely, a function $\gamma \in \Gamma_S$ is randomly chosen, being each $\gamma$ equally likely to be chosen. The coalitions play sequentially (say, for example, in the order $\left(C'_1, C'_2, ..., C'_p\right)$) in the following way: Proposer $\gamma_1$ proposes a feasible payoff, i.e. a vector in $V(S)$. The members of $C'_1 \backslash \gamma_1$ are then asked in some prespecified order to accept or reject the proposal. If one of them rejects the proposal, then we move to the next round where the set of active players is $S$ with probability $\rho$ and $S \backslash \gamma_1$ with probability $1 - \rho$. In the latter case, player $\gamma_1$ gets a final payoff of 0. If all the players accept the proposal, the game moves on to the next coalition, $C'_2$. Then, players of $C'_2$ proceed to repeat the process under the same conditions, and so on. If all the proposals are accepted in each coalition, the proposers are called *representatives*. We denote by $a\left(S, \gamma_q\right) \in V(S)$ the proposal of $\gamma_q$.
>
> In the second stage, a proposal $a\left(S, \gamma_q\right)$ is randomly chosen, each proposal being equally likely to be chosen. We call player $\gamma_q$ the

*representative-proposer*, or simply *r.p.* If all the members of $S^{-q}$ accept a $\left(S, \gamma_q\right)$– they are asked in some prespecified order – then the game ends with these payoffs. If it is rejected by at least one member of $S^{-q}$, then we move to the next round where, with probability $\rho$, the set of active players is again $S$ and, with probability $1-\rho$, the entire coalition $C'_q$ *drops out* and the set of active players becomes $S^{-q}$. In the latter case each member of the dropped coalition $C'_q$ gets a final payoff of 0.

Clearly, given any set of strategies, this mechanism finishes in a finite number of rounds with probability 1.

This mechanism generalizes H&M's for trivial coalition structures. For $\mathcal{C} = \{N\}$, the second stage is trivial, since there is a single r.p. and a single proposal. Moreover, the first stage coincides with H&M's mechanism. For $\mathcal{C} = \{\{1\}, \{2\}, ..., \{n\}\}$, the first stage is trivial. Each player states a proposal, and in the second stage a proposal is randomly selected and voted by the rest of the players/coalitions.

**Remark 1** *The normalization given by property (A.4) does not affect our results, although the bargaining mechanism must be changed as follows: The player* $i \in N$ *who drops out receives an amount* $x^i \in \mathbb{R}$ *such that* $x^i \in V(i)$*. This* $x^i$ *can be considered as a "penalty payoff". Also, the monotonic property must be replaced by* $V(T) \times \left(x^{S \setminus T}\right) \subset V(S)$ *for each* $T \subset S$*.*

The coalitional bargaining mechanism may be interpreted as the mechanism of H&M played in two stages, one of them by the coalitions and the other by the players inside the same coalition. In the second stage, the coalitions play H&M's mechanism. This means that a coalition is randomly chosen to propose a payoff. Disagreement to this payoff by at least one of the other coalitions puts the whole proposing coalition in jeopardy. In order to decide the proposals, the members of each coalition play H&M's mechanism in the first stage. Thus, a player is randomly chosen inside each coalition and proposes a feasible payoff. Only if the rest of the members of his coalition agree to this payoff, the proposal goes on to the second stage. Otherwise, the proposer is in jeopardy. However, once the proposal is presented in the second stage, it is backed by the whole proposing coalition, so that its rejection may imply the whole coalition leaving the game.

In our study, as in H&M's, we consider stationary subgame perfect equilibria. In this context, an equilibrium is stationary if the players' strategies depend only on the set $S$ of active players. They do not depend, however, on the previous history or the number of played rounds.

We also assume, as H&M, that players break ties in favor of quick termination of the game. We must note that this assumption is not needed in H&M's model. However, Example 13 shows that we cannot avoid it in our coalitional mechanism.

From now on, when we say equilibrium, we mean stationary subgame perfect equilibrium satisfying this tie-breaking rule.

Given a set of stationary strategies, let $S$ denote the set of active players.

We denote by $a(S,i) \in V(S)$ the payoff proposed by $i \in C'_q \in \mathcal{C}_S$ when the set of proposers is determined by some $\gamma \in \Gamma_S$ with $\gamma_q = i$. We also define, for a given $\gamma \in \Gamma_S$:

$$a(S)_\gamma := \frac{1}{|\mathcal{C}_S|} \sum_{C'_q \in \mathcal{C}_S} a\left(S, \gamma_q\right).$$

Since $V(S)$ is a convex set and each $a(S,\gamma_q)$ belongs to $V(S)$, their average also belongs to $V(S)$. When all the proposals are accepted, $a(S)_\gamma$ is the expected final payoff when $\gamma$ determines the set of proposers (or representatives).

Given $i \in C'_q \in \mathcal{C}_S$, let $\Gamma_{S,i}$ ($\Gamma_i := \Gamma_{N,i}$) be the subset of functions $\gamma \in \Gamma_S$ such that $\gamma_q = i$. Notice that $|\Gamma_S| = |\Gamma_{S,i}| \left|C'_q\right|$ for all $i \in C'_q \in \mathcal{C}_S$. Then,

$$a\left(S|_i\right) := \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a\left(S, \gamma_r\right) \in V\left(S\right)$$

is the expected final payoff when all the proposals are accepted and player $i$ is the proposer (and representative) of his coalition.

We denote

$$a(S) := \frac{1}{|\Gamma_S|} \sum_{\gamma \in \Gamma_S} a\left(S\right)_\gamma \in V(S)$$

as the expected final payoff when all the proposals are accepted. Given $C'_q \in \mathcal{C}_S$, it is straightforward to prove that $a(S)$ may also be expressed as:

$$a(S) = \frac{1}{|C'_q|} \sum_{i \in C'_q} a\left(S|_i\right).$$

It is also straightforward to prove that

$$a\left(S\right) = \frac{1}{|\mathcal{C}_S|} \sum_{C'_q \in \mathcal{C}_S} \frac{1}{\left|C'_q\right|} \sum_{i \in C'_q} a\left(S, i\right). \tag{4}$$

Proposition 1 in H&M characterizes the proposals corresponding to an equilibrium by (1) $a\left(S,i\right) \in \partial V\left(S\right)$ for all $i \in S$ and (2) $a\left(S,i\right)^j = \rho a\left(S\right)^j + \left(1-\rho\right) a\left(S \backslash i\right)^j$ for all $i,j \in S$.

We now introduce some properties which generalize (1) and (2) in H&M to games with coalition structure.

We consider the following properties:

**(C.1)** $a(S,i) \in \partial V(S)$ for every $i \in S$;

**(C.2)** $a(S|_i)^j = \rho a(S)^j + (1-\rho) a(S \backslash i)^j$ for every $i,j \in C'_q \in \mathcal{C}_S$ with $j \neq i$;

**(C.2')** $a(S,i)^j = \rho a(S)^j + (1-\rho)\left[|\mathcal{C}_S| a(S \backslash i)^j - \sum_{C'_r \in \mathcal{C}_S^{-q}} a(S^{-r})^j\right]$ for every $i,j \in C'_q \in \mathcal{C}_S$;

**(C.3)** $a(S,i)^j = \rho a(S)^j + (1-\rho)a(S^{-q})^j$ for every $i \in C'_q \in \mathcal{C}_S$, $j \notin C'_q$.

Of course (C.1) coincides with Property (1) of Proposition 1 in H&M. Property (2) is split in two properties: (C.2) or (C.2'), and (C.3), following the usual practice found in the literature on games with coalition structure.

**Proposition 2** *If (C.3) holds, then (C.2) is equivalent to (C.2').*

The proof of Proposition 2 is in the Appendix.

**Proposition 3** *If (C.3) holds, then* $a(S)^j = a(S|_i)^j$ *for every* $i \in C'_q \in \mathcal{C}_S$, $j \notin C'_q$.

The proof of Proposition 3 is in the Appendix.

**Proposition 4** *Let* $(N, V, \mathcal{C})$ *be a TU game with coalition structure. Assume a set of strategies* $\left(a(S,i)_{i \in S}\right)_{S \subset N}$ *satisfies (C.1), (C.2) and (C.3). Then,* $(a(S))_{S \subset N}$ *is the Owen value* $\phi$ *for the game* $(N, V, \mathcal{C})$.

The proof of Proposition 4 is located in the Appendix.

By Proposition 2, Proposition 4 also holds if we replace (C.2) by (C.2').

**Proposition 5** *The proposals in any equilibrium of a zero-monotonic TU game or a pure bargaining problem are characterized by (C.1), (C.2) and (C.3). Moreover, all the proposals are accepted and* $a(S) \geq 0^S$ *for all* $S \subset N$.

The proof of Proposition 5 is located in the Appendix.

**Remark 6** *There is a subtle difference between the result given by Proposition 5 and Proposition 1 in H&M. In H&M's model, the proposals* $a(S,i)$ *are nonnegative. In our model, the proposals do not need to be nonnegative; this can be checked in Example 14. However, their (weighted) average* $a(S)$ *is always nonnegative in equilibrium.*

Now, two important corollaries of Proposition 5 are presented.

**Corollary 7** *Let* $(N, V, \mathcal{C})$ *be a zero-monotonic TU game or a pure bargaining problem with coalition structure. Then, a player's expected payoff in equilibrium is independent on who the proposer is in other coalitions, namely:*

$$a(S)^j = a(S|_i)^j \qquad \forall i \in C'_q \in \mathcal{C}_S; j \notin C'_q.$$

The proof of Corollary 7 is immediate from Proposition 3 and 5.

H&M say: "if $\rho$ is close to 1– i.e., the 'cost of delay' is low – then there is little dispersion among individual proposals: all the $a(N,i)$ constitute small deviations of $a(N)$. This implies, first, that $a(N)$ *is almost Pareto optimal* (since the $a(N,i)$ are Pareto optimal). And second, that *there is no substantial advantage or disadvantage to being the proposer*; the 'first-mover' effect vanishes."

H&M denote $a(N,i)$ and $a(N)$ as $a_{N,i}$ and $a_N$, respectively.

The next corollary states that the coalitional bargaining mechanism behaves in the same way.

**Corollary 8** *There exists* $M \in \mathbb{R}$ *such that* $\left|a(N,i)^j - a(N)^j\right| < M(1-\rho)$ *for all* $i, j \in N$.

Corollary 8 is readily implied by (C.2') and (C.3).

In the next proposition we prove the existence of equilibria.

**Proposition 9** *Let* $(N, V, \mathcal{C})$ *be a zero-monotonic TU game or a pure bargaining problem with coalition structure. Then, for each* $\rho \in [0, 1)$*, there exists an equilibrium.*

The proof of Proposition 9 is located in the Appendix.

The next results characterize the equilibrium payoffs.

**Theorem 10** *Let* $(N, V, \mathcal{C})$ *be a zero-monotonic TU game with coalition structure. Then, for each* $\rho \in [0, 1)$*, there exists a unique equilibrium payoff. Furthermore, the equilibrium payoff equals the Owen value of* $(N, V, \mathcal{C})$.

The proof of Theorem 10 is located in the Appendix.
Theorem 10 may be restated as follows:

**Corollary 11** *The coalitional mechanism, when applied to zero-monotonic TU games, implements the Owen value.*

Notice that the coalitional bargaining mechanism implements the Shapley value for zero-monotonic games because the Shapley value coincides with the Owen value when the coalition structure is trivial.

**Theorem 12** *Let* $(N, V, \mathcal{C})$ *be a pure bargaining problem with coalition structure. If* $a_\rho := (a_\rho(S))_{S \subset N}$ *is an equilibrium payoff configuration for each* $\rho$ *and* $a$ *is the limit of* $a_\rho$ *when* $\rho \to 1$*, then* $a$ *is a consistent coalitional payoff configuration of* $(N, V, \mathcal{C})$.

The proof of Theorem 12 is located in the Appendix.

# 4 Concluding remarks

## 4.1 The tie-breaking rule

If we do not assume the tie-breaking rule, the Owen value $\phi$ is still an equilibrium payoff for zero-monotonic TU games. However, there can be other equilibria which do not yield the Owen value, as the next example shows.

**Example 13** *Consider* $(N, v, \mathcal{C})$*, where* $N = \{1, 2, 3, 4\}$*,* $\mathcal{C} = \{C_1, C_2\}$*,* $C_1 = \{1, 2\}$ *and* $C_2 = \{3, 4\}$*. Moreover,* $v$ *is the characteristic function associated to the weighted majority game where the quota is 3 and the weights are 1, 1, 1,* *and 2, respectively. This means that* $v(S) = 1$ *if and only if* $S$ *contains some of the following subsets:* $\{1,2,3\}$, $\{1,4\}$, $\{2,4\}$, *or* $\{3,4\}$.

*It is straightforward to prove that*

$$\begin{aligned}
\phi_N &= \left(0, 0, \frac{1}{2}, \frac{1}{2}\right) \\
\phi_{N\backslash 1} &= \left(-, 0, \frac{1}{4}, \frac{3}{4}\right) \\
\phi_{N\backslash 2} &= \left(0, -, \frac{1}{4}, \frac{3}{4}\right) \\
\phi_{N\backslash 3} &= \left(\frac{1}{4}, \frac{1}{4}, -, \frac{1}{2}\right) \\
\phi_{N\backslash 4} &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, -\right).
\end{aligned}$$

*We now define an equilibrium whose payoff outcome is* $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

*First, we describe the strategies of players* 1 *and* 2. *When one of them is chosen as proposer, his proposal is* $a(N, \gamma_1) = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$. *Moreover, players* 1 *and* 2 *accept an offer if and only if it contributes something positive to them (hence, under these strategies, the proposal is rejected). In the subgame obtained after* $\gamma_1$ *drops out of the game, the strategy of the other player coincides with the strategy with* $\phi_{N\backslash\gamma_1}$ *as payoff outcome. In the subgame obtained after* $C_2$ *drops out of the game, the strategies of players* 1 *and* 2 *coincide with the strategy with* $\phi_{N^{-2}}$ *as payoff outcome.*

*We now describe the strategies of players* 3 *and* 4. *In the subgame obtained after the offer of* $\gamma_1$ *is accepted, the strategies of players* 3 *and* 4 *coincide with the strategies with* $\phi_N$ *as payoff outcome. In the subgame obtained after* $\gamma_1$ *drops out of the game, the strategies of players* 3 *and* 4 *coincide with the strategies with* $\phi_{N\backslash\gamma_1}$ *as payoff outcome. In the subgame obtained after* $C_1$ *drops out of the game, the strategies of players* 3 *and* 4 *coincide with the strategies with* $\phi_{N^{-1}}$ *as payoff outcome.*

*It is not difficult to check that these strategies are an equilibrium.*

*According to these strategies, the offer of player* $\gamma_1$ *is rejected, which means that player* $\gamma_1$ *obtains a final payoff of* 0. *Then, players of* $N \setminus \gamma_1$ *obtain* $\phi_{N\backslash\gamma_1}$ *as final payoff. This means that the final payoff induced by these strategies is* $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

## 4.2 The selected proposal

In the first stage of the coalitional bargaining mechanism, when the players of a coalition accept the proposal of one of their members, they do not know whether this proposal would be selected in the second stage. They only know this proposal would have a chance to be selected and thus be voted by the other coalitions.

This uncertainty is not innocuous and affects in an important way the behavior of the agents. In particular, there may be proposals which assign negative payoffs to some players, as the next example shows.

**Example 14** *Consider* $(N, v, \mathcal{C})$*, where* $N = \{1, 2, 3\}$*,* $\mathcal{C} = \{C_1, C_2\}$*,* $C_1 = \{1, 2\}$ *and* $C_2 = \{3\}$*. Moreover,* $v$ *is the characteristic function associated to the weighted majority game where the quota is 3 and the weights are 2, 1, and 1, respectively. This means that* $v(S) = 1$ *if and only if* $S$ *contains* $\{1, 2\}$ *or* $\{1, 3\}$*. Otherwise,* $v(S) = 0$*.*

*The Owen value for this game is* $\left(\frac{3}{4}, \frac{1}{4}, 0\right)$*.*

*Assume they play the bargaining coalitional mechanism with* $\rho = 0$*. Player* 3 *would propose* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$*, since this is the payoff players in* $C_1$ *would get in absence of him. Player* 2 *would propose* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ *for a similar reason. Player* 1*, however, would propose* $\left(\frac{3}{2}, -\frac{1}{2}, 0\right)$*, and player* 2 *accepts! Notice that, by rejecting, player* 2 *gets* 0*, and by accepting, his final payoff is* $\frac{1}{2}$ *if the r.p. is player* 3*, and* $-\frac{1}{2}$ *if the r.p. is player* 1*. In expected terms, player* 2 *gets* 0*.*

*The expected final payoff is the Owen value:*

$$\frac{1}{4}\left(\frac{3}{2}, -\frac{1}{2}, 0\right) + \frac{1}{4}\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \left(\frac{3}{4}, \frac{1}{4}, 0\right).$$

It may be argued that it makes no sense for a player to agree to take his coalition to the second stage willing to "defend" a negative payoff for him. However, as $\rho$ approaches 1, the proposals also approach the expected equilibrium payoff. In Example 14, for a given $\rho$, player 1 would propose $\left(\frac{6-3\rho}{4}, \frac{3\rho-2}{4}, 0\right)$. Thus, for $\rho > \frac{2}{3}$, the payoff proposed to player 2 is positive.

## 4.3 Voting in the second stage

In the bargaining mechanism, voting in the second stage involves not only the representatives but also the rest of the players. A natural question is what happens when the voting is done only by the representatives. Notice that in this case, proposals in the first stage should include a function which assigns an answer (accept or reject) to each possible proposal made by the other representatives. This allows players to commit to a certain strategy. For example, let $v$ be the two-person normalized game $v(\{1\}) = v(\{2\}) = 0$ and $v(\{1, 2\}) = 1$ with the trivial coalition structure $\{\{1\}, \{2\}\}$. Assume player 1 proposes $(1, 0)$ and to reject any payoff which gives him less than 1; and player 2 proposes $(1, 0)$ and to reject any payoff which gives him less than 0. It is easy to check that these strategies constitute an equilibrium whose final payoff is $(1, 0)$.

Hence, it should be assumed that all the players have *a posteriori* veto on proposals coming from other coalitions. In fact, it seems unreasonable that, in the negotiation within coalitions, players should anticipate every possible proposal of the other potential delegates. Instead, players within a coalition only agree on their own proposal. In the second stage (negotiation among coalitions) the chosen representative may accept another proposal, knowing that his decission is not binding and it may be vetoed by another player. However, as we have shown, this veto does not happen in equilibrium.

### 4.4 The general NTU case

In this paper we concentrate on zero-superadditive TU games and pure bargaining problems. A natural question is whether the mechanism could be applied to general NTU games. In this more general framework, however, we find that a player may not have an acceptable offer to propose. More specifically, property (A.5) (see the proof of Proposition 5) is not satisfied in general.

**Example 15** *Let* $(N, V, \mathcal{C})$ *be such that* $N = \{1,2,3\}$, $\mathcal{C} = \{\{1,2\},\{3\}\}$ *and* $V$ *be defined as follows:*

$V(i) = 0 - \mathbb{R}_+$, $i = 1,2,3$;

$V(\{1,2\}) = V(\{1,3\}) = (0,0) - \mathbb{R}^2_+$;

$V(\{2,3\}) = \left\{(x_2, x_3) : \frac{4}{5}x_2 + 2x_3 \leq 1, x_2 + x_3 \leq \frac{7}{8}\right\}$ *and*

$V(N) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1\}$.

*It can be easily checked that this game is superadditive. It is not smooth, but we can make it smooth by introducing a small modification which does not change our result. Furthermore, if some of the players leave the game, the coalition structure vanishes (i.e. it becomes trivial) and the coalitional bargaining mechanism coincides with the bargaining mechanism of H&M. Thus, it is straightforward to prove that the payoffs in equilibrium are given by:*

$$\begin{aligned}
a(\{i\}) &= 0 \qquad \textit{for all } i = 1,2,3 \\
a(\{1,2\}) &= (0,0,-) \\
a(\{1,3\}) &= (0,-,0) \\
a(\{2,3\}) &= \left(-, \frac{70-55\rho}{24\rho^2 - 160\rho + 160}, \frac{40-34\rho}{24\rho^2 - 160\rho + 160}\right).
\end{aligned}$$

*Assume player* 1 *is chosen as proposer. The only proposal satisfying (C.1), (C.2) and (C.3) is given by*

$$a(N,1) = \left(\frac{12\rho^2 - 25\rho + 10}{12\rho^2 - 80\rho + 80}, \frac{70-55\rho}{12\rho^2 - 80\rho + 80}, 0\right).$$

*But this means that for* $\rho \in \left(\frac{25-\sqrt{145}}{24}, 1\right)$ *player* 1 *is offering himself a negative payoff. Thus, it is optimal for him to make an unacceptable offer and hence Proposition 5 does not hold.*

The author has been unable to find a characterization of the games which satisfy property (A.5). This property is needed so that no player is excluded in equilibrium. When (A.5) holds, the arising final payoff is the consistent coalitional value (see Bergantiños and Vidal-Puga, 2002), which is a generalization of the Owen value for TU games and the consistent value for NTU games with coalition structure.

# 5 Appendix

## 5.1 *Proof* of Proposition 2

Fix $i, j \in C'_q \in \mathcal{C}_S$ with $j \neq i$. It is readily checked that

$$a(S,i)^j = |\mathcal{C}_S| a(S|_i)^j - \sum_{C'_r \in \mathcal{C}_S^{-q}} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S,\gamma_r)^j.$$

We prove that, under (C.3), (C.2) implies (C.2'). The proof for the reciprocal is similar.

By (C.3), we know $a(S,\gamma_r)^j = \rho a(S)^j + (1-\rho) a(S^{-r})^j$ for any $C'_r \neq C'_q$, so

$$a(S,i)^j = |\mathcal{C}_S| a(S|_i)^j - \sum_{C'_r \in \mathcal{C}_S^{-q}} \left[\rho a(S)^j + (1-\rho) a(S^{-r})^j\right]$$

and by (C.2), we know $a(S|_i)^j = \rho a(S)^j + (1-\rho) a(S \backslash i)^j$, hence condition (C.2') holds.■

## 5.2 *Proof* of Proposition 3

We prove the equivalent claim $a(S|_i)^j = a(S|_k)^j$ for all $i, k \in C'_q \in \mathcal{C}_S$; $j \notin C'_q$

$$a(S|_i)^j = \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S^{-q}} \frac{1}{|\Gamma_{S,k}|} \sum_{\gamma \in \Gamma_{S,k}} a(S,\gamma_r)^j + \frac{1}{|\mathcal{C}_S|} a(S,i)^j.$$

By (C.3), $a(S,i)^j = \rho a(S)^j + (1-\rho) a(S^{-q})^j = a(S,k)^j$, so

$$a(S|_i)^j = \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S^{-q}} \frac{1}{|\Gamma_{S,k}|} \sum_{\gamma \in \Gamma_{S,k}} a(S,\gamma_r)^j + \frac{1}{|\mathcal{C}_S|} a(S,k)^j = a(S|_k)^j.$$

■

## 5.3 *Proof* of Proposition 4

We prove the following stronger claim:

**Proposition 4'** *Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure. Assume a set of strategies $\left(a(S,i)_{i \in S}\right)_{S \subset N}$ satisfies (C.1), (C.2) and (C.3). Then, $(a(S))_{S \subset N}$ is the consistent coalitional value for the game $(N, V, \mathcal{C})$.*

We proceed by induction. The case of one player is trivial. Assume the result is true for hyperplane games with less than $n$ players. Assume $V(N) = \left\{x \in \mathbb{R}^N : \lambda \cdot x \leq v(N)\right\}$ for some $\lambda \in \mathbb{R}^N_{++}$.

By Bergantiños and Vidal-Puga (2002), proving that $a(N)$ satisfies (B.1), (B.2), and (B.3) suffices.

We know that $a(N) = \frac{1}{|\mathcal{C}|}\sum_{C_q \in \mathcal{C}} \frac{1}{|C_q|}\sum_{i \in C_q} a(N,i)$. Moreover, $\lambda \cdot a(N,i) = v(N)$ for each $i \in N$ because $a(N,i) \in \partial V(N)$ by (C.1). Then, $\lambda \cdot a(N) = v(N)$ and hence $a(N)$ satisfies (B.1).

We now prove that $a(N)$ satisfies (B.2). For each $\gamma \in \Gamma$ with $\gamma_q = i \in C_q \in \mathcal{C}$, it is easily checked that

$$|\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j_\gamma = \lambda^i a(N,i)^i + \sum_{j \in C_q \setminus i} \lambda^j a(N,i)^j + \sum_{C_r \in \mathcal{C}^{-q}} \left( \sum_{j \in C_q} \lambda^j a(N,\gamma_r)^j \right).$$

By (C.1) and (C.3):

$$\begin{aligned} |\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j_\gamma &= v(N) - \sum_{C_r \in \mathcal{C}^{-q}} \left( \sum_{j \in C_r} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N^{-q})^j\right] \right) \\ &\quad + \sum_{C_r \in \mathcal{C}^{-q}} \left( \sum_{j \in C_q} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N^{-r})^j\right] \right). \end{aligned}$$

This amount is independent of $\gamma$, and thus it coincides with $|\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j$. Since $a(N)$ satisfies (B.1),

$$v(N) = \sum_{j \in N} \lambda^j a(N)^j = \sum_{j \in C_q} \lambda^j a(N)^j + \sum_{C_r \in \mathcal{C}^{-q}} \left( \sum_{j \in C_r} \lambda^j a(N)^j \right).$$

Hence, by doing some computations,

$$\begin{aligned} |\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j &= (1-\rho) \sum_{C_r \in \mathcal{C}^{-q}} \left( \sum_{j \in C_r} \lambda^j \left[a(N)^j - a(N^{-q})^j\right] \right) \\ &\quad - (1-\rho) \sum_{C_r \in \mathcal{C}^{-q}} \left( \sum_{j \in C_q} \lambda^j \left[a(N)^j - a(N^{-r})^j\right] \right) \\ &\quad + |\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j \end{aligned}$$

from where we get (dividing by $(1-\rho)$) property (B.2) when $S = N$ and $\lambda_N = \lambda$.

We now prove that $a(N)$ satisfies (B.3). Given $i \in C_q$, we know that

$$|C_q| \lambda^i a(N)^i = \sum_{j \in C_q} \lambda^i a(N|_j)^i = \lambda^i a(N|_i)^i + \sum_{j \in C_q \setminus i} \lambda^i a(N|_j)^i.$$

Since $a(N|_i) = \frac{1}{|\mathcal{C}|}\sum_{C_r \in \mathcal{C}} \frac{1}{|\Gamma_i|}\sum_{\gamma \in \Gamma_i} a(S,\gamma_r)$ and $a(N,\gamma_r) \in \partial V(N)$ for each

$\gamma \in \Gamma_i$ and $C_r \in \mathcal{C}$, we conclude that $\sum_{j \in N} \lambda^j a\left(N|_i\right)^j = v\left(N\right)$. Then,

$$\left|C_q\right| \lambda^i a(N)^i = v\left(N\right) - \sum_{j \in N^{-q}} \lambda^j a(N|_i)^j - \sum_{j \in C_q \backslash i} \lambda^j a(N|_i)^j + \sum_{j \in C_q \backslash i} \lambda^i a(N|_j)^i$$

since $a\left(N\right) \in \partial V\left(N\right)$

$$\begin{aligned} = \quad & \lambda^i a(N)^i + \sum_{j \in C_q \backslash i} \lambda^j a(N)^j + \underbrace{\sum_{j \in N^{-q}} \lambda^j a(N)^j} \\ & - \underbrace{\sum_{j \in N^{-q}} \lambda^j a(N|_i)^j} - \sum_{j \in C_q \backslash i} \lambda^j a(N|_i)^j + \sum_{j \in C_q \backslash i} \lambda^i a(N|_j)^i. \end{aligned}$$

The terms above the brackets are equal because $a(N)^j = a(N|_i)^j$ for all $j \in N^{-q}$ (Proposition 3)

$$\left|C_q\right| \lambda^i a(N)^i = \lambda^i a(N)^i + \sum_{j \in C_q \backslash i} \lambda^j a(N)^j - \sum_{j \in C_q \backslash i} \lambda^j a(N|_i)^j + \sum_{j \in C_q \backslash i} \lambda^i a(N|_j)^i$$

by (C.2)

$$\begin{aligned} = \quad & \lambda^i a(N)^i + \sum_{j \in C_q \backslash i} \lambda^j a(N)^j \\ & - \sum_{j \in C_q \backslash i} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N \backslash i)^j\right] \\ & + \sum_{j \in C_q \backslash i} \lambda^i \left[\rho a(N)^i + (1-\rho) a(N \backslash j)^i\right] \end{aligned}$$

we add and subtract $\sum_{j \in C_q \backslash i} \lambda^i a(N)^i$ and gather terms to obtain:

$$\begin{aligned} = \quad & \sum_{j \in C_q} \lambda^i a(N)^i + (1-\rho) \sum_{j \in C_q \backslash i} \lambda^j a(N)^j \\ & -(1-\rho) \sum_{j \in C_q \backslash i} \lambda^j a(N \backslash i)^j - (1-\rho) \sum_{j \in C_q \backslash i} \lambda^i a(N)^i + (1-\rho) \sum_{j \in C_q \backslash i} \lambda^i a(N \backslash j)^i. \end{aligned}$$

This first term is $|C_q| \lambda^i a(N)^i$. So, the rest of the terms must equal zero. Dividing by $(1-\rho)$, we obtain property (B.3) when $S = N$ and $\lambda_N = \lambda$.■

### 5.4 *Proof* of Proposition 5

Given $\left(a\left(S,i\right)_{i\in S}\right)_{S\subset N}$ set of proposals, we define the vector $c\left(S,i\right)\in\mathbb{R}^S$ with $S\subset N$ and $i\in C'_q\in\mathcal{C}_S$ as follows:

$$\left.\begin{array}{ll} c\left(S,i\right)^i=-\sum\limits_{C'_r\in\mathcal{C}_S^{-q}} a\left(S^{-r}\right)^i & \\ c\left(S,i\right)^j=\left|\mathcal{C}_S\right| a\left(S\backslash i\right)^j-\sum\limits_{C'_r\in\mathcal{C}_S^{-q}} a\left(S^{-r}\right)^j & \text{for all } j\in C'_q\backslash i \\ c\left(S,i\right)^j=a\left(S^{-q}\right)^j & \text{for all } j\in S^{-q}. \end{array}\right\} \tag{5}$$

We consider the following property:

**(A.5)** For any $\left(a\left(S,i\right)_{i\in S}\right)_{S\subset N}$ set of proposals satisfying (C.1), (C.2), and (C.3), we have that, for every $S\subset N$ and $i\in C_q^{'}\in\mathcal{C}_S$, the vector $c\left(S,i\right)$ belongs to $V\left(S\right)$.

**Lemma 16** *Property (A.5) is satisfied by zero-monotonic TU games and pure bargaining problems with coalition structure.*

**Proof.** Let $(N,v,\mathcal{C})$ be a zero-monotonic TU game with coalition structure. Let $\left(a\left(S,i\right)\right)_{S\subset N,i\in S}$ be a set of proposals satisfying (C.1), (C.2) and (C.3). By Proposition 4, we know that $a\left(S\right)=\phi_S$ for all $S\subset N$.

We use the next result, proven by Vidal-Puga and Bergantiños (2003). Given $S\subset N$ and $i\in C'_q\in\mathcal{C}_S$,

$$\sum_{j\in C'_q}\phi_S^j\geq\sum_{j\in C'_q\backslash i}\phi_{S\backslash i}^j+v(i).$$

By normalization, $v\left(i\right)\geq 0$ and thus

$$\sum_{j\in C'_q}\phi_S^j\geq\sum_{j\in C'_q\backslash i}\phi_{S\backslash i}^j. \tag{6}$$

Now, by (B.2),

$$\begin{aligned}\sum_{j\in S}c\left(S,i\right)^j &= \sum_{j\in C'_q\backslash i}\left|\mathcal{C}_S\right|\phi_{S\backslash i}^j+\sum_{C'_r\in\mathcal{C}_S^{-q}}\left(\sum_{j\in C'_r}\phi_{S^{-q}}^j-\sum_{j\in C'_q}\phi_{S^{-r}}^j\right)\\ &= \sum_{j\in C'_q\backslash i}\left|\mathcal{C}_S\right|\phi_{S\backslash i}^j+\sum_{C'_r\in\mathcal{C}_S^{-q}}\left(\sum_{j\in C'_r}\phi_S^j-\sum_{j\in C'_q}\phi_S^j\right)\\ &= \left|\mathcal{C}_S\right|\left(\sum_{j\in C'_q\backslash i}\phi_{S\backslash i}^j-\sum_{j\in C'_q}\phi_S^j\right)+\sum_{j\in S}\phi_S^j\end{aligned}$$

by (6),

$$\leq \sum_{j \in S} \phi_S^j = v(S),$$

which means that (A.5) holds for $(N, v, \mathcal{C})$.

Assume now that $(N, V, \mathcal{C})$ is a pure bargaining problem with coalition structure. We first prove by induction that, for $S \subsetneq N$, $a(S) = r^S$. By (C.1), the result is trivial for $n = 1$. Assume that $a(T) = r^T$ for all $T \subsetneq S$. Then, given $i \in C'_q \in \mathcal{C}_S$,

- by (C.2'), $a(S,i)^j = \rho a(S)^j + (1-\rho)\left(|\mathcal{C}_S| r^j - \sum_{C'_r \in \mathcal{C}_S^{-q}} r^j\right) = \rho a(S)^j + (1-\rho) r^j$ for all $j \in C'_q \backslash i$;
- by (C.3), $a(S,i)^j = \rho a(S)^j + (1-\rho) r^j$ for all $j \in S^{-q}$.

Thus, $a(S,i)$ coincides with $\rho a(S) + (1-\rho) r$ in all coordinates but (at most) the $i$th. Moreover, both $a(S)$ and $r$ belong to $V(S)$, and so does $\rho a(S) + (1-\rho) r$. Thus, by (C.1), $a(S,i)^i \geq \rho a(S)^i + (1-\rho) r^i$. By averaging over $i$, we have $a(S)^i \geq \rho a(S)^i + (1-\rho) r^i$ and thus $a(S)^i \geq r^i$. We have then $a(S) \geq r^S$. Since $r^S \in \partial V(S)$ and $a(S) \in V(S)$, we conclude that $a(S) = r^S$.

Now, we have

$$c(S,i) = \left(-(|\mathcal{C}| - 1) r^i, r^{S \backslash i}\right).$$

By (A.4), $r^i \geq 0$ and thus $c(S,i) \leq \left(0, r^{S \backslash i}\right)$. By monotonicity, $\left(0, r^{S \backslash i}\right) \in V(S)$. By comprehensiveness, $c(S,i) \in V(S)$. ■

We prove now Proposition 5. We prove the following stronger result:

**Proposition 5'** *The proposals in any equilibrium of an NTU game satisfying (A.5) are characterized by (C.1), (C.2) and (C.3). Moreover, all the proposals are accepted and* $a(S) \geq 0^S$ *for all* $S \subset N$.

We proceed by induction. The result holds trivially when $n = 1$. Assume that it is true when there are at most $n - 1$ players.

Assume we are in an equilibrium. By induction hypothesis, the expected payoff for the players in $S \subsetneq N$ in any equilibrium with $S$ as set of active players is $a(S)$. Let $b_N \in \mathbb{R}^N$ be the expected payoff when $N$ is the set of active players. We must prove that (C.1), (C.2), and (C.3) hold for $S = N$.

We proceed by a series of Claims:

> *Claim (A)*: Given $C_q \in \mathcal{C}$ in the second stage, assume the proposers are determined by $\gamma \in \Gamma$ and the r.p. is $\gamma_q$. Then, all players in $N^{-q}$ accept $\gamma_q$'s proposal if $a\left(N, \gamma_q\right)^i \geq \rho b_N^i + (1-\rho) a\left(N^{-q}\right)^i$ for every $i \in N^{-q}$. Otherwise, the proposal is rejected.

Notice that, in the case of rejection in the second stage, the expected payoff of a player $i \in N^{-q}$ is, by induction hypothesis, $\rho b_N^i + (1-\rho)a(N^{-q})^i$. Thus, by a standard argument in this kind of bargaining, and making use of the tie-breaking rule, we conclude the result.

> *Claim (B)*: Let $\gamma \in \Gamma$ be the correspondence which determines the set of proposers in the first stage. Given $C_q \in \mathcal{C}$, assume we are in the subgame which begins after player $\gamma_q$ makes his proposal. Assume also that all the coalitions which choose representative after $C_q$ are bound to choose their proposer as representative should $\gamma_q$'s proposal be accepted. Let $b_{N,\gamma_q}$ be the expected final payoff in equilibrium if $\gamma_q$'s proposal is accepted. Then, all players in $C_q \backslash \gamma_q$ accept $\gamma_q$'s proposal if $b_{N,\gamma_q}^i \geq \rho b_N^i + (1-\rho)\, a\left(N \backslash \gamma_q\right)^i$ for every $i \in C_q \backslash \gamma_q$. Otherwise, the proposal is rejected.

Notice that, under our hypothesis, in the case of rejection of $\gamma_q$'s proposal in the first stage, the expected payoff to a player $i \in C_q \backslash \gamma_q$ is $\rho b_N^i + (1-\rho)a(N \backslash \gamma_q)^i$. By similar arguments to those used in the proof of Claim (A), we prove the result.

> *Claim (C)*: All the offers in the first stage are accepted.

Assume coalitions play the first stage in the order $(C_1, C_2, ..., C_p)$ and that the mechanism reaches coalition $C_p$, i.e. there has been no previous rejection. Assume the proposal of $\gamma_p$ is rejected. This means the final payoff for player $\gamma_p$ is $\rho b_N^{\gamma_p}$.

We define a new proposal $a\left(N, \gamma_p\right)$ for player $\gamma_p$ as follows: Let $c\left(N, \gamma_p\right)$ be defined as in (5). By (A.5) and induction hypothesis, $c\left(N, \gamma_p\right) \in V(N)$. By convexity, $\rho b_N + (1-\rho)\, c\left(N, \gamma_p\right) \in V(N)$. Let $a\left(N, \gamma_p\right) = \rho b_N + (1-\rho)\, c\left(N, \gamma_p\right)$.

In case of rejection, the expected final payoff for any player $i \in C_p \backslash \gamma_p$ is $\rho b_N^i + (1-\rho)\, a\left(N \backslash \gamma_p\right)^i$.

If all players in $C_p \backslash \gamma_p$ accept $a\left(N, \gamma_p\right)$ and the proposal chosen in the second stage is from $C_r \neq C_p$, then any player $i \in C_p \backslash \gamma_p$ can obtain $\rho b_N^i + (1-\rho)\, a\left(N^{-r}\right)^i$ by rejecting it. If the proposal chosen in the second stage is from $C_p$, then it is accepted (by *Claim (A)*).

Thus, if all players in $C_p \backslash \gamma_p$ accept $a\left(N, \gamma_p\right)$, their expected final payoff is at least

$$
\begin{aligned}
& \frac{1}{|\mathcal{C}|} \sum_{C_r \in \mathcal{C}^{-p}} \left[\rho b_N^{C_p \backslash \gamma_p} + (1-\rho)\, a\left(N^{-r}\right)^{C_p \backslash \gamma_p}\right] + \frac{1}{|\mathcal{C}|} a\left(N, \gamma_p\right)^{C_p \backslash \gamma_p} \\
= \quad & \rho b_N^{C_p \backslash \gamma_p} + (1-\rho)\, a\left(N \backslash \gamma_p\right)^{C_p \backslash \gamma_p}.
\end{aligned}
$$

Thus, by the tie-breaking rule, it is optimal for players in $C_p \backslash \gamma_p$ to accept $a\left(N, \gamma_p\right)$. Furthermore, the expected final payoff for player $\gamma_p$ is not less than

$\rho b_N^{\gamma_p}$. So, by the tie-breaking rule, it is optimal for $\gamma_p$ to change his proposal to $a\left(N,\gamma_p\right)$. This contradiction proves that no proposals are rejected in the first stage in $C_p$. By going backwards, we prove that no proposal is rejected in the first stage in $C_{p-1}, ..., C_1$.

*Claim (D)*: All the offers in the second stage are accepted.

Suppose the proposal of $\gamma_q$ is bound to be rejected in the second stage. Then, the final payoff for the members of $C_q$ (including $\gamma_q$) is 0 with probability $\frac{1}{|\mathcal{C}|} > 0$. By *Claim (B)*, we know that $b^i_{N,\gamma_q} \geq \rho b^i_N + (1-\rho)\, a\left(N\backslash\gamma_q\right)^i$ for all $i \in C_q\backslash\gamma_q$. Assume that $\gamma_q$ changes his strategy and proposes

$$a\left(N,\gamma_q\right) = \left(0^{C_q}, \rho b_N^{N^{-q}} + (1-\rho)\, a\left(N^{-q}\right)^{N^{-q}}\right).$$

By convexity and monotonicity, $a\left(N,\gamma_q\right) \in V(N)$. By *Claim (A)*, this proposal is bound to be accepted should $\gamma_q$ be the r.p. in the second stage. However, $b_{N,\gamma_q}$ remains unaltered. So, by *Claim (B)*, $a\left(N,\gamma_q\right)$ is also accepted in the first stage. Moreover, the expected final payoff for $\gamma_q$ also remains the same. By the tie-breaking rule, we are not in an equilibrium. This contradiction proves that the proposals in the second stage are always accepted.

Since all the proposals are accepted, we can assure that $b_N = a(N)$ and $b_{N,i} = a(N|_i)$ for all $i \in N$.

We show now (C.1), (C.2), and (C.3) hold.

Suppose (C.1) does not hold, i.e., there exists a player $i \in C_q$ such that $a(N,i)$ is not Pareto optimal. Thus, $a(N,i)$ belongs to the interior of $V(N)$; so, there exists $\varepsilon > 0$ such that $d := a(N,i) + \left(\varepsilon, 0^{N\backslash i}\right) \in V(N)$.

Notice that, since the proposal $a(N,i)$ of player $i$ is accepted, by *Claim (B)*, together with *Claim (C)* and *Claim (D)*, we know that $a(N|_i)^j \geq \rho b^j_N + (1-\rho)a(N\backslash i)^j$ for every $j \in C_q\backslash i$ and, by *Claim (A)*, $a(N,i)^j \geq \rho b^j_N + (1-\rho)a(N^{-q})^j$ for every $j \in N^{-q}$. So, if player $i$ changes his proposal to $d$, it is bound to be accepted and his expected final payoff improves by $\dfrac{\varepsilon}{|\mathcal{C}|\,|C_q|} > 0$. This contradiction proves (C.1).

Suppose (C.2) does not hold. Let $j_0 \in C_q\backslash i$ be a player such that $a(N|_i)^{j_0} = \rho a(N)^{j_0} + (1-\rho)a(N\backslash i)^{j_0} + \alpha$ with $\alpha \neq 0$. By *Claim (B)*, $\alpha > 0$.

By comprehensiveness and nonlevelness, we have $a(N,i) - \left(|\mathcal{C}|\,\alpha, 0^{N\backslash j_0}\right)$ belongs to the interior of $V(N)$. Thus, there exists an $\varepsilon > 0$ such that

$$\widehat{a}(N,i) := a(N,i) - \left(|\mathcal{C}|\,\alpha, 0^{N\backslash j_0}\right) + \left(\varepsilon, 0^{N\backslash i}\right)$$

belongs to $V(N)$. Suppose player $i$ changes his proposal to $\widehat{a}(N,i)$. The new value $\widehat{a}(N|_i)$ satisfies the conditions of *Claim (A)* and *Claim (B)*, and thus the new proposal of player $i$ is due to be accepted. Also, player $i$ improves his expected payoff by $\dfrac{\varepsilon}{|\mathcal{C}|\,|C_q|} > 0$. This contradiction proves (C.2).

The reasoning for (C.3) is similar to that for (C.2) so we omit it.

It remains to show that $a(N) \geq 0$. Notice that player $i \in N$ can guarantee himself a payoff of at least 0 by proposing always $0^N$ and accepting only proposals which give him a nonnegative expected payoff. Thus, $a(N) \geq 0$.

Conversely, we show that proposals $(a(S,i)_{i\in S})_{S\subset N}$ satisfying (C.1), (C.2) and (C.3) can be supported as an equilibrium.

First, we prove that $a(S) \geq 0$ for all $S \subset N$. By induction hypothesis, this is true for any $S \subsetneq N$. Given $i \in C_q \in \mathcal{C}$, by (A.5), we have $c(N,i) \in V(N)$. By convexity, $\widetilde{c}(N,i) := \rho a(N) + (1-\rho) c(N,i) \in V(N)$.

Since $a(N,i)$ satisfies (C.2) and (C.3), by Proposition 2, $a(N,i)$ also satisfies (C.2'). Then, $a(N,i)^{N\setminus i} = \widetilde{c}(N,i)^{N\setminus i}$. We now conclude that $a(N,i) \geq \widetilde{c}(N,i)$ because $a(N,i) \in \partial V(N)$ and $\widetilde{c}(N,i) \in V(N)$. Hence,

$$a(N,i)^i \geq \widetilde{c}(N,i)^i = \rho a(N)^i - (1-\rho) \sum_{C_r \in \mathcal{C}^{-q}} a\left(N^{-r}\right)^i.$$

So, by (C.3), $a(N|_i)^i = \rho a(N)^i$.

Furthermore, by (C.2) and $a(N\setminus j) \geq 0$, we have $a(N|_j)^i \geq \rho a(N)^i$ for all $j \in C_q \setminus i$. Thus,

$$a(N)^i = \frac{1}{|C_q|} \sum_{j\in C_q} a(N|_j)^i \geq \frac{1}{|C_q|} \sum_{j\in C_q} \rho a(N)^i = \rho a(N)^i$$

and so $a(N)^i \geq 0$. Moreover,

$$a(N|_i)^i \geq \rho a(N)^i \geq 0.$$

Now, following a similar reasoning to that in Proposition 1 in H&M, it is straighforward to verify that the strategies corresponding to these proposals form an equilibrium.■

## 5.5 *Proof* of Proposition 9

We prove the following stronger result:

**Proposition 9'** *Let $(N, V, \mathcal{C})$ be an NTU game with coalition structure satisfying (A.5). Then, for each $\rho \in [0,1)$, there exists an equilibrium.*

By Proposition 5', we only need to prove that there exist proposals satisfying (C.1), (C.2), and (C.3). We proceed by induction on the number of players. Clearly, the result is true for $n = 1$. Assume now that we have $a(S,i)$ for each $S \subset N$ and each $i \in S$ satisfying (C.1), (C.2), and (C.3) when $S \subsetneq N$. By Proposition 5', $a(S) \geq 0$ for all $S \subsetneq N$.

For each $i \in C_q \in \mathcal{C}$, by property (A.5), the vector $c(N,i)$ belongs to $V(N)$. By property (A.2), there exists a unique $z_i \in \partial V(N)$ such that $z_i^j = c(N,i)^j$ for all $j \in N \backslash i$. We define

$$\begin{aligned}
&\beta_1 := \min\{a^i(S) : i \in S \subsetneq N\} \in \mathbb{R}\\
&\beta_2 := \min\left\{z_j^i : i,j \in N\right\} \in \mathbb{R}\\
&\beta := \min(\beta_1, \beta_2) \in \mathbb{R}\\
&K := \{x \in V(N) : x \geq (\beta, ..., \beta)\}.
\end{aligned}$$

This set $K$ is nonempty ($z_i \in K$ for all $i \in N$), closed (because $V(N)$ is closed) and bounded (by (A.1)). Thus, $K$ is a compact set. Furthermore, $K$ is convex (because $V(N)$ is convex).

We define $n$ functions $\alpha_i : K \to K$ as follows. Given $i \in C_q \in \mathcal{C}$, $\alpha_i^j(x) := \rho x^j + (1-\rho)c(N,i)^j$ for each $j \in N \backslash i$ and $\alpha_i^i(x)$ is defined in such a way that $\alpha_i(x) \in \partial V(N)$.

These functions are well defined, because $y_i := \rho x + (1-\rho)z_i$ belongs to $K$ (by convexity) and $\alpha_i(x)$ equals $y_i$ in all coordinates but $i$'s, which we increase until reaching the boundary of $V(N)$.

Also, because of the smoothness of property (A.2) the functions $\alpha_i$ are continuous. By the convexity of the domain, $\frac{1}{|\mathcal{C}|}\sum_{C_q \in \mathcal{C}} \frac{1}{|C_q|}\sum_{i \in C_q} \alpha_i(x) \in K$ for each $x \in K$. By a standard fix point theorem, there exists a vector $a(N) \in K$ satisfying $a(N) = \frac{1}{|\mathcal{C}|}\sum_{C_q \in \mathcal{C}} \frac{1}{|C_q|}\sum_{i \in C_q} \alpha_i(a(N))$.

We define $a(N,i) = \alpha_i(a(N))$ for each $i \in N$. It is trivial to see that $(a(N,i))_{i \in N}$ satisfies (C.1), (C.2'), and (C.3). By Proposition 2, $(a(N,i))_{i \in N}$ also satisfies (C.2).■

## 5.6 *Proof* of Theorem 10

Notice that the consistent coalitional value coincides with the Owen value in TU games with coalition structure. We prove then the following stronger result:

**Theorem 10'** *Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure satisfying (A.5). Then, for each $\rho \in [0,1)$, there exists a unique equilibrium. Furthermore, the equilibrium payoff configuration equals the unique consistent coalitional payoff configuration of $(N, V, C)$.*

Theorem 10' is an immediate consequence of Proposition 4', 5' and 9'.■

## 5.7 *Proof* of Theorem 12

We prove the following stronger result:

**Theorem 12'** *Let $(N, V, C)$ be an NTU game with coalition structure satisfying (A.5). If $a_\rho := (a_\rho(S))_{S \subset N}$ is an equilibrium payoff configuration for* *each $\rho$ and $a$ is the limit of $a_\rho$ when $\rho \to 1$, then $a$ is a consistent coalitional payoff configuration of $(N, V, C)$.*

By Bergantiños and Vidal-Puga (2002), it is enough to prove that $a = (a(S))_{S \subset N}$ satisfies (B.1), (B.2), and (B.3). By Corollary 8, $a_\rho(S,i) \to a(S)$ for any $i \in S \subset N$. Since $a_\rho(S,i) \in \partial V(S)$ for every $S \subset N$ and every $\rho \in [0,1)$, and $\partial V(S)$ is closed, we conclude that $a(S) \in \partial V(S)$ for every $S \subset N$. Thus, $a$ satisfies property (B.1) of the characterization of the consistent coalitional payoff configuration.

Let $\lambda_S$ be the orthogonal to $\partial V(S)$ unit length vector at $a(S)$ for each $S \subset N$. We associate to each $\rho$ a hyperplane game with coalition structure $(N, V_\rho, \mathcal{C})$ as follows:

Given $\rho \in [0,1)$ and $S \subset N$ with $|S|$ elements, there exists at least one hyperplane on $\mathbb{R}^S$ containing the $|S|$ points $\{a_\rho(S,i) : i \in S\}$. If there are more than one hyperplane, we take the one whose outward orthogonal unit length vector $\lambda_S(\rho)$ is the closest to $\lambda_S$.

We define:

$$V_\rho(S) := \left\{x \in \mathbb{R}^S : \lambda_S(\rho) \cdot x \leq \lambda_S(\rho) \cdot a_\rho(S,i), i \in S\right\}.$$

The half-space $V_\rho(S)$ is well defined because $\lambda_S(\rho) \cdot a_\rho(S,i) = \lambda_S(\rho) \cdot a_\rho(S,j)$ for all $i, j \in S$.

By Corollary 8, $a_\rho(S,i) \to a(S)$. By the smoothness of $\partial V(S)$, $\lambda_S(\rho) \to \lambda_S$. Therefore,

$$V_\rho(S) \to V'(S) := \left\{x \in \mathbb{R}^S : \lambda_S \cdot x \leq \lambda_S \cdot a(S)\right\}.$$

By Proposition 5', the proposals $\{a_\rho(S,i) : S \subset N, i \in S\}$ satisfy (C.1), (C.2), and (C.3) for $(N, V, \mathcal{C})$. But these properties are the same for $(N, V_\rho, \mathcal{C})$. Thus, by Proposition 5', $a_\rho$ is an equilibrium payoff configuration for $(N, V_\rho, \mathcal{C})$. By Theorem 10', this implies that $a_\rho$ is the only consistent coalitional payoff configuration for $(N, V_\rho, \mathcal{C})$.

Hence, each $a_\rho$ satisfies properties (B.1), (B.2), and (B.3) for vectors $\lambda_S(\rho)$. By the continuity of the marginal contributions with respect to the hyperplanes, the vector $a$ satisfies (B.1), (B.2), and (B.3). Hence, we conclude (by the definition of $\Phi$) that $a$ is a consistent coalitional payoff configuration of $(N, V, \mathcal{C})$.■

## References


Bergantiños G., Vidal-Puga J. J.: The NTU consistent coalitional value. Mimeo (2002)

Calvo E., Lasaga J., Winter E.: The principle of balanced contributions and hierarchies of cooperation. Mathematical Social Sciences **31,** 171-182 (1996)

Dasgupta A., Chiu Y. S.: On implementation via demand commitment games. International Journal of Game Theory **27 (2),** 161-189 (1998)

Evans R. A.: Value, consistency, and random coalition formation. Games and Economic Behavior **12,** 68-80 (1996)

Gul F.: Bargaining foundations of the Shapley value. Econometrica **57,** 81-95 (1989)
Harsanyi J. C.: A simplified bargaining model for the n-person cooperative game. International Economic Review **4,** 194-220 (1963)
Hart S., Mas-Colell A.: Bargaining and value. Econometrica **64 (2),** 357-380 (1996)
Hart O., Moore J.: Property rights and the nature of the firm. Journal of Political Economy **98,** 1119-1158 (1990).
Maschler M., Owen G.: The consistent Shapley value for hyperplane games. International Journal of Game Theory **18,** 389-407 (1989)
Maschler M., Owen G.: The consistent Shapley value for games without side payments. In: Selten, R. (ed.) Rational Interaction. pp. 5-12. New York: Springer-Verlag 1992.
Mutuswani S., Pérez-Castrillo, D., Wettstein, D.: Bidding for the surplus: Realizing efficient outcomes in general economic environments. Working Paper, UAB-IAE 479.01 (2002)
Myerson R. B.: Graphs and cooperation in games. Mathematics of Operations Research **2,** 225-229 (1977)
Nash J. F.: The bargaining problem. Econometrica **18,** 155-162 (1950)
Navarro N., Perea A.: Bargaining in networks and the Myerson value. Working paper 01-06, Universidad Carlos III de Madrid (2001)
Owen G.: Values of games with a priori unions. In: Henn R., Moeschlin O. (eds.) Essays in Mathematical Economics and Game Theory. pp. 76-88. Berlin: Springer-Verlag 1977.
Pérez-Castrillo D., Wettstein D.: Bidding for the surplus: A non-cooperative approach to the Shapley value. Journal of Economic Theory **100 (2),** 274-294 (2001)
Shapley S.: A value for n–person games. In: Huhn H. W., Tucker A. W. (eds.) Contributions to the Theory of Games II. Annals of Mathematics Studies **28,** pp. 307-317. Princeton: Princeton University Press 1953
Vidal-Puga J. J., Bergantiños G.: An implementation of the coalitional value. Games and Economic Behavior **44**, 412-427 (2003)
Winter E.: A value for cooperative games with level structure of cooperation. International Journal of Game Theory **18,** 227-240 (1989)
Winter E.: On non-transferable utility games with coalition structure. International Journal of Game Theory **20,** 53-63 (1991)
Winter E.: The demand commitment bargaining and snowballing cooperation. Economic Theory **4,** 255-273 (1994)